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NOTE ON A CLASS OF K-PARANORMAL WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE $\mathcal{F}^p(\mathbb{C})$

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Abstract. In this paper, we characterize the k-paranormal, isometric, spectral radius and the numerical radius of the weighted composition operator $C_{\psi,\phi}$ on the Fock space $\mathcal{F}^p(\mathbb{C})$ for $0 < p \leq \infty$, where ψ is of the form $\psi(0)e^{-\mu\overline{\nu}\zeta}$ and $\mu, \nu, \zeta \in \mathbb{C}$.

1. Introduction

On a space of analytic functions, the composition operator C_ϕ is deifned as $C_\phi(f) = f \circ \phi$ and the weighted composition operator $C_{\psi,\phi}$ is defined as $C_{\psi,\phi}(f) = \psi.f \circ \phi$, where ϕ is an analytic self map on $\mathbb{C}$ and ψ is an holomorphic function on $\mathbb{C}$.

The study of composition operators on function spaces attracted various authors over many decades. A vast number of papers appeared on studying the boundedness, compactness, etc. of composition operators acting on function spaces, namely Hardy spaces, Bergmann spaces etc. For more details, refer to [2, 5, 8].

One of the main objectives of these studies is to find the relationship between the fixed symbol ϕ and the weight function ψ with the operator $C_{\psi,\phi}$.

On a space of analytic functions f, for $p > 0$, the Fock space $\mathcal{F}^p(\mathbb{C})$ defined as a space of analytic functions f on $\mathbb{C}$ such that

$$\|f\|_p := \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |f(\zeta)|^p e^{-\frac{p|\zeta|^2}{2}} dA(\zeta)\Big)^{\frac{1}{p}} < \infty,$$

where dA is the usual Lebesque measure on $\mathbb{C}$. It is well known that for $1 \leq p \leq \infty$, $\mathcal{F}^p(\mathbb{C})$ is a Banach space and for $0 < p < 1$, $\mathcal{F}^p(\mathbb{C})$ is a complete metric space.

When $p = 2$, $\mathcal{F}^2(\mathbb{C})$ is a reproducing kernel Hilbert space (RKHS) with kernel $K_w(\zeta) = e^{\zeta\overline{w}}$ and the norm and the inner product are defined respectively as $\|f\|^2 = \frac{1}{\pi}\int_{\mathbb{C}} |f(\zeta)|^2 e^{-|\zeta|^2} dA(\zeta), \zeta \in \mathbb{C}$ and $\langle f, g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} f(\zeta)\overline{g(\zeta)}e^{-|\zeta|^2} dA(\zeta)$, where dA is the usual Lebesque measure on $\mathbb{C}$. In this case, the normalization of K_w is denoted as k_w (i.e) $k_w = \frac{K_w}{\|K_w\|}$.

In [1], the authors characterized bounded and compact composition operators on the Fock space $\mathcal{F}^2(\mathbb{C}^n)$, the Hilbert space of all holomorphic functions on $\mathbb{C}^n$. In [7], the author gave a simple characterization of bounded and compact weighted composition operators on the Fock space $\mathcal{F}^2$ over $\mathbb{C}$. On the other hand, in [4],

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the authors gave criteria for the boundedness and compactness of the weighted composition operators on the Fock space $\mathcal{F}^p(\mathbb{C})$. To learn more on the Fock space, one can refer to an excellent book [10]. In [9], the author characterized a special class of unitary weighted composition operators $C_{\psi,\phi}$ and their spectrum on the Fock space $\mathcal{F}^2(\mathbb{C})$, where the fixed symbol is of the form $\phi(\zeta) = \mu\zeta - \nu$ and the weighted function is of the form $\psi(\zeta) = \alpha k_{\overline{\mu}\nu}(\zeta)$, with $|\mu| = 1, |\alpha| = 1$.

Inspired by [9], in this paper, we characterize the k-paranormal, isometric, spectral radius and the numerical radius of the weighted composition operators $C_{\psi,\phi}$ on the Fock space $\mathcal{F}^p(\mathbb{C})$, where $\phi(\zeta) = \mu\zeta + \nu, |\mu| \leq 1$ and $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ with $|\mu| = 1$.

For a bounded operator T on a normed space $\mathcal{X}$, we have the following definitions

- T is paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$ for every $x \in \mathcal{X}$.
- T is k-paranormal if there exists an integer $k \geq 1$ such that $\|Tx\|^{k+1} \leq \|T^{k+1}x\|\|x\|$ for every $x \in \mathcal{X}$.
- T is normaloid if $\|T\| = r_\sigma(T)$, spectral radius of T.
- T is spectraloid if $r_\sigma(T) = r_w(T)$, numerical radius of T.

