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ON REAL AND COMPLEX CONVEXITY

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Abstract. We show that the holomorphic differential equation $k''(k+c) = \gamma(k')^2$ is fundamental for the study of a special class of convex and strictly plurisubharmonic functions ($k : \mathbb{C} \to \mathbb{C}$ be holomorphic and $\gamma, c \in \mathbb{C}$). We characterize all the 4 holomorphic non-constant functions $F_1, F_2 : \mathbb{C} \to \mathbb{C}$ and $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that the function u is convex on $\mathbb{C}^n \times \mathbb{C}$, where $u(z,w) = |F_1(w) - g_1(z)|^2 + |F_2(w) - g_2(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

1. Introduction

Let $f : D \to \mathbb{C}$ be a function and $\psi(z,w) = |w - f(z)|^s$, where D is a convex domain of $\mathbb{C}^n$, $n \in \mathbb{N} \setminus \{0\}$ and $s \in \mathbb{R}_+$. The function ψ is convex on $D \times \mathbb{C}$ if and only if f is an affine function on D and $s \geq 1$.

Now using the paper [3], we can establish that ψ is a function of class C^2, convex and strictly plurisubharmonic (psh) on $D \times \mathbb{C}$ if and only if f is an affine function on D, $n = 1, s = 2$ and $\frac{\partial f}{\partial \overline{z}}(z) \neq 0$, for each $z \in D$. Now we can observe that the above two situations are different and the last case is important for studying the complex function theory.

Let $f_1(z) = < z/a > + \overline{< z/b >} + c$, $a, b \in \mathbb{C}^n$, $c \in \mathbb{C}$, $z \in \mathbb{C}^n$. Denote $\psi_1(z,w) = |w - f_1(z)|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then ψ_1 is convex on $\mathbb{C}^n \times \mathbb{C}$ and not strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$ if and only if ($n \geq 2$), or ($n = 1$ and $b = 0$). Therefore, for $b = 0$, the function ψ_1 is convex and not strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$. In one complex variable, we consider the function $\psi_2(z) = x_1^2 + x_2^2 + 4x_1x_2$, for $z = (x_1 + ix_2) \in \mathbb{C}$, $x_1, x_2 \in \mathbb{R}$. Then, ψ_2 is a function of class C^∞ and strictly subharmonic on $\mathbb{C}$, but ψ_2 is not convex at all points of $\mathbb{C}$.

Now put $\psi_3(z,w) = |w - e^{(a_1\overline{z_1}+b_1)}|^2 + \cdots + |w - e^{(a_n\overline{z_n}+b_n)}|^2$ for $(z,w) = (z_1, \ldots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$, where $a_1, \ldots, a_n \in \mathbb{C}\setminus\{0\}$ and $b_1, \ldots, b_n \in \mathbb{C}$. ψ_3 is strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$, but it is not convex in any non-empty euclidean open ball of $\mathbb{C}^n \times \mathbb{C}$ (for the proof, we can consider the function $\psi_3(\overline{z}, w)$).

Moreover, if we consider $\psi_4(z,w) = x_1$, for $z = (z_1, \ldots, z_n)$ and $z_1 = x_1 + iy_1$, where $x_1, y_1 \in \mathbb{R}$, then ψ_4 is convex on $D \times \mathbb{C}$, but ψ_4 is not strictly plurisubharmonic at any point of this domain.

Therefore, there is a great difference between the class of (convex functions) and the class of (strictly plurisubharmonic functions) over any convex domain of $\mathbb{C}^N$,

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for each $N \geq 1$. Therefore, it is clear to study the new special class consisting of convex and strictly plurisubharmonic functions over convex domains of $\mathbb{C}^n$, $n \geq 1$.

Now denote by $\psi_5(z,w) = |w - b\overline{z}|^2$, for $(z,w) \in \mathbb{C}^2$. Then, ψ_5 is C^∞, convex and strictly plurisubharmonic on $\mathbb{C}^2$, but ψ_5 is not strictly convex at any point of $\mathbb{C}^2$. Therefore we have a great difference between the special class (convex and strictly plurisubharmonic functions) and the class (strictly convex functions) over all convex domains of $\mathbb{C}^n$, $n \geq 1$.

Observe that we can prove for each $n \geq 1$, for all convex domains G of $\mathbb{C}^n$ and every function $h : G \to \mathbb{C}$, that the function ψ_6 is not strictly convex in any non-empty euclidean open ball subset of $G \times \mathbb{C}$. $\psi_6(z,w) = |w - h(z)|^2$, for $(z,w) \in G \times \mathbb{C}$.

