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COEFFICIENT ESTIMATES FOR SPECIAL SUBCLASSES OF *k*-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we consider two new subclasses $\mathcal{N}_{\Sigma_k}(\mu,\alpha,\tau)$ and $\mathcal{N}_{\Sigma_k}(\mu,\beta,\tau)$ of Σ_k consisting of analytic and k-fold symmetric bi-univalent functions defined in the open unit disc $\mathcal{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$. For functions belonging to the two classes introduced here, we derive their normalized forms. Furthermore, we find estimates of the initial coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for these functions. Several related classes are also considered and connections to previously known results are made.

1. Introduction

Let $\mathcal{S}$ denote the family of functions analytic in the open unit disc

$$\mathcal{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\},$$

and normalized by the conditions $f(0)=f'(0)-1=0$ and having the form:

$$f(z)=z+\sum_{j=2}^{\infty}a_jz^j. \tag{1.1}$$

Also, let $\mathcal{A}$ denote the subclass of functions in $\mathcal{S}$ which are univalent in $\mathcal{U}$. The Koebe One Quarter Theorem (e.g., see [3]) ensures that the image of $\mathcal{U}$ under every function $f\ (z)\in\mathcal{S}$ contains the disk of radius $\frac{1}{4}$. It is well known that every function f has an inverse f^{-1} satisfying:

$$f^{-1}(f(z))=z,\ (z\in\mathcal{U}) \text{ and } f(f^{-1}(w))=w,\ \left(|w|<r_0(f);\ r_0(f)\geq\tfrac{1}{4}\right),$$

where

$$f^{-1}(w)=g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots. \tag{1.2}$$

A function $f\in\mathcal{S}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of all bi-univalent functions in $\mathcal{U}$. Let Σ denote the class of all bi-univalent functions in $\mathcal{U}$. Examples of functions in class Σ are

$$h_1(z)=\frac{z}{1-z},\ h_2(z)=-\log(1-z),\ h_3(z)=\frac{1}{2}\log\left(\frac{1+z}{1-z}\right),\ z\in\mathcal{U}.$$

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For each function $f \in \mathcal{A}$, the function $h(z) = \sqrt[k]{f(z^k)}, (z \in \mathcal{U},\ k \in \mathbb{N})$ is univalent and maps the unit disc $\mathcal{U}$ into a region with k-fold symmetry. A function is said to be k-fold symmetric (see [7,8]) if it has the following normalized form:

$$f(z) = z + \sum_{j=1}^{\infty} a_{kj+1} z^{kj+1}, \quad (z \in \mathcal{U},\ k \in \mathbb{N}). \tag{1.3}$$

We denote $\mathcal{S}_k$ the class of k-fold symmetric univalent functions in $\mathcal{U}$, which are normalized by the series expansion (1.3). In fact, the functions in the class $\mathcal{A}$ are one-fold symmetric.

Analogously to the concept of k-fold symmetric univalent functions, their study gives some important results, such as the one saying that a function $f \in \Sigma$ generates a k-fold symmetric bi-univalent function for each $k \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), we obtain the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) \quad = \quad & w - a_{k+1} w^{k+1} + [(k+1)a_{k+1}^2 - a_{2k+1}] w^{2k+1} \\ & - \left[\frac{1}{2}(k+1)(3k+2)a_{k+1}^3 - (3k+2)a_{k+1}a_{2k+1} + a_{3k+1}\right] w^{3k+1} + \cdots , \end{aligned} \tag{1.4}$$

where $f^{-1} = g$. We denote by Σ_k the class of k-fold symmetric bi-univalent functions in $\mathcal{U}$. For $k = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ.

Some examples of k-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^k}{1-z^k}\right)^{\frac{1}{k}}, \ \left[-\log(1-z^k)\right]^{\frac{1}{k}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^k}{1-z^k}\right)\right]^{\frac{1}{k}}.$$

Recently, many authors investigated bounds for the various subclasses of k-fold symmetric bi-univalent functions (see [1,2,4,10,11,13]). This work aims to introduce the new subclasses $\mathcal{N}_{\Sigma_k}(\mu,\alpha,\tau)$ and $\mathcal{N}_{\Sigma_k}(\mu,\beta,\tau)$ of Σ_k and find estimates of the coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for functions in each of these new subclasses.

