Math. Appl. **7** (2018), 155–171
DOI: 10.13164/ma.2018.13

THE QUASITOPOS OF B-UNIFORM FILTER SPACES

DIETER LESEBERG AND ZOHREH VAZIRY

Abstract. In this paper a systematic study begins of the category **b-UFIL** of b-uniform filter spaces and b-uniformly continuous mappings. We will show that this construct forms a quasitopos in which quotients are stable under products. Consequently, it is extremely useful for further studies in the realm of Bounded topology.

1. Introduction

This paper continues our treatise on "Categories of several convergences". The present terminology is essentially the one used in the above mentioned paper [10]. Our focus is the consideration of so-called b-uniform filter spaces, which represent a natural generalization of several classical convergences such as uniform convergences, point-convergences, filtermerotopies and Cauchy spaces or suited set-convergences as well, see [1, 2, 4–6, 11, 13]. Moreover, following the idea of establishing a more convenient category with well-behaved properties such as being a quasitopos in the sense of Penon or Preuß [11], respectively, we will show that the category **b-UFIL** of b-uniform filter spaces possesses such desirable features. In addition, we can state that even quotient maps in **b-UFIL** are closed under formation of arbitrary products. Thus, this new established concept makes it easier for topologists to solve their problems since strong topological universes are extremely useful [11].

On the other hand, **b-UFIL** can be nicely embedded into the fundamental construct **b-CONV** of b-convergence spaces and corresponding maps [7–9], which additionally contains all set-convergences and **STOP**, the category of supertopological spaces and continuous maps in the sense of Doitchinov [3]. Hence, **b-UFIL** represents a bridge between all classical convergences and the broader concept of b-convergence in the setting of the fundamental concept called Bounded topology, in which structures on bounded sets or bornologies are defined. Here we should note that classical concepts are essentially working on the power set of a given set X.

MSC (2010): primary 54A20, 54B30; secondary 54E15.
Keywords: set-convergence, point-convergence, b-uniform filter space, b-filter space, completeness, compactness, precompactness, topological construct, strong topological universe, bounded topology.

2. The construct b-UFIL

Definition 2.1. For a set X a pair $(\mathcal{B}^X, \mu)$ consisting of a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ and a non-empty set $\mu \subset FIL(X \times X)$ of uniform filters is called a *b-uniform filter structure* on X, and the triple $(X, \mathcal{B}^X, \mu)$ a *b-uniform filter space* provided that the following axioms are satisfied:

(buf_1) $B_1 \subset B \in \mathcal{B}^X$ imply $B_1 \in \mathcal{B}^X$;
(buf_2) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(buf_3) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \mu$;
(buf_4) $\mathcal{U} \in \mu$ and $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X \times X)$ imply $\mathcal{U}_1 \in \mu$.

Here, $\underline{P}X$ denotes the power set of X, $FIL(X \times X)$ is the set of all filters on $X \times X$ and its elements are called *uniform* filters (on X) and for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\overset{\bullet}{B} := \{A \subset X : A \supset B\}$.

Given a pair of b-uniform filter spaces $(X, \mathcal{B}^X, \mu_X)$, $(Y, \mathcal{B}^Y, \mu_Y)$, a map $f : X \longrightarrow Y$ is called *b-uniformly continuous*, in short *buc*, if f satisfies the following conditions:

(buc_1) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;
(buc_2) $\mathcal{U} \in \mu$ implies $(f \times f)(\mathcal{U}) \in \mu_Y$, where $(f \times f)(\mathcal{U}) := \{V \subset Y \times Y : \exists\, U \in \mathcal{U}$ such that $(f \times f)[U] \subset V\}$ with $(f \times f)[U] := \{(f \times f)(x_1, x_2) : (x_1, x_2) \in U\} = \{(f(x_1), f(x_2)) : (x_1, x_2) \in U\}$.

By **b-UFIL** we will denote the category of b-uniform filter spaces and b-uniformly continuous maps.

Proposition 2.2. b-UFIL *is a topological construct in the sense of* [11].

Proof. For a set X, let I be a class, $(X_i, \mu_i)_{i \in I}$ a family of b–uniform filter spaces and $(f_i : X \longrightarrow X_i)_{i \in I}$ a family of maps. Then, $(\mathcal{B}_I^X, \mu_X^I)$ is the initial **b-UFIL** structure on X, where $\mathcal{B}_I^X := \{B \subset X : \forall\, i \in I, f_i[B] \in \mathcal{B}^{X_i}\}$ and $\mu_X^I := \{\mathcal{U} \in FIL(X \times X) : \forall i \in I,\ (f_i \times f_i)(\mathcal{U}) \in \mu_i\}$.

Evidently, $(\mathcal{B}_I^X, \mu_X^I)$ satisfies the axioms (buf_1), (buf_2), and (buf_4), respectively.

To (buf_3): For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ we are getting $(f_i \times f_i)(\overset{\bullet}{B} \times \overset{\bullet}{B}) = f_i(\overset{\bullet}{B}) \times f_i(\overset{\bullet}{B})$ $= f[\overset{\bullet}{B}] \times f_i[\overset{\bullet}{B}] \in \mu_i$ for each $i \in I$, hence, $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \mu_X^I$ follows.

Here, in general, for $\mathcal{U}_1, \mathcal{U}_2 \in FIL(X \times X)$, their *cross-product* is defined by setting:

$$\mathcal{U}_1 \times \mathcal{U}_2 := \{R \subset X \times X : \exists U_1 \in \mathcal{U}_1, \exists U_2 \in \mathcal{U}_2 \quad \text{such that} \quad R \supset U_1 \times U_2\}.$$

By definition of $(\mathcal{B}_I^X, \mu_X^I)$, each f_i is b-uniformly continuous. Now let

$$(Y, \mathcal{B}^Y, \mu_Y)$$

be a b-uniform filter space and $g : Y \longrightarrow X$ be a mapping such that, for each $i \in I$, $f_i \circ g : (Y, \mathcal{B}^Y, \mu_Y) \longrightarrow (X_i, \mathcal{B}^{X_i}, \mu_i)$ is b-uniformly continuous. We have to show that $g : (Y, \mathcal{B}^Y, \mu_Y) \longrightarrow (X, \mathcal{B}_I^X, \mu_X^I)$ is b-uniformly continuous.

To (buc_1): $B \in \mathcal{B}^Y$ implies $(f_i \circ g)[B] = f_i[g[B]] \in \mathcal{B}^{X_i}$ for each $i \in I$ by the assumption. Hence, $g[B] \in \mathcal{B}_I^X$ follows.

To (buc_2): $\mathcal{U} \in \mu_Y$ implies $(f_i \times f_i)((g \times g)(\mathcal{U})) = ((f_i \circ g) \times (f_i \circ g))(\mathcal{U}) \in \mu_i$ for each $i \in I$ by the hypothesis. Consequently, $(g \times g)(\mathcal{U}) \in \mu_X^I$ results, showing that $g : (Y, \mathcal{B}^Y, \mu_Y) \longrightarrow (X, \mathcal{B}_I^X, \mu_X^I)$ is b-uniformly continuous. Since $(\mathcal{B}^X, \mu) \in \underline{P}(\underline{P}X) \times \underline{P}(FIL(X \times X))$ is valid, the class of all b-uniform filter structures on X is a set.

Finally, the only b-uniform filter structure on a set X with *card* $X = 1$ is the pair $(\{\emptyset, \{x\}\}, \{\dot{x} \times \dot{x}, \underline{P}(\{x\} \times \{x\})\})$, where x denotes the element of X. If X is empty, then $(\{\emptyset\}, \{\{\emptyset\}\})$ represents the only b-uniform filter structure on X. □

Remark 2.3. As already observed in our former paper [10], preuniform convergence spaces and b-uniform filter spaces essentially coincide if and only if the assumed boundedness $\mathcal{B}^X$ is discrete, meaning that $\mathcal{B}^X = \{\emptyset\} \cup \{\{x\} : x \in X\} := \mathcal{D}^X$ (compare with Definition 2.12). Further, we repeat that the corresponding named category **DISb-UFIL** is bicoreflective in **b-UFIL** and itself forms a strong topological universe, see [11]. On the other hand, some important set-convergences are in a one to one correspondence to b-uniform filter spaces and, finally, generalized filter merotopies introduced as b-filter spaces can also be nicely embedded into b-UFIL.

Thus, our introduced concept can be regarded as a suitable tool for studying all the mentioned constructs in common. Moreover, in this context let us still mention the fact that basic properties of spaces such as compactness, precompactness and completeness can be newly defined in b-UFIL, and fundamental theorems, as for example that of Tychonoff, find corresponding expressions [10].

Remark 2.4. Since **b-UFIL** is a topological construct, the set of all b-uniform filter structures on X forms a complete lattice. So the following definition makes sense.

Definition 2.5. Let X be a set and let $(\mathcal{B}_1^X, \mu_1)$, $(\mathcal{B}_2^X, \mu_2)$ be b-uniform filter structures on X. Then we are setting:

$$(\mathcal{B}_1^X, \mu_1) \leq (\mathcal{B}_2^X, \mu_2) \text{ if and only if } \mathcal{B}_1^X \subset \mathcal{B}_2^X \text{ and } \mu_1 \subset \mu_2.$$

$(\mathcal{B}_1^X, \mu_1)$ is said to be *finer* than $(\mathcal{B}_2^X, \mu_2)$ and $(\mathcal{B}_2^X, \mu_2)$ is said to be *coarser* than $(\mathcal{B}_1^X, \mu_1)$.

Remark 2.6. The initial b-uniform filter structure on a set X with respect to $(X, f_i, (X_i, \mathcal{B}_i^X, \mu_i), I)$ is the *coarsest* b-uniform filter structure on X such that f_i is b-uniformly continuous for each $i \in I$.