Problem 1.1. Find exactly all the holomorphic non-constant functions

$$F_1, F_2, F_3, F_4 : \mathbb{C} \to \mathbb{C}$$

and all the holomorphic functions $\varphi_1, \varphi_2, \varphi_3, \varphi_4 : \mathbb{C}^n \to \mathbb{C}$, such that v_1 and v_2 are convex on $\mathbb{C}^n \times \mathbb{C}$ and v is strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$.

$$\begin{aligned} v_1(z,w) &= |F_1(w) - \varphi_1(z)|^2 + |F_2(w) - \varphi_2(z)|^2, \\ v_2(z,w) &= |F_3(w) - \varphi_3(z)|^2 + |F_4(w) - \varphi_4(z)|^2 \end{aligned}$$

and $v = (v_1 + v_2)$, where $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

In this paper, we consider the case where we characterize $F_1, \varphi_1, F_2, \varphi_2$ such that v_1 is convex and u is strictly plurisubharmonic on $\mathbb{C}^n \times \mathbb{C}$, $u(z,w) = v_1(z, \overline{w})$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Remark 1.2. We have the following result for holomorphic functions. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Assume that there exist

$$(A_1, \ldots, A_N), (B_1, \ldots, B_N) \in \mathbb{C}^N \setminus \{0\}, \quad (\alpha_1, \ldots, \alpha_N), (\beta_1, \ldots, \beta_N) \in \mathbb{C}^N$$

such that

$$\frac{A_1\overline{\alpha_1} + \cdots + A_N\overline{\alpha_N}}{\|(A_1, \ldots, A_N)\|^2} \neq \frac{B_1\overline{\beta_1} + \cdots + B_N\overline{\beta_N}}{\|(B_1, \ldots, B_N)\|^2}.$$

Assume that

$$|A_1 f - \alpha_1|^2 + \cdots + |A_N f - \alpha_N|^2$$

and

$$|B_1 f - \beta_1|^2 + \cdots + |B_N f - \beta_N|^2$$

are convex functions on $\mathbb{C}^N$. Using the holomorphic differential equation $k''(k + c) = \gamma(k')^2$, ($\gamma, c \in \mathbb{C}$ and $k : \mathbb{C} \to \mathbb{C}$ be analytic) and the paper [2], we prove that f is an affine function over $\mathbb{C}^n$. Observe that the complex structure is fundamental in this situation. Because if we consider $F(z) = x_1^4 + 1$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $z_1 = x_1 + iy_1$, $(x_1, y_1) \in \mathbb{R}^2$. F is real analytic on $\mathbb{C}^n$, $|F|^2$ and $|F+1|^2$ are convex functions on $\mathbb{C}^n$, but F is not an affine function on $\mathbb{C}^n$.

Let U be a domain of $\mathbb{R}^d, (d \geq 2)$. We denote by sh(U) the subharmonic functions on U. m_d is the Lebesgue measure on $\mathbb{R}^d$. Let $f : U \to \mathbb{C}$ be a function. $|f|$ is the modulus of f, $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are respectively the real and imaginary parts of f.

Let $g : D \to \mathbb{C}$ be an analytic function, D a domain of $\mathbb{C}$. We denote by $g^{(0)} = g, g^{(1)} = g'$ the holomorphic derivative of g over D. $g^{(2)} = g'', g^{(3)} = g'''$. In general, $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ is the holomorphic derivative of g of order m, for all $m \in \mathbb{N}$. Let $z \in \mathbb{C}^n, z = (z_1, \dots, z_n), n \geq 1$. If $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$, we denote $< z/\xi >= z_1\overline{\xi_1} + \dots + z_n\overline{\xi_n}$ and $B(\xi, r) = \{\zeta \in \mathbb{C}^n / \|\zeta - \xi\| < r\}$ for $r > 0$, where $\|\xi\| = \sqrt{< \xi/\xi >}$.

$C(U) = \{\varphi : U \to \mathbb{C}/\varphi$ is continuous on $U\}$.

$C^k(U) = \{\varphi : U \to \mathbb{C}/\varphi$ is of class C^k on $U\}, k \in \mathbb{N} \cup \{\infty\}$ and $k \geq 1$.

Let $\varphi : U \to \mathbb{C}$ be a function of class C^2. $\Delta(\varphi)$ is the Laplacian of φ.