For a bounded linear operator T on a Hilbert space $\mathcal{H}$, we have

- Numerical range is defined as $W(T) = \{\langle Th, h\rangle | h \in \mathcal{H}, \|h\| = 1\}$.
- T is normal if $T^*T = TT^*$, i.e. T^* and T commute.

It is also well known that the following inclusion relationship is true for a bounded operator: Normal $\subset$ Quasinormal $\subset$ Subnormal $\subset$ Hyponormal $\subset$ Paranormal $\subset$ k-Paranormal $\subset$ Normaloid $\subset$ Spectraloid.

2. Preliminary Results

In this section, we list the well known results on weighted composition operators on $\mathcal{F}^p(\mathbb{C})$.

Theorem 2.1. ([1], Theorem 1) *Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is an analytic function.*

(a) *If C_ϕ is bounded on $\mathcal{F}^2(\mathbb{C})$, then $\phi(\zeta) = \mu\zeta + \nu$, where $\mu, \nu \in \mathbb{C}$, $|\mu| \leq 1$ and if $|\mu| = 1$, then $\nu = 0$.*

(b) *If C_ϕ is compact on $\mathcal{F}^2(\mathbb{C})$, then $\phi(\zeta) = \mu\zeta + \nu, |\mu| < 1$.*

By ([1], Theorem 2), the converse of the above theorem is also true.

Lemma 2.2. ([7], Proposition 2.1) *Let ψ and ϕ be two entire functions on $\mathbb{C}$ such that $\psi \not\equiv 0$. Suppose there is a positive constant K such that*

$$|\psi(\zeta)|^2 e^{|\phi(\zeta)|^2 - |\zeta|^2} \leq K$$

for all $\zeta \in \mathbb{C}$. Then $\phi(\zeta) = \mu\zeta + \nu$ for $|\mu| \leq 1$. If $|\mu| = 1$, then $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$.

Theorem 2.3. ([7], Theorem 2.2) *Suppose ψ, ϕ are analytic functions on $\mathbb{C}$ such that ψ is not identically zero. Then, $C_{\psi,\phi}$ is bounded iff ψ belongs to $\mathcal{F}^2(\mathbb{C})$, $\phi(\zeta) = \phi(0) + \lambda\zeta$ with $|\lambda| \leq 1$ and $M(\psi, \phi) := sup\{|\psi|^2 exp(|\phi(\zeta)|^2 - |\zeta|^2); \zeta \in \mathbb{C}\} < \infty$.*

Proposition 2.4. ([7], Proposition 3.1) *Let $\phi(\zeta) = \mu\zeta - \nu, |\mu| = 1$ and ν is an arbitrary complex number. Then $C_{k_{\overline{\mu}\nu},\phi}$ is an unitary operator on $\mathcal{F}^2(\mathbb{C})$.*

3. Main Results

In this section, we assume that for analytic functions ψ, ϕ on $\mathbb{C}$, there exists a positive constant M such that $|\psi(\zeta)|^2 e^{|\phi(\zeta)|^2-|\zeta|^2} \leq M$. Then, by Lemma 2.3, ϕ is of the form $\phi(\zeta) = \mu\zeta + \nu, |\mu| \leq 1$. Moreover, in this case when $|\mu| = 1$, we have $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$.

We begin the proof of our main results by defining the operator $S_{\mu,\nu}f(\zeta) := e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu\zeta + \nu)$.

Proposition 3.1. *For $n \geq 1$, let $\phi(\zeta) = \mu^n\zeta + \nu$ and $\psi(\zeta) = \psi(0)e^{-\mu^n\overline{\nu}\zeta}$ such that $|\mu| = 1$. Then $S_{\mu^n,\nu}f(\zeta) = e^{-\mu^n\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu^n\zeta + \nu)$ is an isometry on $\mathcal{F}^p(\mathbb{C})$.*

Proof. For $f \in \mathcal{F}^p(\mathbb{C})$, consider

$$\begin{aligned}
\|S_{\mu^n,\nu}f\| &= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |e^{-\mu^n\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu^n\zeta+\nu)|^p e^{-\frac{p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |e^{-\mu^n\overline{\nu}\zeta-\frac{|\nu|^2}{2}}|^p |f(\mu^n\zeta+\nu)|^p e^{-\frac{p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \qquad (3.1) \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |e^{-\mu^n\overline{\nu}\zeta-\frac{|\nu|^2}{2}-\frac{|\zeta|^2}{2}}|^p |f(\mu^n\zeta+\nu)|^p dA(\zeta)\Big)^{1/p}.
\end{aligned}$$