In the event of I being the empty class, $(\mathcal{B}_I^X, \mu_X^I) = (\underline{P}X, FIL(X \times X))$, and it is called an *indiscrete* b-uniform filter structure on X.

Proposition 2.7. *Let $(X, \mathcal{B}^X, \mu)$ be a b-uniform filter space and $A \subset X$. Then, $(\mathcal{B}^A, \mu_A)$ is a b-uniform filter structure on A, where $\mathcal{B}^A := \{B \cap A : B \in \mathcal{B}^X\}$ and $\mu_A := \{\mathcal{U}_A : \mathcal{U} \in \mu\}$ with $\mathcal{U}_A := \{U \cap A \times A : U \in \mathcal{U}\}$, such that $(A, \mathcal{B}^A, \mu_A)$ represents the b-uniform filter subspace of $(X, \mathcal{B}^X, \mu)$ in* **b-UFIL**, *meaning that $(\mathcal{B}^A, \mu_A)$ is the initial b-uniform filter structure on A with respect to $(X, i, (X, \mathcal{B}^X, \mu))$, where $i : A \longrightarrow X$ denotes the inclusion map.*

Proof. First let us denote by $i : A \longrightarrow X$ the corresponding inclusion map. Evidently, $\mathcal{B}^A$ is not empty.

To (buf_1): Let $B_1 \subset B \cap A$ for some $B \in \mathcal{B}^X$. Then, $B_1 \in \mathcal{B}^X$ with $B_1 = B_1 \cap A$, and $B_1 \in \mathcal{B}^A$ follows.

To (buf$_2$): $x \in A$ implies $\{x\} \in \mathcal{B}^X$ with $\{x\} = \{x\} \cap A$, and $\{x\} \in \mathcal{B}^A$ results. μ_A is not empty, since $\mu \neq \emptyset$.

First, we show $\mathcal{U}_A \in FIL(A \times A)$. $\mathcal{U}_A \neq \emptyset$, since $\mathcal{U} \neq \emptyset$. Now let $U_1 \cap A \times A$ for some $U_1 \in \mathcal{U}$ and $U_2 \cap A \times A$ for some $U_2 \in \mathcal{U}$ be given. Then, $(U_1 \cap A \times A) \cap (U_2 \cap A \times A) = (U_1 \cap U_2) \cap (A \times A)$ with $U_1 \cap U_2 \in \mathcal{U}$, so that the intersection of the given elements of $\mathcal{U}_A$ belongs to $\mathcal{U}_A$.

Finally, let $(U_1 \cap A \times A) \cap (U_2 \cap A \times A)$ be an element of $\mathcal{U}_A$ with $U_1, U_2 \in \mathcal{U}$. We have to show that $U_1 \cap A \times A$ as well as $U_2 \cap A \times A$ belongs to $\mathcal{U}_A$. By the hypothesis, we are getting $(U_1 \cap A \times A) \cap (U_2 \cap A \times A) = U \cap A \times A$ for some $U \in \mathcal{U}$, hence, $U \cup U_1 \in \mathcal{U}$ and $U \cup U_2 \in \mathcal{U}$ follows. But then, $U_1 \cap A \times A = (U \cup U_1) \cap A \times A$ and $U_2 \cap A \times A = (U \cup U_2) \cap A \times A$ can be easily deduced showing the claim.

To (buf$_3$): For $D \in \mathcal{B}^A \backslash \{\emptyset\}$, we have to verify that $\overset{\bullet}{D} \times \overset{\bullet}{D} \in \mu_A$ holds. By the definition of $\mathcal{B}^A$, we are getting $D = B \cap A$ for some $B \in \mathcal{B}^X$, hence, $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \mu$ follows, and $\overset{\bullet}{D} \times \overset{\bullet}{D} = \{U \cap A \times A : U \in \overset{\bullet}{B} \times \overset{\bullet}{B}\} = (\overset{\bullet}{B} \times \overset{\bullet}{B})_A \in \mu_A$ results.

To " $\leq$ ": $R \in \overset{\bullet}{D} \times \overset{\bullet}{D}$ implies $R \supset D \times D$, hence, $R \supset (B \cap A) \times (B \cap A) = (B \times B) \cap (A \times A) =: R_1$. Consequently, $R_1 \in (\overset{\bullet}{B} \times \overset{\bullet}{B})_A$ is true, and $R \in (\overset{\bullet}{B} \times \overset{\bullet}{B})_A$ follows.

To " $\geq$ ": Conversely, let $R \in (\overset{\bullet}{B} \times \overset{\bullet}{B})_A$. Then $R = U \cap A \times A$ for some $U \in \overset{\bullet}{B} \times \overset{\bullet}{B}$, hence, $R \supset (B \times B) \cap (A \times A) = (B \cap A) \times (B \cap A) = D \times D$, and $R \in \overset{\bullet}{D} \times \overset{\bullet}{D}$ is valid.

To (buf$_4$): Now $\mathcal{V} \in FIL(A \times A)$ with $\mathcal{V} \supset \mathcal{U}_A$ for some $\mathcal{U} \in \mu$ are implying $(i \times i)(\mathcal{V}) \supset \mathcal{U}$ since $R \in \mathcal{U}$ implies $R \cap A \times A \in \mathcal{U}_A$, hence, $R \cap A \times A \in \mathcal{V}$ follows by the hypothesis, and $R \in (i \times i)(\mathcal{V})$ results. Consequently, $(i \times i)(\mathcal{V}) \in \mu$ follows. But $\mathcal{V} = ((i \times i)(\mathcal{V}))_A$ because $R \in \mathcal{V}$ implies $R = A \times A \cap (i \times i)[R]$, hence, $R \in ((i \times i)(\mathcal{V}))_A$ is true.

Conversely, $R \in ((i \times i)(\mathcal{V}))_A$ implies $R = S \cap A \times A$ for some $S \in ((i \times i)(\mathcal{V})$. Consequently, we can find $V \in \mathcal{V}$ such that $S \supset (i \times i)[V] = V$ holds. But then, $S \cap A \times A \supset V$, and $R \in \mathcal{V}$ follows. Evidently, $\{i[D] : D \in \mathcal{B}^A\} =: i\mathcal{B}^A \subset \mathcal{B}^X$ holds, so that $i : (A, \mathcal{B}^A, \mu_A) \longrightarrow (X, \mathcal{B}^X, \mu)$ satisfies (buc$_1$).

To (buc$_2$): Now let $\mathcal{U}_A \in \mu_A$ for some $\mathcal{U} \in \mu$. We will show that $\mathcal{U} \subset (i \times i)(\mathcal{U}_A)$ can be deduced. $U \in \mathcal{U}$ implies $U \cap A \times A \in \mathcal{U}_A$, hence, $U \cap A \times A = (i \times i)[U \cap A \times A] \in (i \times i)(\mathcal{U}_A)$ is valid, and $U \in (i \times i)(\mathcal{U}_A)$ results. Finally, let $(\mathcal{E}^A, \eta)$ be b-uniform filter structure on A with $i : (A, \mathcal{E}^A, \eta) \longrightarrow (X, \mathcal{B}^X, \mu)$ is buc. Our goal is to verify $(\mathcal{E}^A, \eta) \leq (\mathcal{B}^A, \mu_A)$. First, let $D \in \mathcal{E}^A$. Then, by the hypothesis, $i[D] \in \mathcal{B}^X$ is true. But $D = i[D] = D \cap A$, and $D \in \mathcal{B}^A$ results.

Next, let $\mathcal{V} \in \eta$, we will show that $\mathcal{V} = \mathcal{U}_A$ for some $\mathcal{U} \in \mu$. By the assumption, we know that $(i \times i)(\mathcal{V}) \in \mu$ is true, which means that, by the former proof, $\mathcal{V} = ((i \times i)(\mathcal{V}))_A$ results. This statement concludes the proof. $\square$

Remark 2.8. Since **b-UFIL** forms a topological construct, there exist arbitrary final structures. The final b-uniform filter structures on a set X with respect to $((X_i, \mathcal{B}^{X_i}, \mu_i), f_i, X, I)$, where $f_i : X_i \longrightarrow X$ are mappings for each $i \in I$, denoted by $(\mathcal{B}^I_X, \mu^X_I)$ is the *finest* b-uniform filter structure on X such that f_i

is b-uniformly continuous for each $i \in I$, (compare with Definition 2.5 and Remark 2.6, respectively). If I is the empty class, then $(\mathcal{B}_X^I, \mu_I^X) = (\mathcal{D}^X, \{\mathcal{V} \in FIL(X \times X) : \exists x \in X \text{ such that } \overset{\bullet}{x} \times \overset{\bullet}{x} \subset \mathcal{V}\} \cup \{\underline{P}(X \times X)\})$.

Remark 2.9. In general, let $\underline{C}$ be a category. Then a family $(f_i : X_i \longrightarrow X)_{i \in I}$ of $\underline{C}$-morphisms indexed by some class I, in short a *sink* in $\underline{C}$, is called an *epi-sink* in $\underline{C}$ provided that, for any pair $X \overset{\alpha}{\to} Y$ and $X \overset{\beta}{\to} Y$ of $\underline{C}$-morphisms such that $\alpha \circ f_i = \beta \circ f_i$ for each $i \in I$, it follows that $\alpha = \beta$.

If $\underline{C}$ is a topological construct with structured sets (X, μ) as objects, then $f_i : (X_i, \mu_i) \longrightarrow (X, \mu)$ is called *final* provided that μ is the final $\underline{C}$-structure on X with respect to the given data.

Motivation 2.10. In a topological construct $\underline{C}$, final epi-sinks play an important role. So it is possible to describe the extensionality of $\underline{C}$ (meaning that each $\underline{C}$-object has a one-point extension in $\underline{C}$) by the fact that in $\underline{C}$ final epi-sinks are hereditary. Moreover, a construct is Cartesian closed, meaning that in $\underline{C}$ natural function space structures are available, if and only if for each $\underline{C}$-object Y and for any final epi-sink $(f_i : X_i \longrightarrow X)_{i \in I}$ in $\underline{C}$ $(1_Y \times f_i : Y \times X_i \longrightarrow Y \times X)_{i \in I}$ is a final epi-sink, i.e., the functor "$Y \times -$" preserves final epi-sinks, see [11].