Let D be a domain of $\mathbb{C}^n$ $(n \geq 1)$, $psh(D)$ and $prh(D)$ the class of plurisubharmonic and pluriharmonic functions on D, respectively. For all $a \in \mathbb{C}, |a|$ is the modulus of a. $\mathrm{Re}(a)$ is the real part of a. $D(a, r) = \{z \in \mathbb{C}/|z - a| < r\}$.

If p is an analytic polynomial over $\mathbb{C}$, $\deg(p)$ is the degree of p.

For the study of the theory and extension problems of analytic and plurisubharmonic functions we cite the references [1, 5–10, 12, 14, 16–19, 22–24, 27–30, 32, 33, 35–39]. The class of n-harmonic functions is first introduced by Rudin in [36]. There are several investigations of the plurisubharmonic functions announced in [2, 13, 21, 25, 26, 31, 32, 34] and [11].

Good references for the study of convex functions in complex convex domains are [15, 20, 24].

2. The real and complex convexity

We begin this section by the following lemma.

Lemma 2.1. *Consider the holomorphic functions k_1, k_2, k_3 and k_4 defined by $k_1(z) = z^2$, $k_2(z) = -z^2$, $k_3(z) = Az^s$, $k_4(z) = -Az^s$, for $z \in \mathbb{C}$, where $A \in \mathbb{C}\setminus\{0\}$, $s \in \mathbb{N} \setminus \{0, 1\}$. Define*

$$\begin{aligned}
u(z, w) &= |w - z^2|^2 + |w + z^2|^2 + |w - z|^2,\\
v(z, w) &= |w - Az^s|^2 + |w + Az^s|^2 + |w - (az + b)|^2,\\
\varphi(z, w) &= |w - Az^s|^2 + |w + Az^s|^2,\\
\varphi_1(z, w) &= |w - Az^s|^2 + |w + Az^s|^2 + |w - A(z + 1)^s|^2 + |w + A(z + 1)^s|^2,\\
\varphi_2(z, w) &= |w - Az^s|^2 + |w + Az^s|^2 + |w - A(\overline{z + 1})^s|^2 + |w + A(\overline{z + 1})^s|^2,\\
\varphi_3(z, w) &= |w - z - Az^2|^2 + |w - z + Az^2|^2\\
&\quad + |w - A(z + 1)^2|^2 + |w + A(z + 1)^2|^2,
\end{aligned}$$

$(z, w) \in \mathbb{C}^2$, $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Then, we have the following 5 properties.

(1) *u and v are strictly convex functions on $\mathbb{C}^2$, but k_j is not an affine function for all $j \in \{1, 2, 3, 4\}$.*

(2) *φ is convex on $\mathbb{C}^2$ but k_j is not an affine holomorphic function for all $j \in \{3, 4\}$. Moreover, φ is strictly convex on $\mathbb{C} \setminus \{0\} \times \mathbb{C}$, but φ is not strictly convex on $\mathbb{C}^2$.*

(3) *φ_1 is strictly convex on $\mathbb{C}^2$, but f_j, k_j are not affine functions for all $j \in \{3, 4\}$, where $f_3(z) = k_3(z + 1)$ and $f_4(z) = k_4(z + 1)$ for all $z \in \mathbb{C}$.*

(4) φ_2 *is strictly convex on* $\mathbb{C}^2$. *But* k_3, k_4, $\overline{f_3}$ *and* $\overline{f_4}$ *are not affine functions over* $\mathbb{C}$.
(5) φ_3 *is strictly convex on* $\mathbb{C}^2$, *but* g_j *and* f_j *are not affine functions for all* $j \in \{3,4\}$, *with* $g_3(z) = z + Az^2$ *and* $g_4(z) = z - Az^2$ *for* $z \in \mathbb{C}$.

Proof. Let $\psi : D \to \mathbb{R}$ be a function of class C^2, D is a convex domain of $\mathbb{C}^n$, $n \geq 1$. Recall that ψ is convex on D if and only if

$$\Big|\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big| \leq \sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k},$$

for each $z = (z_1, \ldots, z_n) \in D$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$. Now ψ is strictly convex on D if and only if

$$\Big|\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big| < \sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k},$$

for each $z \in D$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n \setminus \{0\}$.