2. Main Results

Definition 2.1. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}(\mu,\alpha,\tau)$ if the following conditions are satisfied:

$$\left|\arg\left(1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(zf'(z) - f(z)\right)}{(1-\mu)f(z) + \mu z f'(z)}\right]\right)\right| < \frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left|\arg\left(1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(wg'(w) - g(w)\right)}{(1-\mu)g(w) + \mu w g'(w)}\right]\right)\right| < \frac{\alpha\pi}{2} \tag{2.2}$$

$$(0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \tau \in \mathbb{C}\backslash\{0\};\ z,\ w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.4).

Definition 2.2. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ if the following conditions are satisfied:

$$Re\left(1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(zf'(z) - f(z)\right)}{(1-\mu)f(z) + \mu z f'(z)}\right]\right) > \beta \tag{2.3}$$

and

$$Re\left(1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(wg'(w) - g(w)\right)}{(1-\mu)g(w) + \mu w g'(w)}\right]\right) > \beta \tag{2.4}$$

$$(0 \leq \beta < 1;\ 0 \leq \mu < 1;\ \tau \in \mathbb{C}\backslash\{0\};\ z,\ w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.4).

Lemma 2.3. (See [6]) *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in $\mathcal{U}$, for which*

$$Re(h(z)) > 0,\ (z \in \mathcal{U})$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots,\ (z \in \mathcal{U}).$$

Theorem 2.4. *Let $f \in \mathcal{N}_{\Sigma_k}(\mu, \alpha, \tau)$ $(0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \tau \in \mathbb{C}\backslash\{0\})$ be of the form* (1.3)*. Then*

$$|a_{k+1}| \leq \frac{2\alpha|\tau|}{k(1-\mu)\sqrt{|1 + \alpha(2\tau - 1)|}} \tag{2.5}$$

and

$$|a_{2k+1}| \leq \frac{2\alpha^2|\tau|^2(k+1)}{k^2(1-\mu)^2} + \frac{\alpha|\tau|}{k(1-\mu)}.$$

Proof. It follows from (2.1) and (2.2) that

$$1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(zf'(z) - f(z)\right)}{(1-\mu)f(z) + \mu z f'(z)}\right] = [p(z)]^{\alpha} \tag{2.6}$$

and

$$1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(wg'(w) - g(w)\right)}{(1-\mu)g(w) + \mu w g'(w)}\right] = [q(w)]^{\alpha}, \tag{2.7}$$

where the functions $p(z)$ and $q(w)$ are in $\mathcal{P}$ and have the following series representations:

$$p(z) = 1 + p_k z^k + p_{2k} z^{2k} + p_{3k} z^{3k} + ... \tag{2.8}$$

and

$$q(w) = 1 + q_k w^k + q_{2k} w^{2k} + q_{3k} w^{3k} + ... \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we obtain

$$\frac{k(1-\mu)}{\tau} a_{k+1} = \alpha p_k, \tag{2.10}$$

$$\frac{(1-\mu)k}{\tau}[2a_{2k+1} - (1 + k\mu)a_{k+1}^2] = \alpha p_{2k} + \frac{\alpha(\alpha-1)}{2} p_k^2, \tag{2.11}$$

and

$$-\frac{k(1-\mu)}{\tau} a_{k+1} = \alpha q_k, \tag{2.12}$$

$$\frac{(1-\mu)k}{\tau}[2(k+1)a_{k+1}^2 - 2a_{2k+1} - (1 + k\mu)a_{k+1}^2] = \alpha q_{2k} + \frac{\alpha(\alpha-1)}{2} q_k^2. \tag{2.13}$$

From (2.10) and (2.12), we find

$$p_k = -q_k$$

and

$$2\frac{k^2(1-\mu)^2 a_{k+1}^2}{\tau^2} = \alpha^2(p_k^2 + q_k^2). \tag{2.14}$$

From (2.11), (2.13) and (2.14), we get

$$\begin{aligned}\frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau} &= \alpha(p_{2k} + q_{2k}) + \frac{\alpha(\alpha-1)}{2}\left(p_k^2 + q_k^2\right)\\ &= \alpha(p_{2k} + q_{2k}) + \frac{(\alpha-1)k^2(1-\mu)^2}{\alpha\tau^2}a_{k+1}^2.\end{aligned}$$

Therefore, we have

$$a_{k+1}^2 = \frac{\alpha^2\tau^2(p_{2k} + q_{2k})}{k^2(1-\mu)^2\left[1 + \alpha(2\tau - 1)\right]}.$$