Proposition 2.11. *Let X be a set, $(X, \mathcal{B}^{X_i}, \mu_i)_{i \in I}$ a family of b-uniform filter spaces and $(f_i : X_i \longrightarrow X)_{i \in I}$ a family of maps. If $(f_i : X_i \longrightarrow X)_{i \in I}$ is an epi-sink in b-UFIL, then $(\mathcal{B}_X^I, \mu_I^X)$ is the final b-uniform filter structure on X with respect to the given data, where $\mathcal{B}_X^I := \{B \subset X : \exists\, i \in I\ \exists B_i \in \mathcal{B}^{X_i}$ such that $B \subset f_i[B_i]\}$ and $\mu_I^X := \{\mathcal{U} \in FIL(X \times X) : \exists\, i \in I\ \exists\, \mathcal{U}_i \in \mu_i$ such that $(f_i \times f_i)(\mathcal{U}_i) \subset \mathcal{U}\}$.*

Proof. It suffices to show that the equation $X = \bigcup_{i \in I} f_i[X_i]$ holds, then the remainder is clear. If $card\ X < 2$, then the assertion is trivial. Let $card\ X \geq 2$. If $X \neq \bigcup_{i \in I} f_i[X_i]$, then there would be $x_0 \in \bigcup_{i \in I} f_i[X_i]$ and $x_1 \in X \setminus \bigcup_{i \in I} f_i[X_i]$. If $\{x_0, x_1\}$ is endowed with the indiscrete b-uniform filter structure, see Remark 2.6, then one obtains an object in b-UFIL. Hence, $\alpha : X \longrightarrow Z$ defined by $\alpha(x) := x_0$ for each $x \in X$ and $\beta : X \longrightarrow Z$ defined by

$$\beta(x) := \begin{cases} x_0 & \text{for } x \in \cup_{i \in I} f_i[X_i]; \\ x_1 & \text{otherwise} \end{cases}$$

are b-uniformly continuous maps such that $\alpha \circ f_i = \beta \circ f_i$ for each $i \in I$. Obviously, $\alpha \neq \beta$ in contradiction to the fact that $(f_i)_{i \in I}$ is an epi-sink. Consequently, $X = \bigcup_{i \in I} f_i[X_i]$ follows.

Next, we will demonstrate how b-UFIL can be nicely embedded into b-CONV, the topological construct of b-convergence spaces and b-continuous maps, see [7] and the following definition. This superconstruct contains not only the neighborhood spaces of Tozzi and Wyler [12] or the supertopological spaces in the sense of Doitchinov [3] but also the set-convergences as defined by Wyler [13]. Hence, convergences in all their facets are now being involved and can be examined for their prevailing aspects. □

Definition 2.12. A triple $(X, \mathcal{B}^X, \tau)$ consisting of a set X, a boundedness $\mathcal{B}^X$ and a convergence function $\tau : \mathcal{B}^X \longrightarrow \underline{P}(FIL(X \times X))$ is called *b-convergence space* provided that τ satisfies the below conditions:

(bc$_1$) $B \in \mathcal{B}^X, \mathcal{U} \in \tau(B)$ and $\mathcal{U} \subset \mathcal{V} \in FIL(X \times X)$ imply $\mathcal{V} \in \tau(B)$;
(bc$_2$) $\tau(\emptyset) = \{\underline{P}(X \times X)\}$;
(bc$_3$) $x \in X$ implies $\overset{\bullet}{x} \times \overset{\bullet}{x} \in \tau(\{x\})$.

Here, a boundedness $\mathcal{B}^X$ is a non-empty subset of $\underline{P}X$ that satisfies the following conditions:

(b$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;
(b$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

Given two b-convergence spaces $(X, \mathcal{B}^X, \tau_X)$, $(Y, \mathcal{B}^Y, \tau_Y)$ a function $f : X \longrightarrow Y$ is called *b-continuous* if it is bounded, which means $\{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$ and, in addition, we have that f preserves uniform filters in the sense that $B \in \mathcal{B}^X$ and $\mathcal{U} \in \tau_X(B)$ imply $(f \times f)(\mathcal{U}) \in \tau_Y(f[B])$.

Moreover, by **b-CONV** we will denote the corresponding category.

Remark 2.13. There exist some interesting subcategories of **b-CONV** that can be described as follows. Let us call a b-convergence space $(X, \mathcal{B}^X, \tau)$

(i) *equiform* if τ satisfies the condition
(e) $B_1, B_2 \in \mathcal{B}^X \setminus \{\emptyset\}$ imply $\tau(B_1) = \tau(B_2)$;
(ii) *set-pointed* if τ satisfies the condition
(sp) $B \in \mathcal{B}^X$ implies $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \tau(B)$.

For both definitions, there exist special fundamental convergences independent of each other as presented in [10]. Now, let us call a b-convergence space $(X, \mathcal{B}^X, \tau)$ *set-pointed equiform* provided τ satisfies both the above conditions. By **SETeb-CONV** we will denote the full subcategory of **b-CONV** whose objects are the set-pointed equiform b-convergence spaces.

Theorem 2.14. *The categories* **b-UFIL** *and* **SETeb-CONV** *are isomorphic.*

Proof. Let $(X, \mathcal{B}^X, \mu)$ be a b-uniform filter space. Then define a convergence function τ_μ by setting:

$\tau_\mu(\emptyset) := \{\underline{P}(X \times X)\}$ and
$\tau_\mu(B) := \mu$ for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$.

Evidently, τ_μ satisfies all axioms (bc$_1$) To (bc$_3$), (sp) and (e), respectively. Conversely, if assuming a set-pointed equiform b-convergence space $(Y, \mathcal{B}^Y, t)$, we put:

$$\eta_t := \{\mathcal{U} \in FIL(Y \times Y) : \exists\ B \in \mathcal{B}^Y \text{ such that } \mathcal{U} \in t(B)\}.$$

Hence, $(Y, \mathcal{B}^Y, \eta_t)$ defines a b-uniform filter space so that the following equations hold:

(i) $\eta_{\tau_\mu} = \mu$;
(ii) $\tau_{\eta_t} = t$.

To (i) "$\geq$": Let $X = \emptyset$ and $\mathcal{U} \in \mu$, hence, $\mathcal{U} = \underline{P}(X \times X) = \{\emptyset\}$, according to Proposition 2.2 consequently, $\mathcal{U} \in \tau(\emptyset)$ follows, and $\mathcal{U} \in \eta_{\tau_\mu}$ results.

"$\leq$" Conversely, $\mathcal{U} \in \eta_{\tau_\mu}$ implies $\mathcal{U} \in \tau_\mu(B)$ for some $B \in \mathcal{B}^X$. Since by the hypothesis $X = \emptyset$, $\mathcal{B}^X = \{\emptyset\}$ follows, and $\mathcal{U} = \underline{P}(X \times X)$ by the definition of τ_μ. But $\mu \neq \emptyset$, and therefore $\mathcal{U} \supset \mathcal{V}$ for some $\mathcal{V} \in \mu$, showing that $\mathcal{U} \in \mu$ is true. Now let $X \neq \emptyset$.

To (i) " $\geq$ ": Let $\mathcal{U} \in \mu$, hence, we can find $x \in X$, and $\{x\} \in \mathcal{B}^X \backslash \{\emptyset\}$ follows. Thus by definition $\mathcal{U} \in \tau_\mu(\{x\})$ is valid, and $\mathcal{U} \in \eta_{\tau_\mu}$ results.

" $\leq$ ": Conversely, $\mathcal{U} \in \eta_{\tau_\mu}$ implies $\mathcal{U} \in \tau_\mu(B)$ for some $B \in \mathcal{B}^X$. If $B \neq \emptyset$, then $\mathcal{U} \in \mu$ follows. If $B = \emptyset$, $\mathcal{U} = \underline{P}(X \times X)$ is valid, and $\mathcal{U} \in \mu$ results because $\mu \neq \emptyset$.

To (ii) $\geq$: Without restriction let $B \in \mathcal{B}^X \backslash \{\emptyset\}$. $\mathcal{U} \in t(B)$ implies $\mathcal{U} \in \eta_t$, hence, $\mathcal{U} \in \tau_{\eta_t}(B)$ follows.

" $\leq$ ": $\mathcal{U} \in \tau_{\eta_t}(B)$ implies $\mathcal{U} \in \eta_t$. Then, there exists $D \in \mathcal{B}^X$ such that $\mathcal{U} \in t(D)$. If $D \neq \emptyset$, $\mathcal{U} \in t(B)$ follows. If $D = \emptyset$, $\mathcal{U} = \underline{P}(X \times X)$ holds and $\overset{\bullet}{B} \times \overset{\bullet}{B} \in t(B)$, this implies $\mathcal{U} \in t(B)$ according To (bc$_1$). Evidently, the former established correspondence is functorial, meaning that for set-pointed equiform b-convergence spaces $(X, \mathcal{B}^X, \tau_X)$, $(Y, \mathcal{B}^Y, \tau_Y)$ and every map $f : X \longrightarrow Y$ the following statements are equivalent:

(i) $f : (X, \mathcal{B}^X, \tau_X) \longrightarrow (Y, \mathcal{B}^Y, \tau_Y)$ is b-continuous;
(ii) $f : (X, \mathcal{B}^X, \eta_{\tau_X}) \longrightarrow (Y, \mathcal{B}^Y, \eta_{\tau_Y})$ is b-uniformly continuous.

□

Remark 2.15. Now, we pointed out that **SETeb-CONV** can be even regarded as a bireflective subcategory of **b-CONV**.