(1) We have u a function of class C^∞ on $\mathbb{C}^2$.

$$u(z,w) = 2|w|^2 + 2|z|^4 + |w-z|^2,$$

for $(z,w) \in \mathbb{C}^2$. Therefore,

$$\begin{aligned}
&\Big|\frac{\partial^2 u}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u}{\partial z \partial w}(z,w)\alpha\beta\Big| \\
=&4|z|^2|\alpha|^2 < \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w)\beta\overline{\beta} + 2\mathrm{Re}\Big(\frac{\partial^2 u}{\partial z \partial \overline{w}}(z,w)\alpha\overline{\beta}\Big) \\
=&|\beta - 2z\alpha|^2 + |\beta + 2z\alpha|^2 + |\beta - \alpha|^2,
\end{aligned}$$

for each $(z,w) \in \mathbb{C}^2$ and $(\alpha,\beta) \in \mathbb{C}^2 \setminus \{0\}$. Consequently, u is strictly convex on $\mathbb{C}^2$. Using the above 2 inequalities, we obtain the proof of the lemma. ☐

The above result suggests the following natural problem.

Question 2.2. Find all the holomorphic functions $k_1, k_2 : \mathbb{C} \to \mathbb{C}$ such that φ is strictly convex on $\mathbb{C}^2$, $\varphi(z,w) = |w - k_1(z)|^2 + |w - k_2(z)|^2$, for $(z,w) \in \mathbb{C}^2$. Is the number of holomorphic functions fundamental in this situation?

We have an answer to this question which is given by

Theorem 2.3. *Let* $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ *be* 2 *holomorphic functions. Given* $A_1, A_2 \in \mathbb{C} \setminus \{0\}$, $c_1, c_2 \in \mathbb{C}$, *put*

$$\begin{aligned}
u(z,w) &= |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2, \\
v(z,w) &= |A_1 w - g_1(z) + c_1|^2 + |A_2 w - g_2(z) + c_2|^2,
\end{aligned}$$

$(z,w) \in \mathbb{C}^2$. *The following are equivalent*

(I) u *is strictly convex on* $\mathbb{C}^2$;
(II) g_1 *and* g_2 *are affine holomorphic functions,* $g_1(z) = a_1 z + b_1$, $g_2(z) = a_2 + b_2$, *for* $z \in \mathbb{C}$, *where* $a_1, b_1, a_2, b_2 \in \mathbb{C}$ *with* $(a_1 A_2 \neq a_2 A_1)$;
(III) v *is strictly convex on* $\mathbb{C}^2$.

Proof. (I) implies (II). Note that u is a function of class C^∞ on $\mathbb{C}^2$. Since u is strictly convex on $\mathbb{C}^2$, we have

$$\Big|\frac{\partial^2 u}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u}{\partial z\partial w}(z,w)\alpha\beta\Big| < \frac{\partial^2 u}{\partial z\partial \overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w\partial \overline{w}}(z,w)\beta\overline{\beta} + 2\mathrm{Re}\Big(\frac{\partial^2 u}{\partial z\partial \overline{w}}(z,w)\alpha\overline{\beta}\Big),$$

for all $(z,w)\in\mathbb{C}^2$ and for all $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$. Fix $z\in D$. Then, we have

$$|(g_1''(z)\overline{g_1}(z) - \overline{A_1}\overline{w}g_1''(z) + g_2''(z)\overline{g_2}(z) - \overline{A_2}\overline{w}g_2''(z))\alpha^2| < |A_1\beta - g_1'(z)\alpha|^2 + |A_2\beta - g_2'(z)\alpha|^2, \tag{1}$$

for all $w\in\mathbb{C}$ and (α,β) fixed in $\mathbb{C}^2$ with $\alpha\neq 0$. Now if $(\overline{A_1}g_1''(z)+\overline{A_2}g_2''(z))\neq 0$, then the subset $\mathbb{C}$ is bounded. We get a contradiction. Consequently, $\overline{A_1}g_1''+\overline{A_2}g_2''=0$ over $\mathbb{C}$. We have

$$|g_1''(z)\overline{g_1}(z)+g_2''(z)\overline{g_2}(z)||\alpha|^2 < |A_1\beta - g_1'(z)\alpha|^2 + |A_2\beta - g_2'(z)\alpha|^2,$$