Since $|\mu| = 1$, we can derive that $e^{-\mu^n\overline{\nu}\zeta-\frac{|\nu|^2}{2}-\frac{|\zeta|^2}{2}} = e^{\frac{-|\mu^n\zeta+\nu|^2}{2}}$. Substituting this in (3.1), we get

$$\begin{aligned}
\|S_{\mu^n,\nu}f\| &= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |f(\mu^n\zeta+\nu)|^p |e^{\frac{-|\mu^n\zeta+\nu|^2}{2}}|^p dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |f(\mu^n\zeta+\nu)|^p e^{\frac{-p|\mu^n\zeta+\nu|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \|f\|.
\end{aligned}$$

Thus, $S_{\mu^n,\nu}$ is an isometry on $\mathcal{F}^p(\mathbb{C})$. □

Corollary 3.2. *Let $\phi(\zeta) = \mu\zeta + \nu$ and $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ such that $|\mu| = 1$. Then $S_{\mu,\nu}f(\zeta) = e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu\zeta + \nu)$ is a unitary on Hilbert Fock space $\mathcal{F}^2(\zeta)$.*

Proof. The proof follows from Proposition 2.4 with a change of the sign of μ in $C_{k_{\overline{\mu}\nu},\phi}$. □

Theorem 3.3. *Let $\phi(\zeta) = \mu\zeta + \nu$ and $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ such that $|\mu| = 1$. Then, $C_{\psi,\phi}$ is a k-paranormal operator on the Fock space $\mathcal{F}^p(\mathbb{C})$.*

Proof. For $f \in \mathcal{F}^p(\mathbb{C})$, consider

$$\begin{aligned}
C^k_{\psi,\phi} f(\zeta) &= C_{\psi,\phi} C_{\psi,\phi} \dots C_{\psi,\phi} f(\zeta) \\
&= \psi(\zeta) \cdot (\psi \circ \phi(\zeta)) \cdot (\psi \circ \phi \circ \phi(\zeta)) \dots (\psi \circ \phi \dots \phi(\zeta)) \cdot f(\phi \circ \phi \dots \phi(\zeta)) \\
&= \psi(\zeta) \cdot \psi(\phi(\zeta)) \cdot \psi(\phi^2(\zeta)) \dots \psi(\phi^{k-1}(\zeta)) \cdot f(\phi^k(\zeta)) \\
&= \psi(0) e^{-\mu\overline{\nu}\zeta} \cdot \psi(0) e^{-\mu\overline{\nu}\phi(\zeta)} \cdot \psi(0) e^{-\mu\overline{\nu}\phi^2(\zeta)} \dots \psi(0) e^{-\mu\overline{\nu}\phi^{k-1}(\zeta)} \cdot f(\phi^k(\zeta) \\
&= \psi(0)^k e^{-\overline{\nu}\zeta(\mu+\mu^2+\mu^3+\cdots+\mu^k)} e^{-|\nu|^2(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu)} \\
&\quad f(\mu^k\zeta + \nu(1+\mu+\mu^2+\cdots+\mu^{k-1})).
\end{aligned} \tag{3.2}$$

Taking $\rho = \nu(1+\mu+\mu^2+\cdots+\mu^{k-1})$ along with the fact $\mu^k\overline{\rho} = \overline{\nu}(\mu+\mu^2+\cdots+\mu^k)$ in (3.2), we get

$$\begin{aligned}
C^k_{\psi,\phi} f(\zeta) &= \psi(0)^k e^{-|\nu|^2(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu)} e^{-\mu^k\overline{\rho}\zeta} f(\mu^k\zeta+\rho) \\
&= \psi(0)^k e^{-|\nu|^2(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu)} e^{-\mu^k\overline{\rho}\zeta - \frac{|\rho|^2}{2}} e^{\frac{|\rho|^2}{2}} f(\mu^k\zeta+\rho) \\
&= \psi(0)^k e^{-|\nu|^2(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu)} e^{\frac{|\rho|^2}{2}} S_{\mu^k,\rho} f(\zeta).
\end{aligned}$$