Proof. For a b-convergence space $(X, \mathcal{B}^X, \tau)$, let us consider the b-convergence space $(X, \mathcal{B}^X, ((\overset{\bullet}{\tau_a})_e))$, where τ_a, τ_e are defined as in [7] and in general

$\overset{\bullet}{\tau}(\emptyset) := \{\underline{P}(X \times X)\}$ and
$\overset{\bullet}{\tau}(B) := \tau(B) \cup \{\mathcal{U} \in FIL(X \times X) : \exists\, D \in \mathcal{B}^X \backslash \{\emptyset\}$ such that $\overset{\bullet}{D} \times \overset{\bullet}{D} \subset \mathcal{U}\}$
for each $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and some b-convergence operator τ .

It is easy to verify that $(X, \mathcal{B}^X, ((\overset{\bullet}{\tau_a})_e))$ defines a set-pointed equiform b-convergence space such that

$$1_X : (X, \mathcal{B}^X, \tau) \longrightarrow (X, \mathcal{B}^X, ((\overset{\bullet}{\tau_a})_e))$$

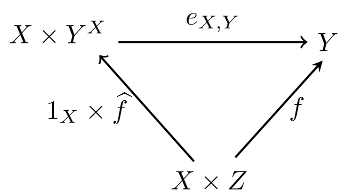
is b-continuous. Now, let $(Y, \mathcal{B}^Y, \tau_Y)$ be a set-pointed equiform b-convergence space and $f : (X, \mathcal{B}^X, \tau) \longrightarrow (Y, \mathcal{B}^Y, \tau_Y)$ be b-continuous map. We have to verify that $f : (X, \mathcal{B}^X, ((\overset{\bullet}{\tau_a})_e)) \longrightarrow (Y, \mathcal{B}^Y, \tau_Y)$ is b-continuous, too. For $B \in \mathcal{B}^X \backslash \{\emptyset\}$ let $\mathcal{U} \in ((\overset{\bullet}{\tau_a})_e)\ (B)$. In case of $\mathcal{U} \in ((\overset{\bullet}{\tau_a})_e)\ (B)$, we can find $x \in X$ such that $\mathcal{U} \in \tau_a(\{x\})$. Hence, by definition $\mathcal{U} \in \tau(D)$ for some $D \in \mathcal{B}^X \backslash \{\emptyset\}$ with $x \in D$. By the hypothesis we are getting $(f \times f)(\mathcal{U}) \in \tau_Y(f[D]) = \tau_Y(f[B])$. Note also that $\{f[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^Y$ holds. If $\overset{\bullet}{D} \times \overset{\bullet}{D} \subset \mathcal{U}$ for some $D \in \mathcal{B}^X \backslash \{\emptyset\}$, we are getting $(f \times f)(\overset{\bullet}{D} \times \overset{\bullet}{D}) = f[\overset{\bullet}{D}] \times f[\overset{\bullet}{D}] \in \tau_Y(f[D]) = \tau_Y(f[B])$, and $(f \times f)(\mathcal{U}) \in \tau_Y(f[B])$ results, concluding the proof. □

3. On the Cartesian closedness

Cartesian closedness, i.e., the existence of natural function spaces is useful, e.g. in homotopy theory (fundamental groups) or for the constructing of completions [11]. In particular, it plays an important role in topological constructs. Moreover it should be noted that this property may be defined by means of a pair of adjoint functors $(\mathcal{F}_1, \mathcal{F}_2)$, where neither $\mathcal{F}_1$ nor $\mathcal{F}_2$ is an inclusion functor as in the preceding section dealing with reflections and coreflections.

Definition 3.1. A category $\underline{C}$ is called *Cartesian closed* provided that the following conditions are satisfied [11].

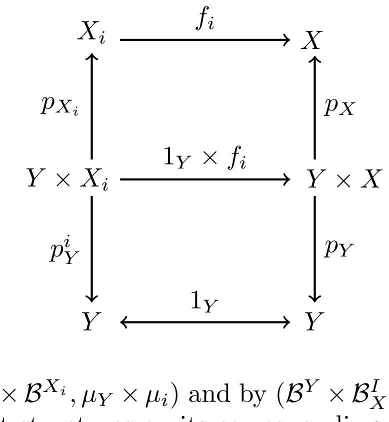
(1) For each pair (X, Y) of $\underline{C}$-objects there exists a product $X \times Y$ in $\underline{C}$;
(2) For each $\underline{C}$-objects X, the following holds: for each $\underline{C}$-object Y there exists some $\underline{C}$-object Y^X (called *power object*) and some $\underline{C}$ morphism $e_{X,Y} : X \times Y^X \longrightarrow Y$ (called *evaluation* morphism) such that, for each $\underline{C}$-object Z and each $\underline{C}$-morphism $f : X \times Z \longrightarrow Y$, there exists a unique $\underline{C}$-morphism $\widehat{f} : Z \longrightarrow Y^X$ such that the following diagram commutes.



Remark 3.2. In topological constructs, the condition (1) is automatically fulfilled. Now it will be useful that in a topological construct $\underline{C}$ the property of being Cartesian closed is equivalent to the following statement: for each $\underline{C}$-object Y and for any final epi-sink $(f_i : X_i \longrightarrow X)_{i\in I}$ in $\underline{C}$, $(1_Y \times f_i : Y \times X_i \longrightarrow Y \times X)_{i\in I}$ is a final epi-sink, i.e., "$Y \times -$" preserves final epi-sinks.

Theorem 3.3. *Let* $(f_i : (X_i, \mathcal{B}^{X_i}, \mu_i) \longrightarrow (X, \mathcal{B}^X, \mu))_{i\in I}$ *be an epi-sink in* **b-UFIL**. *Then, for any b-uniform filter space* $(Y, \mathcal{B}^Y, \mu_Y)$, $(1_Y \times f_i : (Y \times X_i, \mathcal{B}^Y \times \mathcal{B}^{X_i}, \mu_Y \times \mu_i) \longrightarrow (Y \times X, \mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I))_{i\in I}$ *is a final epi-sink in* **b-UFIL**, *see Proposition 2.2, Remark 2.9 and Proposition 2.11, respectively.*

Proof. First let us consider the following diagram:



We will denote by $(\mathcal{B}^Y \times \mathcal{B}^{X_i}, \mu_Y \times \mu_i)$ and by $(\mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I)$ the prevailing b-uniform filter product structures on its corresponding sets:

Now we will show that $(1_Y \times f_i : (Y \times X_i, \mathcal{B}^Y \times \mathcal{B}^{X_i}, \mu_Y \times \mu_i) \longrightarrow (Y \times X, \mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I))_{i\in I}$ is an epi-sink in **b-UFIL**.

If $\alpha, \beta : (Y \times X, \mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I) \longrightarrow (Z, \mathcal{B}^Z, \mu_Z)$ are b-uniformly continuous maps such that $\alpha \circ (1_Y \times f_i) = \beta \circ (1_Y \times f_i)$ for each $i \in I$ and if $(y, x) \in Y \times X$ then since $X = \bigcup_{i\in I} f_i[X_i]$ (see Proposition 2.11), there is some $i \in I$ and some $x_i \in X_i$ with $f_i(x_i) = x$. Hence, $\alpha((y, x)) = \alpha((y, f_i(x_i))) = \alpha((1_Y \times f_i)(y, x_i)) = \beta((1_Y \times f_i)((y, x_i))) = \beta((y, f_i(x_i)) = \beta((y, x))$ and, consequently, $\alpha = \beta$ results.

Next, we will show that, for each $i \in I$, $1_Y \times f_i : (Y \times X_i, \mathcal{B}^Y \times \mathcal{B}^{X_i}, \mu_Y \times \mu_i) \longrightarrow (Y \times X, \mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I)$ satisfies (buc$_1$). So let for $i \in I$, $B^*_i \in \mathcal{B}^Y \times \mathcal{B}^{X_i}$. We have to verify that $(1_Y \times f_i)[B^*_i] \in \mathcal{B}^Y \times \mathcal{B}^I_X$ holds, which means $p_Y[(1_Y \times f_i)[B^*_i]] \in \mathcal{B}^Y$ and $p_X[(1_Y \times f_i)[B^*_i]] \in \mathcal{B}^I_X$. By the hypothesis, we get $p^i_Y[B^*_i] \in \mathcal{B}^Y$ and $p_{X_i}[B^*_i] \in \mathcal{B}^{X_i}$. But $p_Y[(1_Y \times f_i)[B^*_i]] \subset p^i_Y[B^*_i]$ and $p_X[(1_Y \times f_i)[B^*_i] \subset f_i[p_{X_i}[B^*_i]]$ hold since both parts of the diagram are commutative. Now the claim follows.

Next, the inclusion $\mathcal{B}^Y \times \mathcal{B}^I_X \subset \mathcal{B}^I_{Y\times X}$ is valid. $B^* \in \mathcal{B}^Y \times \mathcal{B}^I_X$ implies $p_Y[B^*] \in \mathcal{B}^Y$ and $p_X[B^*] \in \mathcal{B}^I_X$. Hence, there exists $i \in I$ and $B_i \in \mathcal{B}^{X_i}$ such that $p_X[B^*] \subset f_i[B_i]$. We set $B^*_i := {P^i_Y}^{-1}[p_Y[B^*]] \cap p^{-1}_{X_i}[B_i]$. Then, $B^*_i \in \mathcal{B}^Y \times \mathcal{B}^{X_i}$, since $p^i_Y[B^*_i] = p^i_Y[{p^i_Y}^{-1}[p_Y[B^*] \cap p^{-1}_{X_i}[B_i]] \subset p^i_Y[{p^i_Y}^{-1}[p_Y[B^*]] = p_Y[B^*] \in \mathcal{B}^Y$. $p_{X_i}[B^*_i] = p_{X_i}[p_Y[B^*] \cap p^{-1}_{X_i}[B_i]] \subset p_{X_i}[p^{-1}_{X_i}[B_i]] = B_i \in \mathcal{B}^{X_i}$. So it remains to verify that $B^* \subset (1_Y \times f_i)[B^*_i]$.