$\forall z\in\mathbb{C}$, $\forall(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$. Put $A=\frac{A_1}{A_2}\in\mathbb{C}\setminus\{0\}$. Thus, we have

$$|g_1''(\overline{g_1}-\overline{A}\overline{g_2})||\alpha|^2 < (|A_1|^2+|A_2|^2)|\beta|^2 + |g_1'\alpha|^2 + |g_2'\alpha|^2 - 2\mathrm{Re}\left(g_1'\overline{A_1}\alpha\overline{\beta} + g_2'\overline{A_2}\alpha\overline{\beta}\right)$$

over $\mathbb{C}$ and for all $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$. It follows that

$$(|A_1|^2+|A_2|^2)|\beta|^2 + [|g_1'|^2+|g_2'|^2 - |g_1''(\overline{g_1}-\overline{A}\overline{g_2})|]|\alpha|^2 - 2\mathrm{Re}\left((g_1'\overline{A_1}+g_2'\overline{A_2})\alpha\overline{\beta}\right) > 0$$

over $\mathbb{C}$, for all $(\alpha,\beta)\in\mathbb{C}^2\setminus\{0\}$. By the below Lemma 2.7 we have

$$|g_1'\overline{A_1}+g_2'\overline{A_2}|^2 < (|A_1|^2+|A_2|^2)[|g_1'|^2+|g_2'|^2 - |g_1''(g_1-Ag_2)|],$$

over $\mathbb{C}$. Thus,

$$|A_1|^2|g_1'|^2 + |A_2|^2|g_2'|^2 + 2\mathrm{Re}\left(g_1'\overline{A_1}\overline{g_2'}A_2\right) < (|A_1|^2+|A_2|^2)[|g_1'|^2+|g_2'|^2] - (|A_1|^2+|A_2|^2)|g_1''(g_1-Ag_2)|,$$

over $\mathbb{C}$. Then,

$$|A_2|^2|g_1'|^2 + |A_1|^2|g_2'|^2 - 2\mathrm{Re}\left(g_1'A_2\overline{g_2'A_1}\right) - (|A_1|^2+|A_2|^2)|g_1''(g_1-Ag_2)| > 0,$$

over $\mathbb{C}$. Therefore, $|A_2g_1'-A_1g_2'|^2 - (|A_1|^2+|A_2|^2)|g_1''(g_1-Ag_2)|>0$, on $\mathbb{C}$. Then, we have

$$|g_1''(g_1-Ag_2)| < \frac{|A_2|^2}{|A_1|^2+|A_2|^2}|g_1'-Ag_2'|^2, \tag{2}$$

over $\mathbb{C}$. Replace now g_1'' by $\frac{-\overline{A_2}}{\overline{A_1}} g_2'' = \frac{-1}{\overline{A}} g_2''$. It follows that $|g_2''(\frac{1}{\overline{A}} g_1 - g_2)| < \frac{|A_1|^2}{|A_1|^2+|A_2|^2} |g_2' - \frac{1}{A} g_1'|^2$, over $\mathbb{C}$, and then

$$\frac{1}{|A|^2} |Ag_2''(g_1 - Ag_2)| < \frac{|A_1|^2 |\frac{1}{A}|^2}{|A_1|^2 + |A_2|^2} |g_1' - Ag_2'|^2,$$

over $\mathbb{C}$. Then,

$$(3) \quad |-Ag_2''(g_1 - Ag_2)| < \frac{|A_1|^2}{|A_1|^2 + |A_2|^2} |g_1' - Ag_2'|^2,$$

over $\mathbb{C}$. The sum of (2) and (3) implies that

$$|g_1''(g_1 - Ag_2)| + |-Ag_2''(g_1 - Ag_2)| < \frac{|A_2|^2 + |A_1|^2}{|A_1|^2 + |A_2|^2} |g_1' - Ag_2'|^2,$$

over $\mathbb{C}$. By the triangle inequality, we have then $|(g_1''-Ag_2'')(g_1-Ag_2)| < |g_1'-Ag_2'|^2$, over $\mathbb{C}$.

Now put $k = g_1 - Ag_2$. Observe that the function k is holomorphic on $\mathbb{C}$, k is not constant and k satisfies the holomorphic differential inequality $|k''k| < |k'|^2$ over $\mathbb{C}$. Therefore, $k''(z)k(z) = \gamma(k'(z))^2$, $\forall z \in \mathbb{C}$, where $\gamma \in \mathbb{C}$ and $|\gamma| < 1$ (the above strict inequality is very important in this situation). By [2, Théorème 21] it follows that $|k|^2$ is convex on $\mathbb{C}$ and $\gamma \in \{\frac{s-1}{s}, 1/s \in \mathbb{N} \setminus \{0\}\}$. Indeed, we have $k(z) = (az + b)^s$, for each $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$ and $s \in \mathbb{N}$, or $k(z) = e^{(cz+d)}$, for each $z \in \mathbb{C}$, where $c, d \in \mathbb{C}$. Put $k_1(z) = e^{(cz+d)}$, for $z \in \mathbb{C}$. We see that k_1 is analytic on $\mathbb{C}$, and $|k_1''(z)k_1(z)| = |k_1'(z)|^2$, for each $z \in \mathbb{C}$. Consequently, $k(z) = (az + b)^s$, where $a, b \in \mathbb{C}$ and $s \in \mathbb{N}$. Observe that $s \neq 0$ because k is not constant on $\mathbb{C}$. $g_1(z) - Ag_2(z) = (az+b)^s$, for all $z \in \mathbb{C}$. Suppose that $s \geq 2$. Then $g_1'(-\frac{b}{a}) - Ag_2'(-\frac{b}{a}) = 0$. But we have