Applying Lemma 2.3, for the coefficients p_{2k} and q_{2k}, we have

$$|a_{k+1}| \le \frac{2\alpha|\tau|}{k(1-\mu)\sqrt{|1 + \alpha(2\tau - 1)|}}.$$

This gives the desired bound for $|a_{k+1}|$ as asserted in (2.5). In order to find the bound on $|a_{2k+1}|$, by subtracting (2.13) from (2.11), we get

$$\frac{2k(1-\mu)}{\tau}\left[2a_{2k+1} - (k+1)a_{k+1}^2\right] = \alpha(p_{2k} - q_{2k}) + \frac{\alpha(\alpha-1)}{2}\left(p_k^2 - q_k^2\right). \tag{2.15}$$

It follows from (2.14) and (2.15) that

$$a_{2k+1} = \frac{\alpha^2\tau^2\left(p_k^2 + q_k^2\right)(k+1)}{4k^2(1-\mu)^2} + \frac{\alpha\tau(p_{2k} - q_{2k})}{4k(1-\mu)}.$$

Applying Lemma 2.3 once again for the coefficients p_k, p_{2k}, q_k and q_{2k}, we readily obtain

$$|a_{2k+1}| \le \frac{2\alpha^2|\tau|^2(k+1)}{k^2(1-\mu)^2} + \frac{\alpha|\tau|}{k(1-\mu)}.$$

The following theorem finds the estimates of the coefficients $|a_{k+1}|$ and $|a_{2k+1}|$ for functions in the class $\mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$. □

Theorem 2.5. *Let* $f \in \mathcal{N}_{\Sigma_k}(\mu, \beta, \tau)$ $(0 < \beta \le 1;\ 0 \le \mu < 1;\ \tau \in \mathbb{C}\backslash\{0\})$ *be of the form* (1.3). *Then*

$$|a_{k+1}| \le \frac{\sqrt{2|\tau|(1-\beta)}}{k(1-\mu)}, \tag{2.16}$$

$$|a_{2k+1}| \le \frac{|\tau|^2(1-\beta)^2(k+1)}{2k^2(1-\mu)^2} + \frac{|\tau|(1-\beta)}{k(1-\mu)}.$$

Proof. It follows from (2.3) and (2.4) that

$$1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(zf'(z) - f(z)\right)}{(1-\mu)f(z) + \mu zf'(z)}\right] = \beta + (1-\beta)p(z) \tag{2.17}$$

and

$$1 + \frac{1}{\tau}\left[\frac{(1-\mu)\left(wg'(w) - g(w)\right)}{(1-\mu)g(w) + \mu wg'(w)}\right] = \beta + (1-\beta)q(w), \tag{2.18}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. By suitably comparing the coefficients in (2.17) and (2.18), we get

$$\left[\frac{k(1-\mu)}{\tau}\right] a_{k+1} = (1-\beta)p_k, \tag{2.19}$$

$$\frac{2k(1-\mu)a_{2k+1} - k(1-\mu)(1+k\mu)a_{k+1}^2}{\tau} = (1-\beta)p_{2k}, \tag{2.20}$$

and

$$-\left[\frac{k(1-\mu)}{\tau}\right] a_{k+1} = (1-\beta)q_k, \tag{2.21}$$

$$\frac{2k(1-\mu)\left[(k+1)a_{k+1}^2 - a_{2k+1}\right] - k(1-\mu)(1+k\mu)a_{k+1}^2}{\tau} = (1-\beta)q_{2k}. \tag{2.22}$$

From (2.19) and (2.21), we find

$$p_k = -q_k \tag{2.23}$$

and

$$\frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau^2} = (1-\beta)^2\left(p_k^2 + q_k^2\right). \tag{2.24}$$

Adding (2.20) and (2.22), we have

$$\frac{2k^2(1-\mu)^2 a_{k+1}^2}{\tau} = (1-\beta)\left(p_{2k} + q_{2k}\right).$$

Applying Lemma 2.3, we obtain

$$|a_{k+1}| \leq \frac{\sqrt{2|\tau|(1-\beta)}}{k(1-\mu)}.$$

This is the bound on $|a_{k+1}|$ asserted in (2.16). In order to find the bound on $|a_{2k+1}|$, by subtracting (2.22) from (2.20), we get

$$\frac{2k(1-\mu)\left[2a_{2k+1} - (k+1)a_{k+1}^2\right]}{\tau} = (1-\beta)\left(p_{2k} - q_{2k}\right),$$

or equivalently,

$$a_{2k+1} = \frac{(k+1)a_{k+1}^2}{2} + \frac{\tau(1-\beta)\left(p_{2k} - q_{2k}\right)}{4k(1-\mu)}.$$