Let $(y, x) \in B^*$ imply $p_X(y, x) = x \in f_i[B_i]$. Hence, there exists $x_i \in B_i$ with $x = f_i(x_i)$. Now we put $z^i := (y, x_i)$, consequently, $(1_Y \times f_i)(z^i) = (1_Y \times f_i)(y, x_i)$ implying $(y, f_i(x_i)) = (y, x)$. On the other hand, $z^i \in B^*_i$ is true, because $p^i_Y(z^i) = p^i_Y(y, x_i) = y$ and $p_{X_i}(z^i) = p_{X_i}(y, x_i) = x_i$. Thus, our assumed inclusion holds. In addition $1_Y \times f_i : (Y \times X_i, \mathcal{B}^Y \times \mathcal{B}^{X_i}, \mu_Y \times \mu_i) \longrightarrow (Y \times X, \mathcal{B}^Y \times \mathcal{B}^I_X, \mu_Y \times \mu^X_I)$ satisfies (buc$_2$) for each $i \in I$. So let for $i \in I$, $\mathcal{U}^* \in \mu_Y \times \mu_i$. We have to verify that $((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}^*) \in \mu_Y \times \mu^X_I$ is valid. By the hypothesis, we get $(p^i_Y \times p^i_Y)(\mathcal{U}^*) \in \mu_Y$ and $(p_{X_i} \times p_{X_i})(\mathcal{U}^*) \in \mu_i$. Now we will show that

(i) $(p^i_Y \times p^i_Y)(\mathcal{U}^*) \subset (p_Y \times p_Y)((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}^*)$ and
(ii) $(f_i \times f_i)((p_{X_i} \times p_{X_i})(\mathcal{U}^*) \subset (p_X \times p_X)((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}^*)$

are true. Then, the claim immediately follows.

To (i): $R^* \in (p^i_Y \times p^i_Y)(\mathcal{U}^*)$ implies $R^* \supset (p^i_Y \times p^i_Y)[U^*]$ for some $U^* \in \mathcal{U}^*$. It remains to prove the inclusion $(p^i_Y \times p^i_Y)[U^*] \supset (p_Y \times p_Y)[((1_Y \times f_i) \times (1_Y \times f_i))[U^*]]$.

$z \in (p_Y \times p_Y)[((1_Y \times f_i) \times (1_Y \times f_i))[U^*]]$ implies $z = (p_Y \times p_Y)(s)$ for some $s \in ((1_Y \times f_i) \times (1_Y \times f_i))[U^*]$, hence, $s = ((1_Y \times f_i) \times (1_Y \times f_i))(u^*)$ for some $u^* \in U^*$. Consequently, $u^* = (u, v)$ for elements $u, v \in Y \times X_i$. It follows that $u = (y_1, x^1_i)$ and $v = (y_2, x^2_i)$ for $y_1, y_2 \in Y$ and $x^1_i, x^2_i \in X_i$. Thus, we get $s = ((1_Y \times f_i) \times (1_Y \times f_i))(u^*) = ((1_Y \times f_i) \times (1_Y \times f_i))(u, v) = ((1_Y \times f_i) \times (1_Y \times f_i))((y_1, x^1_i), (y_2, x^2_i)) = ((1_Y \times f_i)(y_1, x^1_i)), (1_Y \times f_i)(y_2, x^2_i)) = ((y_1, f_i(x^1_i)), (y_2, f_i(x^2_i))$. Hence,

$$\begin{aligned} z = (p_Y \times p_Y)(s) &= (p_Y \times p_Y)((y_1, f_i(x^1_i)), (y_2, f_i(x^2_i)) \\ &= (p_Y(y_1, f_i(x^1_i), p_Y(y_2, f_i(x^2_i)) = (y_1, y_2). \end{aligned}$$

On the other hand, we have $(p^i_Y \times p^i_Y)(u^*) \in (p^i_Y \times p^i_Y)[U^*]$ with $(p^i_Y \times p^i_Y)(u^*) = (p^i_Y \times p^i_Y)((u, v)) = (p^i_Y(u), p^i_Y(v)) = (p^i_Y(y_1, x^1_i), p^i_Y(y_2, x^2_i) = (y_1, y_2)$.

To (ii): $R \in (f_i \times f_i)((p_{X_i} \times p_{X_i})(\mathcal{U}^*))$ implies $R \supset (f_i \times f_i)[S]$ for some $S \in (p_{X_i} \times p_{X_i})(\mathcal{U}^*)$. Consequently, $S \supset (p_{X_i} \times p_{X_i})[U^*]$ for some $U^* \in \mathcal{U}^*$.

Hence$(f_i \times f_i)[S] \supset (f_i \times f_i)[(p_{X_i} \times p_{X_i})][U^*]] = (p_X \times p_X)[((1_Y \times f_i) \times (1_Y \times f_i))[U^*]] \in (p_X \times p_X)((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}^*))$, because of the commutative diagram, and the claim follows.

Next, we will show that the inclusion $\mu_Y \times \mu_I^X \subset \mu_I^{Y\times X}$ holds.

$\mathcal{U}^* \in \mu_Y \times \mu_I^X$ implies $(p_Y \times p_Y)(\mathcal{U}^*) \in \mu_Y$ and $(p_X \times p_X)(\mathcal{U}^*) \in \mu_I^X$. Hence, we can find $i \in I$ and $\mathcal{U}_i \in \mu_i (f_i \times f_i)(\mathcal{U}_i) \subset (p_X \times p_X)(\mathcal{U}^*)$. Now we put : $\mathcal{U}_i^* := \{R \subset (Y \times X_i) \times (Y \times X_i) : \exists\, S \in (p_Y \times p_Y)(\mathcal{U}^*), \exists\, U \in \mathcal{U}_i$ such that $R \supset (p_Y^i \times p_Y^i)^{-1}[S] \cap (p_{X_i} \times p_{X_i})^{-1}[U]\}$. Then, $\mathcal{U}_i^* \in \mu_Y \times \mu_i$, since the following inclusions hold:

(1) $(p_Y^i \times p_Y^i)(\mathcal{U}_i^*) \supset (p_Y \times p_Y)(\mathcal{U}^*)$ and
(2) $(p_{X_i} \times p_{X_i})(\mathcal{U}_i^*) \supset \mathcal{U}_i$.

To (1): $S \in (p_Y \times p_Y)(\mathcal{U}^*)$ implies $S \supset (p_Y \times p_Y)[U^*]$ for some $U^* \in \mathcal{U}^*$. Choose $U \in \mathcal{U}_i$, hence, $(p_Y^i \times p_Y^i)^{-1}[(p_Y \times p_Y)[U^*]] \cap (p_{X_i} \times p_{X_i})^{-1}[U] \in \mathcal{U}_i^*$. Consequently, $(p_Y^i \times p_Y^i)[(p_Y^i \times p_Y^i)^{-1}[(p_Y \times p_Y)[U^*]] \in \mathcal{U}_i^*$ follows, and $(p_Y \times p_Y)[U^*] \in \mathcal{U}_i^*$ results, showing that $S \in \mathcal{U}_i^*$ is valid.

To (2): For $U \in \mathcal{U}_i$, choose $U^* \in \mathcal{U}^*$, hence, $(p_Y \times p_Y)[U^*] \in (p_Y \times p_Y)(\mathcal{U}^*)$ and, consequently, $(p_Y^i \times p_Y^i)^{-1}[(p_Y \times p_Y)[U^*]] \cap (p_{X_i} \times p_{X_i})^{-1}[U] \in \mathcal{U}_i^*$. But then $U = (p_{X_i} \times p_{X_i})[(p_{X_i} \times p_{X_i})^{-1}[U]] \in (p_{X_i} \times p_{X_i})(\mathcal{U}_i^*)$ follows.