$$0 \leq \left|\left(g_1''\left(-\frac{b}{a}\right) - Ag_2''\left(-\frac{b}{a}\right)\right)\left(g_1\left(-\frac{b}{a}\right) - Ag_2\left(-\frac{b}{a}\right)\right)\right| < \left|g_1'\left(-\frac{b}{a}\right) - Ag_2'\left(-\frac{b}{a}\right)\right|^2 = 0.$$

This is a contradiction. It follows that $s = 1$. Therefore, $g_1(z) - Ag_2(z) = az + b$, for all $z \in \mathbb{C}$. In this case, we have $g_1'' - Ag_2'' = 0$ on $\mathbb{C}$. Thus, $g_1'' - \frac{A_1}{A_2} g_2'' = 0$ and then $A_2 g_1'' - A_1 g_2'' = 0$ on $\mathbb{C}$.

Now recall that we have $\overline{A_1} g_1'' + \overline{A_2} g_2'' = 0$ on $\mathbb{C}$. Finally, we have the system of holomorphic differential equations over $\mathbb{C}$

$$\begin{cases} A_2 g_1'' - A_1 g_2'' = 0 \\ \overline{A_1} g_1'' + \overline{A_2} g_2'' = 0. \end{cases}$$

Now since $(|A_1|^2 + |A_2|^2) > 0$, we have $g_1'' = g_2'' = 0$ over $\mathbb{C}$. It follows that g_1 and g_2 are holomorphic affine functions on $\mathbb{C}$. $g_1(z) = a_1 z + b_1$, $g_2(z) = a_2 z + b_2$ for all $z \in \mathbb{C}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$. Since now the function u is strictly convex on $\mathbb{C}^2$, $0 < |A_1\beta - a_1\alpha|^2 + |A_2\beta - a_2\alpha|^2$, $\forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{0\}$. Therefore, the system

$$\begin{cases} A_1\beta - a_1\alpha = 0 \\ A_2\beta - a_2\alpha = 0 \end{cases}$$

for $(\alpha, \beta) \in \mathbb{C}^2$ has only one solution $(\alpha, \beta) = (0, 0)$. It follows that $(-A_1 a_2 + a_1 A_2) \neq 0$ and, consequently, $A_1 a_2 \neq a_1 A_2$.

(II) implies (I) and (II) implies (III) are in fact classical cases. (III) implies (II). In this situation we consider the above proof and replace g_1 by $(g_1 - c_1)$ and g_2 by $(g_2 - c_2)$, we conclude this proof. □

Remark 2.4. Let $g_1, \dots, g_N : \mathbb{C} \to \mathbb{C}$ be N holomorphic functions, $N \geq 2$. Put

$$u(z, w) = |w - g_1(z)|^2 + \cdots + |w - g_N(z)|^2, \quad \text{for} \quad (z, w) \in \mathbb{C}^2.$$

We prove that if the number N of the holomorphic functions satisfies $N \geq 3$, the above theorem is false. Define $g_1(z) = z^2$, $g_2(z) = -z^2$, $g_3(z) = z, \dots, g_N(z) = z$, for $z \in \mathbb{C}$. g_1 and g_2 are not affine functions over $\mathbb{C}$. But u is strictly convex on $\mathbb{C}^2$ because, if $N \geq 4$, the function u_1 is strictly convex on $\mathbb{C}^2$, where $u_1(z, w) = |w - z^2|^2 + |w + z^2|^2 + |w - z|^2$, for $(z, w) \in \mathbb{C}^2$. The function u_2 is convex on $\mathbb{C}^2$, where $u_2(z, w) = (N-3)|w - z|^2$. Therefore, $u = (u_1 + u_2)$ is strictly convex on $\mathbb{C}^2$. If $N = 3$, $u = u_1$ is strictly convex on $\mathbb{C}^2$.