This implies

$$\begin{aligned}
\|C^k_{\psi,\phi} f(\zeta)\| &= |\psi(0)|^k |e^{-|\nu|^2(\mu^{k-1}+2\mu^{k-2}+\cdots+(k-1)\mu)}| |e^{\frac{|\rho|^2}{2}}| \|S_{\mu^k,\rho} f(\zeta)\| \\
&= |\psi(0)|^k e^{\frac{-|\nu|^2}{2}[\mu^{k-1}+\overline{\mu}^{k-1}+2(\mu^{k-2}+\overline{\mu}^{k-2})+\cdots+(k-1)(\mu+\overline{\mu})]+\frac{|\rho|^2}{2}} \\
&\quad \|S_{\mu^k,\rho} f(\zeta)\|
\end{aligned} \tag{3.3}$$

since

$$\begin{aligned}
|\rho|^2 &= |\nu|^2 |1+\mu+\mu^2+\cdots+\mu^{k-1}|^2 \\
&= |\nu|^2 (1+\mu+\mu^2+\cdots+\mu^{k-1})(\overline{1+\mu+\mu^2+\cdots+\mu^{k-1}}) \\
&= |\nu|^2 (1+\mu+\mu^2+\cdots+\mu^{k-1})(1+\overline{\mu}+\overline{\mu}^2+\cdots+\overline{\mu}^{k-1}) \\
&= |\nu|^2 [\mu^{k-1}+\overline{\mu}^{k-1}+2(\mu^{k-2}+\overline{\mu}^{k-2})+\cdots+(k-1)(\mu+\overline{\mu})+k].
\end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.3) and using Proposition 3.1, we have

$$\|C^k_{\psi,\phi} f\| = |\psi(0)|^k e^{\frac{k|\nu|^2}{2}} \|f\|. \tag{3.5}$$

On the other hand,

$$\begin{aligned}
\|C_{\psi,\phi}f\| &= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(\zeta).f\circ\phi(\zeta)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p |e^{-\mu\overline{\nu}\zeta}|^p |f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p e^{\frac{p|\nu|^2}{2}} |e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}}|^p |f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p e^{\frac{p|\nu|^2}{2}} |e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= |\psi(0)|e^{\frac{|\nu|^2}{2}} \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |S_{\mu,\nu}(f(\zeta)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= |\psi(0)|e^{\frac{|\nu|^2}{2}} \|S_{\mu,\nu}f\|.
\end{aligned} \tag{3.6}$$

Applying Proposition 3.1 in (3.6), we get

$$\|C_{\psi,\phi}f\|^k = |\psi(0)|^k e^{\frac{k|\nu|^2}{2}} \|f\|^k. \tag{3.7}$$

From (3.5) and (3.7), we have $\|C_{\psi,\phi}f\|^k = \|C^k_{\psi,\phi}f\|\|f\|^{k-1}$. Hence, $C_{\psi,\phi}$ is a k-paranormal operator on $\mathcal{F}^p(\mathbb{C})$. □

Theorem 3.4. *Let $\phi(\zeta) = \mu\zeta + \nu$ and $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ such that $|\mu| = 1$. Then $C_{\psi,\phi}$ is an isometric operator on the Fock space $\mathcal{F}^p(\mathbb{C})$ if and only if $|\psi(0)| = e^{\frac{-|\nu|^2}{2}}$.*

Proof. Assume that $|\psi(0)| = e^{\frac{-|\nu|^2}{2}}$. For $f \in \mathcal{F}^p(\mathbb{C})$,

$$\begin{aligned}
\|C_{\psi,\phi}f\| &= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(\zeta).f\circ\phi(\zeta)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p |e^{-\mu\overline{\nu}\zeta}|^p |f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p e^{\frac{p|\nu|^2}{2}} |e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}}|^p |f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |\psi(0)|^p e^{\frac{p|\nu|^2}{2}} |e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}} f(\mu\zeta+\nu)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= |\psi(0)|e^{\frac{|\nu|^2}{2}} \Big(\frac{p}{2\pi}\int_{\mathbb{C}} |S_{\mu,\nu}(f(\zeta)|^p e^{\frac{-p|\zeta|^2}{2}} dA(\zeta)\Big)^{1/p} \\
&= |\psi(0)|e^{\frac{|\nu|^2}{2}} \|S_{\mu,\nu}f\|.
\end{aligned} \tag{3.8}$$

Using Corollary 3.2 and $|\psi(0)| = e^{\frac{-|\nu|^2}{2}}$ in (3.8), we get

$$\|C_{\psi,\phi}f\| = \|f\|.$$

Thus, $C_{\psi,\phi}$ is an isometric operator on the Fock space $\mathcal{F}^p(\mathbb{C})$. □

Next, we derive the spectrum of the weighted composition operators $C_{\psi,\phi}$ on the Fock space $\mathcal{F}^p(\mathbb{C})$.