Now if we can show that $((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}_i^*) \subset \mathcal{U}^*$ is true, the proposed claim follows. $V \in ((1_Y \times f_i) \times (1_Y \times f_i))(\mathcal{U}_i^*)$ implies $V \supset ((1_Y \times f_i) \times (1_Y \times f_i))[R]$ for some $R \in \mathcal{U}_i^*$. Hence, we can find $S \in (p_Y \times p_Y)(\mathcal{U}^*)$ and $U \in \mathcal{U}_i$ such that $R \supset (p_Y^i \times p_Y^i)^{-1}[S] \cap (p_{X_i} \times p_{X_i})^{-1}[U]$. Then, $(f_i \times f_i)[U] \in (p_X \times p_X)(\mathcal{U}^*)$ is valid according To (2) and, consequently, $(f_i \times f_i)[U] \supset (p_X \times p_X)[U^*]$ for some $U^* \in \mathcal{U}^*$. $S \supset (p_Y \times p_Y)[V^*]$ for some $V^* \in \mathcal{U}^*$ and, thus, $S^* := U^* \cap V^* \in \mathcal{U}^*$ follows. Now, it remains to verify that the inclusion $S^* \subset ((1_Y \times f_i) \times (1_Y \times f_i))[R]$ holds. $z \in S^*$ implies $z \in U^* \cap V^*$, hence, $z = (r, v)$ for $r, v \in (Y \times X) \times (Y \times X)$ so that $r = (y_1, x_1)$ and $v = (y_2, x_2)$ for elements $y_1, y_2 \in Y$ and $x_1, x_2 \in X$. Hence, $(p_X \times p_X)(z) = (p_X \times p_X)(r, v) = (p_X(r), p_X(v)) = (p_X(y_1, x_1), p_X(y_2, x_2)) = (x_1, x_2) \in (f_i \times f_i)[U]$ follows. But then we can find $w \in U$ such that $(x_1, x_2) = (f_i \times f_i)(w)$ for $w = (z_i^1, z_i^2)$ with $z_i^1, z_i^2 \in X_i \times X_i$. $(f_i \times f_i)(w) = (f_i \times f_i)(z_i^1, z_i^2) = (f_i(z_i^1), f_i(z_i^2))$, and $x_1 = f_i(z_i^1)$, $x_2 = f_i(z_i^2)$ are resulting. Consequently, $(p_Y \times p_Y)(z) = (p_Y(r), p_Y(v)) = (p_Y(y_1, x_1), p_Y(y_2, x_2)) = (y_1, y_2) \in S$ follows. On the other hand, $((y_1, z_i^1), (y_2, z_i^2)) \in (p_Y^i \times p_Y^i)^{-1}[S] \cap (p_{X_i} \times p_{X_i})^{-1}[U]$ is true because $(p_Y^i \times p_Y^i)((y_1, z_i^1), (y_2, z_i^2)) = (p_Y^i(y_1, z_i^1), p_Y^i(y_2, z_i^2)) = (y_1, y_2) \in S$ by the hypothesis, since $(p_{X_i} \times p_{X_i})((y_1, z_i^1), (y_2, z_i^2)) = (p_{X_i}(y_1, z_i^1), p_{X_i}(y_2, z_i^2)) = (z_i^1, z_i^2) \in U$ implying $((y_1, z_i^1), (y_2, z_i^2)) \in R$ and, consequently, $((1_Y \times f_i) \times (1_Y \times f_i))((y_1, z_i^1), (y_2, z_i^2)) \in ((1_Y \times f_i) \times (1_Y \times f_i))[R]$. But $z = ((1_Y \times f_i) \times (1_Y \times f_i))((y_1, z_i^1), (y_2, z_i^2))$. Taking our results into account, we get $\mathcal{B}^Y \times \mathcal{B}_X^I \subset \mathcal{B}_{Y\times X}^I$ and $\mu_Y \times \mu_I^X \subset \mu_I^{Y\times X}$, which means $(\mathcal{B}^Y \times \mathcal{B}_X^I, \mu_Y \times \mu_I^X) \leq (\mathcal{B}_{X\times Y}^I, \mu_I^{Y\times X})$. But then the equality holds, since $(\mathcal{B}_{X\times Y}^I, \mu_I^{Y\times X})$ is the *finest* b-uniform filter structure on $Y \times X$ with respect to the given data, see Proposition 2.11. □

Remark 3.4. Since **b-UFIL** is a Cartesian closed topological construct, it follows that quotient maps are finitely productive, but not necessarily productive (i.e., not closed under the formation of arbitrary products). Later, we will see that, in **b-UFIL**, this nice property holds in addition.

4. On one-point extensions

As seen above, final epi-sinks play an important role if one considers Cartesian closedness. In what follows, they also are of interest in connection with the so-called extensionality of a topological construct $\underline{C}$ because, in $\underline{C}$, final epi-sinks are hereditary if and only if $\underline{C}$ is extensional, see [11].

Definition 4.1. A topological construct $\underline{C}$ is called *extensional* (hereditary) provided that every $\underline{C}$-object X has a *one-point extension* X^* as $\underline{C}$-object, i.e., every $\underline{C}$-object X can be embedded via the addition of a single point ∞ into a $\underline{C}$-object X^* such that, for every $\underline{C}$-morphism $f : A \longrightarrow X$ whose domain is a subobject of an $\underline{C}$-object Y, the map $f^* : Y \longrightarrow X^*$ defined by

$$f^*(y) := \begin{cases} f(y), & \text{if} \quad y \in A; \\ \infty, & \text{if} \quad y \in Y \backslash A. \end{cases}$$

is a $\underline{C}$-morphism , i.e., the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*} \\ X & \xrightarrow{\;i\;} & X^* := X \cup \{\infty\} \end{array}$$

Theorem 4.2. *In* ***b-UFIL*** *every b-uniform filter space* $(X, \mathcal{B}^X, \mu)$ *has a one-point extension.*

Proof. For a b-uniform filter space $(X, \mathcal{B}^X, \mu)$, we put $X^* := X \cup \{\infty\}$ with $\infty \notin X$. Now, we set $\mathcal{B}^{X^*} := \mathcal{B}^X \cup \{\{\infty\}\}$ and $\mu^* := \{\mathcal{V} \in FIL(X^* \times X^*) : \exists\, \mathcal{U} \in \mu$ such that $\mathcal{V} \supset \mathcal{U}^*\}$, where $\mathcal{U}^* := \{U^* : U \in \mathcal{U}\}$ with $U^* := U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$. Then, we claim that $(X^*, \mathcal{B}^{X^*}, \mu^*)$ is the one-point extension of $(X, \mathcal{B}^X, \mu)$ in **b-UFIL**. Evidently, $(\mathcal{B}^{X^*}, \mu^*)$ satisfies the axioms (buf_1), (buf_2) and (buf_4) for being a b-uniform filter structure on X^*.

To (buf_3): Let $B \in \mathcal{B}^{X^*} \backslash \{\emptyset\}$. In the first case, if $B \in \mathcal{B}^X \backslash \{\emptyset\}$, then $\dot{B} \times \dot{B} \in \mu$ is valid. But $\dot{B} \times \dot{B} \supset (\dot{B} \times \dot{B})^*$ because $U^* \in (\dot{B} \times \dot{B})^*$ implies $U^* = U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$ with $U \in \dot{B} \times \dot{B}$. Hence, $U \supset B \times B$, and $U^* \in \dot{B} \times \dot{B}$ follows. In the second case, if $B = \{\infty\}$, then choose $\mathcal{U} \in \mu$, hence, $\dot{B} \times \dot{B} = \dot{\infty} \times \dot{\infty} \supset \mathcal{U}^*$, since $U^* \in \mathcal{U}^*$ implies $U^* = U \cup (X^* \times \{\infty\}) \cup (\{\infty\} \times X^*)$ for some $U \in \mathcal{U}$. Consequently, $U^* \supset \{\infty\} \times \{\infty\}$, and $U^* \in \dot{B} \times \dot{B}$ is true.

Further, we indicate that $(X, \mathcal{B}^X, \mu)$ is a b-uniform filter subspace of

$$(X^*, \mathcal{B}^{X^*}, \mu^*)$$

which means that $(\mathcal{B}^X, \mu)$ is the coarsest b-uniform filter structure on X such that $i : (X, \mathcal{B}^X, \mu) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu^*)$ is b-uniformly continuous. Evidently, the inclusion map $i : X \longrightarrow X^*$ is buc. Now, let $(\mathcal{A}^X, \mu_X)$ be a b-uniform filter structure on X such that $i : (X, \mathcal{A}^X, \mu_X) \longrightarrow (X^*, \mathcal{B}^X, \mu^*)$ is buc, hence, $\mathcal{A}^X \subset$

$\mathcal{B}^X$. Since $A \in \mathcal{A}^X$ implies $i(A) = A \neq \{\infty\}$, $A \in \mathcal{B}^X$ follows. Now let $\mathcal{V} \in \mu_X$, then, by the hypothesis, $(i \times i)(\mathcal{V}) \in \mu^*$. At once we can find $\mathcal{U} \in \mu$ with $(i \times i)(\mathcal{V}) \supset \mathcal{U}^*$. Our goal is to verify $\mathcal{U} \subset \mathcal{V}$. $U \in \mathcal{U}$ implies $U^* \in (i \times i)(\mathcal{V})$, hence, $U^* \supset (i \times i)[V]$ for some $V \in \mathcal{V}$. But $(i \times i)[V] = V$, and $(x_1, x_2) \in V$ implies $x_1 \neq \infty \neq x_2$. Consequently, $U \supset V$ follows, and $U \in \mathcal{V}$ results. Taking our two results into account, we get $(\mathcal{A}^X, \mu_X) \leq (\mathcal{B}^X, \mu)$, which proves the claim.

Next, let $(Y, \mathcal{B}^Y, \mu_Y)$ be a b-uniform filter space, $(A, \mathcal{B}^A, \mu_A)$ b-uniform filter subspace of $(Y, \mathcal{B}^Y, \mu_Y)$ and $f : (A, \mathcal{B}^A, \mu_A) \longrightarrow (X, \mathcal{B}^X, \mu)$ b-uniformly continuous map, then we claim that $f^* : (Y, \mathcal{B}^Y, \mu_Y) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu^*)$ is buc, where f^* is defined by setting:

$$f^*(y) := \begin{cases} f(y) & \text{for each} \quad y \in A; \\ \infty & \text{for each} \quad y \in Y \backslash A. \end{cases}$$

To (buc_1): Let $B \in \mathcal{B}^Y$. If $f^*[B] = \{\infty\}$, then there is nothing to show. In the case of $\infty \notin f^*[B]$, $B \cap A \in \mathcal{B}^A$, and, by the hypothesis, $f[B \cap A] \in \mathcal{B}^X$ follows. We will show that $f^*[B] \subset f[B \cap A]$ holds. $z \in f^*[B]$ implies $z = f^*(y)$ for some $y \in B$. If assuming $y \in Y \backslash A$, $f^*(y) = \infty$ follows, which is a contradiction. Hence, the claim is true.

To (buc_2): Let $\mathcal{U} \in \mu_Y$, hence, $\mathcal{U}_A \in \mu_A$, see Proposition 2.7. By the hypothesis, we get $(f \times f)(\mathcal{U}_A) \in \mu$. Now we will show that $(f^* \times f^*)(\mathcal{U}) \supset ((f \times f)(\mathcal{U}_A))^*$ is true. Let R^* for $R \in (f \times f)(\mathcal{U}_A)$ be given, hence, $R \supset (f \times f)[V]$ for $V \in \mathcal{U}_A$. Then, $R^* = R \cup (X^* \times \{\infty\}) \cup (\{\infty\} \cup X^*) \supset (f \times f)(V)$ with $V = U \cap A \times A$ for some $U \in \mathcal{U}$ and, consequently, $(f^* \times f^*)[U] \in (f^* \times f^*)(\mathcal{U})$ is valid. It remains to verify $(f^* \times f^*)[U] \subset R^*$. Let $a \in (f^* \times f^*)[U]$, hence, $a = (f^* \times f^*)(b)$ for some $b \in U$, hence, $b = (y_1, y_2)$ for some pair $(y_1, y_2) \in U$. Consequently, $a = (f^*(y_1), f^*(y_2))$ follows. In the case of $f^*(y_1) = \infty = f^*(y_2)$, $a \in R^*$ results.