We have

Lemma 2.5. *Let $n \geq 2$. For $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^n \times \mathbb{C}$, we define*

$$v(z, w) = |w - (z_1 + \cdots + z_n)^2|^2 + |w + (z_1 + \cdots + z_n)^2|^2 + |w - z_1|^2 + \cdots + |w - z_n|^2.$$

Then, v is strictly convex on $\mathbb{C}^n \times \mathbb{C}$.

Proof. Note that v is the sum of $(n+2)$ functions, $v = v_1 + \cdots + v_{n+2}$, where $v_1(z, w) = |w - (z_1 + \cdots + z_n)^2|^2 = |w - g_1(z)|^2$, $v_2(z, w) = |w + (z_1 + \cdots + z_n)^2|^2 = |w - g_2(z)|^2$, $v_3(z, w) = |w - z_1|^2 = |w - g_3(z)|^2, \dots, v_{n+2}(z, w) = |w - z_n|^2 = |w - g_{n+2}(z)|^2$. $g_1(z) = (z_1 + \cdots + z_n)^2$, $g_2(z) = -(z_1 + \cdots + z_n)^2$, $g_3(z) = z_1, \dots, g_n(z) = z_n$, for each $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. g_1 and g_2 are not affine functions over $\mathbb{C}^n$. But v is strictly convex on $\mathbb{C}^n$, because, if we put $w = z_{n+1} \in \mathbb{C}$, we have

$$\Big| \sum_{j,k=1}^{n+1} \frac{\partial^2 v}{\partial z_j \partial z_k}(z, z_{n+1}) \alpha_j \alpha_k \Big| < \sum_{j,k=1}^{n+1} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z, z_{n+1}) \alpha_j \overline{\alpha_k}$$

for each $(\alpha_1, \dots, \alpha_n, \alpha_{n+1}) = (\alpha_1, \dots, \alpha_n, \beta) \in \mathbb{C}^{n+1} \setminus \{0\}$. Thus, we have

$$\begin{aligned} 4|z_1 + z_2 + \cdots + z_n|^2 |\alpha_1 + \alpha_2 + \cdots + \alpha_n|^2 \\ < 2|\beta|^2 + 4|z_1 + z_2 + \cdots + z_n|^2 |\alpha_1 + \alpha_2 + \cdots + \alpha_n|^2 \\ + |\beta - \alpha_1|^2 + |\beta - \alpha_2|^2 + \cdots + |\beta - \alpha_n|^2. \end{aligned}$$

Therefore,

$$\varphi(\alpha_1, \alpha_2, \dots, \alpha_n, \beta) = 2|\beta|^2 + |\beta - \alpha_1|^2 + |\beta - \alpha_2|^2 + \cdots + |\beta - \alpha_n|^2 > 0.$$

In fact, if $\beta = 0$, then $\varphi(\alpha_1, \alpha_2, \dots, \alpha_n, \beta) = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 > 0$. Now, if $\beta \neq 0$, then $\varphi(\alpha_1, \alpha_2, \dots, \alpha_n, \beta) > 0$. Consequently, v is strictly convex on $\mathbb{C}^n \times \mathbb{C}$. The number of functions in this case is $(n+2)$. □

Proposition 2.6. *Over $\mathbb{C}^2 \times \mathbb{C}$, there exist 3 functions $g_1, g_2, g_3 : \mathbb{C}^2 \to \mathbb{C}$ such that g_j is holomorphic not affine over $\mathbb{C}$, for all $j \in \{1, 2, 3\}$ and the function u is convex on $\mathbb{C}^2 \times \mathbb{C}$; $u(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2 + |w - g_3(z)|^2$, for*

$(z, w) \in \mathbb{C}^2 \times \mathbb{C}$. *But u is not strictly convex at any point of $\mathbb{C}^3$ (then u is not strictly convex in any not empty Euclidean open ball subset of $\mathbb{C}^3$).*

Proof. For $z = (z_1, z_2) \in \mathbb{C}^2$, we define $g_1(z) = (z_1 + z_2)^2$, $g_2(z) = (z_1 + z_2)^2$ and $g_3(z) = -2(z_1 + z_2)^2$. Then, g_1, g_2 and g_3 satisfy the condition of the above proposition 1. Observe that, in this case, the number of functions is 3. □

The following lemma is fundamental in this paper:

Lemma 2.7. *Let $a, b, c \in \mathbb{C}$. Then*

(1) $a\alpha\overline{\alpha} + b\beta\overline{\beta} + 2Re(c\alpha\overline{\beta}) > 0$ *for each* $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ *if and only if* $a > 0, b > 0$ *and* $|c|^2 < ab$.