It follows from (2.23) and (2.24) that

$$a_{2k+1} = \frac{\tau^2(1-\beta)^2(k+1)\left(p_k^2 + q_k^2\right)}{4k^2(1-\mu)^2} + \frac{\tau(1-\beta)\left(p_{2k} - q_{2k}\right)}{4k(1-\mu)}.$$

Applying Lemma 2.3 once again for the coefficients p_k, p_{2k}, q_k and q_{2k}, we easiliy obtain

$$|a_{2k+1}| \leq \frac{|\tau|^2(1-\beta)^2(k+1)}{2k^2(1-\mu)^2} + \frac{|\tau|(1-\beta)}{k(1-\mu)}.$$

For one-fold symmetric bi-univalent functions and $\tau = 1$, Theorem 2.4 and Theorem 2.5 reduce to Corollary 2.8 and Corollary 2.9, respectively, which were proven very recently by Frasin [5] (see also [9]). □

Definition 2.6. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma}(\mu, \alpha)$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right)\right| < \frac{\alpha\pi}{2}$$

and

$$\left|\arg\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right)\right| < \frac{\alpha\pi}{2}$$

$$(0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ z,\ w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.2).

Definition 2.7. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{N}_{\Sigma}(\mu, \beta)$ if the following conditions are satisfied:

$$Re\left(\frac{zf'(z)}{(1-\mu)f(z)+\mu zf'(z)}\right) > \beta$$

and

$$Re\left(\frac{wg'(w)}{(1-\mu)g(w)+\mu wg'(w)}\right) > \beta$$

$$(0 \leq \beta < 1;\ 0 \leq \mu < 1;\ z,\ w \in \mathcal{U}),$$

where the function $g = f^{-1}$ is given by (1.2).

Corollary 2.8. *Let* $f \in \mathcal{N}_{\Sigma}(\mu, \alpha)$ $(0 < \alpha \leq 1;\ 0 \leq \mu < 1)$ *be of the form* (1.1). *Then*

$$|a_2| \leq \frac{2\alpha}{(1-\mu)\sqrt{1+\alpha}}$$

and

$$|a_3| \leq \frac{4\alpha^2}{(1-\mu)^2} + \frac{\alpha}{1-\mu}.$$

Corollary 2.9. *Let* $f \in \mathcal{N}_{\Sigma}(\mu, \beta)$ $(0 \leq \beta < 1;\ 0 \leq \mu < 1)$ *be of the form* (1.1). *Then*

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{1-\mu}$$

and

$$|a_3| \leq \frac{(1-\beta)^2}{(1-\mu)^2} + \frac{1-\beta}{1-\mu}.$$

If we set $\mu = 0$ and $\tau = 1$ in Theorem 2.4 and Theorem 2.5, then the classes $\mathcal{N}_{\Sigma_k}(\mu, \alpha)$ and $\mathcal{N}_{\Sigma_k}(\mu, \beta)$ reduce to the classes $\mathcal{N}^{\alpha}_{\Sigma_k}$ and $\mathcal{N}^{\beta}_{\Sigma_k}$ investigated recently by Srivastava et al. ([12]).

Definition 2.10. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}^{\alpha}_{\Sigma_k}$ $(0 < \alpha \leq 1)$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (w \in \mathcal{U})$$

and where the function g is given by (1.4).

Definition 2.11. A function $f \in \Sigma_k$ given by (1.3) is said to be in the class $\mathcal{N}^{\beta}_{\Sigma_k}$ $(0 \leq \beta < 1)$ if the following conditions are satisfied:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad (z \in \mathcal{U})$$

and

$$Re\left(\frac{wg'(w)}{g(w)}\right) > \beta \quad (w \in \mathcal{U}),$$

and where the function g is given by (1.4).

Corollary 2.12. *Let $f \in \mathcal{N}^{\alpha}_{\Sigma_k}$ $(0 < \alpha \leq 1)$ be of the form* (1.3)*. Then*

$$|a_{k+1}| \leq \frac{2\alpha}{k\sqrt{1+\alpha}}$$

and

$$|a_{2k+1}| \leq \frac{2\alpha^2(k+1)}{k^2} + \frac{\alpha}{k}.$$

Corollary 2.13. *Let $f \in \mathcal{N}^{\beta}_{\Sigma_k}$ $(0 \leq \beta < 1)$ be of the form* (1.3)*. Then*

$$|a_{k+1}| \leq \frac{\sqrt{2(1-\beta)}}{k}$$

and

$$|a_{2k+1}| \leq \frac{(1-\beta)^2(k+1)}{2k^2} + \frac{1-\beta}{k}.$$

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