In the case of $(y_1, y_2) \in U \cap A \times A$, $(f^*(y_1), f^*(y_2)) = (f(y_1), f(y_2)) = (f \times f)(y_1, y_2) \in (f \times f)[V] \subset R^*$, implying $a \in R^*$. If $f^*(y_1) = f(y_1)$ and $f^*(y_2) = \infty$, then $a = (f(y_1), \infty) \in X^* \times \{\infty\} \subset R^*$. In the case of $f^*(y_1) = \infty$ and $f^*(y_2) = f(y_2)$, $a = (\infty, f(y_2)) \in \{\infty\} \times X^* \subset R^*$.

Now, to sum up, we can state that **b-UFIL** forms a quasitopos as indicated. □

5. On the productivity of quotient maps

In a topological construct $\underline{C}$ a $\underline{C}$-morphisms $f : (Y, \eta) \longrightarrow (X, \xi)$, where (Y, η) and (X, ξ) denote $\underline{C}$-objects is an extremal epimorphism in $\underline{C}$ if and only if f is a quotient map, i.e., $f : X \longrightarrow Y$ is surjective, and ξ is the final $\underline{C}$-structure on X, meaning that $(f : Y \longrightarrow X)$ is a final epi-sink. By transforming this fact into **b-UFIL**, we can state that ξ consists of the pair $(\mathcal{B}_X^f, \mu_f^X)$ with $\mathcal{B}_X^f = \{B \subset X : \exists\ D \in \mathcal{B}^Y$ such that $f[D] \supset B\}$ and $\mu_f^X = \{\mathcal{U} \in FIL(X \times X) : \exists\ \mathcal{V} \in \mu_Y$ such that $(f \times f)(\mathcal{V}) \subset \mathcal{U}\}$, where $(Y, \mathcal{B}^Y, \mu_Y)$ denotes the proposed b-uniform filter space. Then, in **b-UFIL** the following proposition holds:

Proposition 5.1. *In* **b-UFIL***, products of quotient maps are quotient maps again.*

Proof. In order to prove this statement in **b-UFIL**, let $(f_i : (Y_i, \mathcal{B}^{Y_i}, \mu_{Y_i}) \longrightarrow (X_i, \mathcal{B}^{X_i}, \mu_{X_i}))_{i\in I}$ be a non-empty family of quotient maps in **b-UFIL** and consider the following product diagram in **b-UFIL**, where $Y := \prod_{i\in I} Y_i$ and $X = \prod_{i\in I} X_i$, and p_{Y_i}, p_{X_i}, respectively, denote the prevailing projections, see Proposition 2.2.

$$\begin{array}{ccc} (Y, \mathcal{B}_I^Y, \mu_Y^I) & \xrightarrow{\prod f_i} & (X, \mathcal{B}_I^X, \mu_X^I) \\ \downarrow{\scriptstyle p_{Y_i}} & & \downarrow{\scriptstyle p_{X_i}} \\ (Y_i, \mathcal{B}^{Y_i}, \mu_{Y_i}) & \xrightarrow{f_i} & (X_i, \mathcal{B}^{X_i}, \mu_{X_i}) \end{array}$$

Since all f_i are surjective, $\prod f_i$ is surjective. For each $i \in I$, $\mathcal{B}^{X_i} = \mathcal{B}^{f_i}_{X_i}$ and $\mu_{X_i} = \mu^{X_i}_{f_i}$, because f_i is a quotient map. Now, we consider the pair $(\mathcal{E}^X, \mu_X)$, where $\mathcal{E}^X := \{D \subset X : \exists G \in \mathcal{B}_I^Y$ such that $D \subset \prod f_i[G]\}$ and $\mu_X := \{\mathcal{V} \in FIL(X \times X) : \exists \mathcal{W} \in \mu_Y^I$ such that $(\prod f_i \times \prod f_i)(\mathcal{W}) \subset \mathcal{V}\}$. Then, the following equation holds: $(\mathcal{B}_I^X, \mu_X^I) = (\mathcal{E}^X, \mu_X)$, which indicates that $\prod f_i$ is a quotient map.

To "$\leq$" : $B \in \mathcal{B}_I^X$ implies $p_{X_i}[B] \in \mathcal{B}^{X_i}$ for each $i \in I$. Thus, for each $i \in I$, there is some $B_i \in \mathcal{B}^{Y_i}$ with $p_{X_i}[B] \subset f_i[B_i]$. Then, $D := \prod_{i\in I} B_i \in \mathcal{B}_I^Y$ with $B \subset \prod f_i[D]$, because $x \in B$ implies $p_{X_i}(x) = x_i \in f_i[B_i]$. Hence, $x_i = f_i(z_i)$ for some $z_i \in B_i$ for each $i \in I$, and consequently $z = (z_i)_{i\in I} \in D$ with $\prod f_i(z) = z$ by using the commutativity of the product diagram. Consequently, $\mathcal{B}_I^X \subset \mathcal{E}^X$ is valid. Now let $\mathcal{U} \in \mu_X^I$, then, for each $i \in I$, $(p_{X_i} \times p_{X_i})(\mathcal{U}) \in \mu_{X_i}$. Hence, we can find $\mathcal{U}_i \in \mu_{Y_i}$ with $(f_i \times f_i)(\mathcal{U}_i) \subset (p_{X_i} \times p_{X_i})(\mathcal{U})$. If $j : \prod_{i\in I}(Y_i \times Y_i) \longrightarrow \prod_{i\in I} Y_i \times \prod_{i\in I} Y_i$ denotes the canonical isomorphism, i.e., $j((y_i, z_i)) = ((y_i), (z_i))$, and $\prod_{i\in I} \mathcal{U}_i$ the product filter on $\prod_{i\in I}(Y_i \times Y_i)$, then $j(\prod_{i\in I} \mathcal{U}_i)$ is a filter on $\prod_{i\in I} Y_i \times \prod_{i\in I} Y_i$ with $(p_{Y_i} \times p_{Y_i})(j(\prod_{i\in I} \mathcal{U}_i)) = \mathcal{U}_i$ for each $i \in I$. Thus $j(\prod_{i\in I} \mathcal{U}_i) \in \mu_Y^I$. If $\widehat{j} : \prod(X_i \times X_i) \longrightarrow \prod_{i\in I} X_i \times \prod_{i\in I} X_i$ denotes the canonical isomorphism, then $\widehat{j}^{-1}(\prod f_i \times \prod f_i)(j(\prod_{i\in I} \mathcal{U}_i)) \subset \prod_{i\in I}(f_i \times f_i)(\mathcal{U}_i) \subset \prod_{i\in I}(p_{X_i} \times p_{X_i})(\mathcal{U}) \subset \widehat{j}^{-1}(\mathcal{U})$. Hence, $(\prod f_i \times \prod f_i)(j(\prod_{i\in I} \mathcal{U}_i)) \subset \mathcal{U}$, which means that $\mathcal{U} \in \mu_X$. Thus, together we get $(\mathcal{B}_I^X, \mu_X^I) \leq (\mathcal{E}^X, \mu_X)$. Conversely, let $D \in \mathcal{E}^X$. Hence, we can find $G \in \mathcal{B}_I^Y$ with $D \subset \prod f_i[G]$. Then, $p_{Y_i}[G] \in \mathcal{B}^{Y_i}$ for each $i \in I$. Consequently, $f_i[p_Y[G]] \in \mathcal{B}^{f_i}_{X_i}$ follows for each $i \in I$. Since the diagram commutes, the equation $f_i[p_Y[G]] = p_{X_i}[\prod f_i[G]]$ holds for each $i \in I$. But the latter implies $\prod f_i[G] \in \mathcal{B}_I^X$, and $\mathcal{E}^X \subset \mathcal{B}_I^X$ follows. At last if $\mathcal{V} \in \mu_X$, then there exists $\mathcal{W} \in \mu_Y^I(\prod f_i \times \prod f_i)(\mathcal{W}) \subset \mathcal{V}$. Furthermore, $(p_{X_i} \times p_{X_i})((\prod f_i \times \prod f_i)(\mathcal{W})) = (f_i \times f_i)((p_{Y_i} \times p_{Y_i})(\mathcal{W})) \subset (p_{X_i} \times p_{X_i})(\mathcal{V})$ for each $i \in I$. Consequently, $(p_{X_i} \times p_{X_i})(\mathcal{V}) \in \mu_{X_i}$ is true for each $i \in I$, and $\mathcal{V} \in \mu_X^I$ results, which means that $\mu_X \subset \mu_X^I$ is valid and, thus, concludes the proof. □

Remark 5.2. Following the terminology in [11], we can state that the category **b-UFIL** now forms a strong topological universe and, therefore, makes an important contribution to the *convenient topology.*

Next, we will look at the behavior of certain subconstructs of **b-UFIL**, i.e., that of **sb-UFIL**, the full and isomorphism-closed subconstruct of **b-UFIL** whose objects are the symmetric b-uniform filter spaces and that of **SETb-UFIL**, the full and isomorphism-closed subconstruct of **b-UFIL** whose objects are the setconvergent b-uniform filter spaces, see [10].

6. The topological constructs sb-UFIL, b-FIL and SETb-UFIL

First, let us recall the definitions for b-uniform filter spaces to be symmetric and setconvergent.