Theorem 3.5. *Let* $\phi(\zeta) = \mu\zeta + \nu$ *and* $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ *such that* $|\mu| = 1$. *Then the spectral radius of* $C_{\psi,\phi}$ *is* $|\psi(0)|e^{\frac{|\nu|^2}{2}}$.

Proof. By Theorem 3.3, $C_{\psi,\phi}$ is a k-paranormal operator on the Fock space $\mathcal{F}^p(\mathbb{C}), p > 0$. Since every k-paranormal operator on a normed space is normaloid [6], we have $r_\sigma(C_{\psi,\phi}) = \|C_{\psi,\phi}\|$. From the proof of Theorem 3.4, we get

$$\|C_{\psi,\phi}\| = |\psi(0)|e^{\frac{|\nu|^2}{2}}\|S_{\mu,\nu}\|. \tag{3.9}$$

By equation (3.9) and Corollary 3.2, we get $r_\sigma(C_{\psi,\phi}) = \|C_{\psi,\phi}\| = |\psi(0)|e^{\frac{|\nu|^2}{2}}$. □

In the following result, we will characterize the spectraloid weighted composition operators on the Hilbert Fock space $\mathcal{F}^2(\mathbb{C})$.

Theorem 3.6. *Let* $\phi(\zeta) = \mu\zeta + \nu$ *and* $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ *such that* $|\mu| = 1$. *Then the numerical radius* $C_{\psi,\phi}$ *is* $|\psi(0)|e^{\frac{|\nu|^2}{2}}$.

Proof. For $f \in \mathcal{F}^2(\mathbb{C})$ with $\|f\| = 1$, we consider

$$\begin{aligned} C_{\psi,\phi}f(\zeta) = \psi(\zeta)f(\phi(\zeta)) &= \psi(0)e^{-\mu\overline{\nu}\zeta}f(\mu\zeta+\nu) \\ &= \psi(0)e^{\frac{|\nu|^2}{2}}e^{-\mu\overline{\nu}\zeta-\frac{|\nu|^2}{2}}f(\mu\zeta+\nu) \\ &= \psi(0)e^{\frac{|\nu|^2}{2}}S_{\mu,\nu}f(\zeta). \end{aligned} \tag{3.10}$$

Using (3.10), it follows that

$$\begin{aligned} \langle C_{\psi,\phi}f, f\rangle &= \langle \psi(0)e^{\frac{|\nu|^2}{2}}S_{\mu,\nu}f, f\rangle \\ &= \psi(0)e^{\frac{|\nu|^2}{2}}\langle S_{\mu,\nu}f, f\rangle. \end{aligned} \tag{3.11}$$

By Corolloary 3.2, $S_{\mu,\nu}$ is unitary on the Hilbert Fock space $\mathcal{F}^2(\mathbb{C})$. This implies that $S_{\mu,\nu}$ is normal. Hence, by Theorem 1.4–2 of [3], $r_\sigma(S_{\mu,\nu}) = r_w(S_{\mu,\nu}) = \|S_{\mu,\nu}\| = 1$.

Next, we will calculate the numerical radius of $C_{\psi,\phi}$ using (3.11),

$$\begin{aligned} W(C_{\psi,\phi}) &= \{\langle C_{\psi,\phi}f, f\rangle : f \in \mathcal{F}^2(\mathbb{C})\} \\ &= \{\psi(0)e^{\frac{|\nu|^2}{2}}\langle S_{\mu,\nu}f, f\rangle\}. \end{aligned} \tag{3.12}$$

From (3.12), we get $r_w(C_{\psi,\phi}) = |\psi(0)|e^{\frac{|\nu|^2}{2}}r_w(S_{\mu,\nu}) = |\psi(0)|e^{\frac{|\nu|^2}{2}}$. □

Corollary 3.7. *Let* $\phi(\zeta) = \mu\zeta + \nu$ *and* $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}$ *such that* $|\mu| = 1$. *Then* $r_\sigma(C_{\psi,\phi}) = r_w(C_{\psi,\phi}) = \|C_{\psi,\phi}\|$.

Conclusion

On the Fock space $\mathcal{F}^p(\mathbb{C})$ for $0 < p \leq \infty$, where ψ is of the form $\psi(\zeta) = \psi(0)e^{-\mu\overline{\nu}\zeta}, \mu, \nu, \zeta \in \mathbb{C}$, we obtained our main results that show under what conditions the weight composition operator $C_{\psi,\phi}$ is a k-paranormal operator and isometric. Also, we derived the spectral and numerical radius of the weighted composition operator $C_{\psi,\phi}$.

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