Definition 6.1. A b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called

(i) *symmetric*, provided that μ satisfies the following condition:
  (s) $\mathcal{U} \in \mu$ implies $\mathcal{U}^{-1} \in \mu$, where $\mathcal{U}^{-1} := \{R^{-1} : R \in \mathcal{U}\}$ with $R^{-1} := \{(x,z) \in X \times X : (z,x) \in R\}$;

(ii) *setconvergent*, provided that $(\mathcal{B}^X, \mu)$ satisfies the following condition:
  (sc) $\mathcal{U} \in \mu$ implies the existence of $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{F} \in FIL(X)$ such that $\overset{\bullet}{B} \times \mathcal{F} \in \mu$ with $\overset{\bullet}{B} \times \mathcal{F} \subset \mathcal{U}$.

By **sb-UFIL** and **SETb-UFIL**, we will denote the full subconstructs of **b-UFIL**, whose objects are the symmetric and setconvergent b-uniform filter spaces, respectively.

Remark 6.2. As seen in [10], **sb-UFIL** is bireflective as well as bicoreflective in **b-UFIL** and contains, in particular, the category of semiuniform convergence spaces in the sense of Preuß and the category **b-FIL** of b-filter spaces as bireflective and bicoreflective subconstruct in **sb-UFIL**. The last mentioned candidates can be regarded as spaces where the structures $(\mathcal{B}^X, \mu)$ are generated by all their μ-Cauchy filters and, thus, play an important role if one considers the *completeness* of spaces.

In some special cases, filter merotopic spaces or filter spaces [5], respectively, can be recovered so that the Cauchy spaces or, more specifically, proximity spaces are integrated as well. In addition, **SETb-UFIL** is bicoreflective in **b-UFIL**, and it is isomorphic to the full subcategory **RO-SETCONV** of **SETCONV**, whose objects are the reordered set-convergence spaces. Let us mention that **RO-SETCONV** is bireflective in **SETCONV**. Reordered set-convergence spaces are coming into play if one considers *point-convergence* on arbitrary $\underline{B}$-sets. In the *classical* case point-convergence on a set, X can be regarded as a relation $q \subset FIL(X) \times X$ satisfying certain conditions. Hence, the pair (X, q) and the set-convergence space $(X, \mathcal{D}^X, \tau_q)$ are essentially the same. Here we note that $\mathcal{D}^X := \{\emptyset\} \cup \{\{x\} : x \in X\}$, and τ_q is defined by setting:

$\mathcal{F}\ \tau_q\ \emptyset$ if and only if $\mathcal{F} = \underline{P}X$;
$B \in \mathcal{D}^X \backslash \{\emptyset\}$ implies $\mathcal{F}\ \tau_q\ B$ if and only if $\mathcal{F}\ q\ x$ for each $x \in B$.

Evidently $(X, \mathcal{D}^X, \tau_q)$ defines a reordered set-convergence space, which is *repointed* by satisfying the following more extended definition:

A set-convergence $(\mathcal{B}^X, \tau)$, where $\mathcal{B}^X$ is $\underline{B} - set$ and $\tau \subset FIL(X) \times \mathcal{B}^X$, is called *repointed* if $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\mathcal{F}\ \tau\ B$ if $\mathcal{F}\ \tau\ \{x\}$ for each $x \in B$.

Evidently, each repointed set-convergence is reordered. But now, under this more general premise, we are able to consider point-convergence even on arbitrary $\underline{B}$-sets not only restricted to the discrete one. Thus, for example, point-convergence on the set of finite subsets, compact subsets or totally bounded subsets on a set X, respectively, can be now described and examined in addition to [4].

By the way, we also note that the full subconstruct **rp-SETCONV** of **RO-SETCONV**, whose objects are the repointed set-convergence spaces is bireflective in **RO-SETCONV**. As essence we keep hold that **b-UFIL** is also a suitable candidate for studying point-convergence on a more general level than the classical one. Here we should note that the considered convergence in [4] can be equivalently described by certain discrete b-uniform filter structures and vice versa. In fact, let a convergence space (X, ξ) be given in the sense of [4] then the space $(X, \mathcal{D}^X, \mu_\xi)$, where $\mu_\xi := \{\mathcal{U} \in FIL(X \times X) : \exists\ \mathcal{F} \in FIL(X)\ \exists\ x \in X\ ((\mathcal{F}, x) \in \xi$ and $\mathcal{U} \supset \mathcal{F} \times \overset{\bullet}{x}\}$ defines a specific discrete b-uniform filter space.

Conversely, if such a specific discrete b-uniform filter space $(X, \mathcal{D}^X, s)$ is given, then (X, η_s) is a convergence space, where $(\mathcal{F}, x) \in \eta_s$ if and only if $\mathcal{F} \times \overset{\bullet}{x} \in s$. The just defined assignments are functorial and, thus, determine the proposed isomorphism.

Theorem 6.3. *The topological construct* **sb-UFIL** *is Cartesian closed.*

Proof. This statement can be proved by purely categorical arguments, see Remark 6.2. and [1]. □

Theorem 6.4. *The topological construct* **b-FIL** *is Cartesian closed.*

Proof. See the above stated references. □

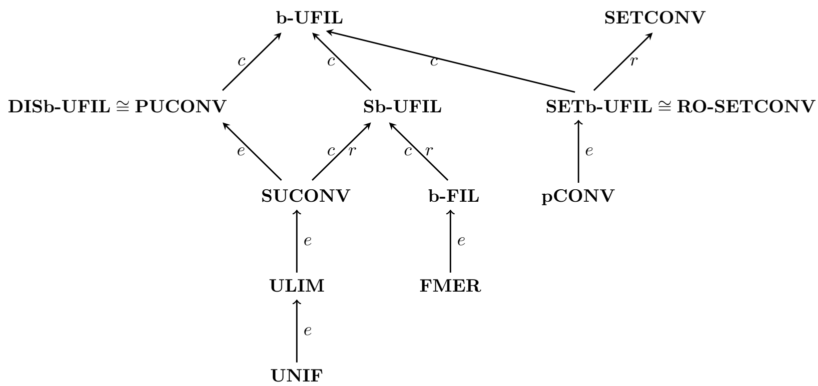
Theorem 6.5. *The topological constructs* **sb-UFIL** *and* **b-FIL** *are both hereditary.*

Proof. Since **b-UFIL** is hereditary and **sb-UFIL** a bicoreflective subconstruct of **b-UFIL**, respectively **b-FIL** a bicoreflective subconstruct of **sb-UFIL**, which are both closed under formation of subspaces in its prevailing supercategories, hence, the statements made are true by applying purely categorical arguments. □

Theorem 6.6. *In the topological constructs* **sb-UFIL** *and* **b-FIL***, the products of quotient maps are quotient maps again.*

Proof. Since **sb-UFIL** is bicoreflective in **b-UFIL**, closed under formation of products in **b-UFIL** and in **b-UFIL** quotients are productive, the claim follows by purely categorical arguments. Since **b-FIL** is bireflective in **sb-UFIL** closed under formation of quotient objects in **sb-UFIL** and in **sb-UFIL** quotients are productive, then, by using the above mentioned arguments, the claim results. □

Corollary 6.7. *The constructs* **sb-UFIL** *and* **b-FIL** *are forming strong topological universes.*



Legend:
e := embedding
c := bicoreflection
r := bireflection
≅: isomorphism
UFIL := Category of uniform limit spaces
UFIL:= Category of uniform spaces
FMER:= Category of filtermorotopic spaces
pCONV:= Categories of point-convergence spaces, e.g., **KENT**-convergence spaces, limit spaces, pseudotopological spaces, pretopological spaces, topological spaces, etc.

References

[1] A. Behling, *Einbettung uniformer Räume in topologische Universen*, Dissertation, Freie Universität Berlin, Berlin, 1991.

[2] H. L. Bentley, H. Herrlich and E. Lowen-Colebunders, *Convergence*, Journal of Pure and Applied Algebra **68** (1990), 27–45.

[3] D. Doitchinov, *A unified theory of topological, proximal and uniform spaces*, Doklady Akademii Nauk SSSR **156** (1964), 21–24 (Russian), Soviet Mathematics Dokl. **5** (1964) 595–598 (English Translation).

[4] S. Dolecki and F. Mynard, *Convergence Foundations of Topology*, World Scientific, 2016.

[5] M. Katětov, *On continuity structures and spaces of mappings*, Commentationes Mathematicae Universitatis Carolinae **6** (1965), 275–278.

[6] D. C. Kent, *Convergence functions and their related topologies*, Fundamenta Mathematicae **54** (1964), 125–133.

[7] D. Leseberg, *On topologically induced b-convergences*, Topology Proceedings **37** (2011), 293–313.

[8] D. Leseberg, *A new concept of convergence space*, Mathematica Pannonica **19** (2008), 291–303.

[9] D. Leseberg, *Strict Topological Extensions and Power-Objects for B-Convergence*, LAP Lambert Academic Publishing, Saarbrücken, 2015.

[10] D. Leseberg and Z. Vaziry, *Categories of several convergences*, Submitted to Journal of the American Mathematical Society.

[11] G. Preuß, *Foundation of Topology, An Approach to Convenient Topology*, Kluwer, 2002.

[12] A. Tozzi and O. Wyler, *On categories of supertopological spaces.* Acta Universitatis Carolinae. Mathematica et Physica **28** (1987), 137–149.

[13] O. Wyler, *On convergence of filters and ultrafilters to subsets*, in: H. Ehrig, H. Herrlich, H.-J. Kreowski and G. Preuß (eds.), Categorical Methods in Computer Science With Aspects from Topology, Lecture Notes in Computer Science **393**, 1988, 340–350.

Dieter Leseberg, Ernst Reuter Gesellschaft Berlin, Germany
e-mail: leseberg@zedat.fu-berlin.de

Zohreh Vaziry, Department of Mathematics, Islamic Azad University, Karaj Branch, Karaj, Iran
e-mail: z_m_vaziry@yahoo.co.in