(2) $a\alpha\overline{\alpha} + b\beta\overline{\beta} + 2Re(c\alpha\overline{\beta}) \geq 0$ *for all* $(\alpha, \beta) \in \mathbb{C}^2$ *if and only if* $a \geq 0, b \geq 0$ *and* $|c|^2 \leq ab$.

Proof. A proof of this lemma can be found in [2]. □

Remark 2.8. Let $D = D(2, \frac{1}{4}) = \{z \in \mathbb{C}/|z - 2| < \frac{1}{4}\}$. Define $g(z) = z^2 + 1$, $z \in D$. g is a holomorphic function on $\mathbb{C}$. Put $g_1 = g$ and $g_2 = -g$. Let $u(z, w) = |w - g(z)|^2 + |w + g(z)|^2$, $(z, w) \in D \times \mathbb{C}$. We can verify that u is strictly convex on $D \times \mathbb{C}$, but g_1 and g_2 are not affine functions over D. This proves that, in Theorem 2.3, it is fundamental to consider the subset $\mathbb{C}$ globally. On the other hand, we also conclude that the above Theorem 2.3 is false on all convex domains in the form $G \times \mathbb{C}$, where G is a non-empty bounded convex domain of $\mathbb{C}$.

For future analysis, we have the following question.

Question 2.9. Characterize exactly all the analytic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ such that the function u is convex on $\mathbb{C}^2$ $(u(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2, (z, w) \in \mathbb{C}^2)$.

Characterize exactly all the analytic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ such that the function u is strictly psh and convex on $\mathbb{C}^2$.

Characterize exactly all the analytic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ such that the function u is strictly psh and not convex on $\mathbb{C}^2$.

Characterize exactly all the analytic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ such that the function u is convex and not strictly psh on $\mathbb{C}^2$ (or convex and not strictly psh on all not empty euclidean open ball subsets of $\mathbb{C}^2$, convex strictly psh and not strictly convex on all not empty euclidean open ball subsets of $\mathbb{C}^2$, convex and not strictly convex on all not empty euclidean open ball subsets of $\mathbb{C}^2$).

Using the holomorphic differential equation $k''(k + c) = \gamma(k')^2$ ($k : \mathbb{C} \to \mathbb{C}$ is a holomorphic function and $(\gamma, c) \in \mathbb{C}^2$), we have the following characterization in complex analysis.

Theorem 2.10. *Given two holomorphic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$, let $v(z, w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, for $(z, w) \in \mathbb{C}^2$. Then, the following assertions are equivalent*

(I) *v is convex on $\mathbb{C}^2$;*

(II) *$g_1(z) = az + b + (cz + d)^m$ and $g_2(z) = az + b - (cz + d)^m$ for each $z \in \mathbb{C}$, with (a, b, c and $d \in \mathbb{C}$), or $g_1(z) = a_1 z + b_1 + e^{(c_1 z + d_1)}$ and $g_2(z) = a_1 z + b_1 - e^{(c_1 z + d_1)}$ for each $z \in \mathbb{C}$, where (a_1, b_1, c_1 and $d_1 \in \mathbb{C}$).*

Proof. (I) implies (II). Note that u is a function of class C^∞ on $\mathbb{C}^2$. Since u is convex on $\mathbb{C}^2$, we have

$$\left|\frac{\partial^2 u}{\partial z^2}(z,w)\alpha^2+\frac{\partial^2 u}{\partial w^2}(z,w)\beta^2+2\frac{\partial^2 u}{\partial z\partial w}(z,w)\alpha\beta\right|$$
$$\leq \frac{\partial^2 u}{\partial z\partial \overline{z}}(z,w)\alpha\overline{\alpha}+\frac{\partial^2 u}{\partial w\partial \overline{w}}(z,w)\beta\overline{\beta}+2\mathrm{Re}\left(\frac{\partial^2 u}{\partial z\partial \overline{w}}(z,w)\alpha\overline{\beta}\right),$$

for all $(z,w)\in\mathbb{C}^2$ and for all $(\alpha,\beta)\in\mathbb{C}^2$.

Fix $z\in D$. Then, we have

$$(1)\quad |(g_1''(z)\overline{g_1}(z)-\overline{w}g_1''(z)+g_2''(z)\overline{g_2}(z)-\overline{w}g_2''(z))\alpha^2|$$
$$\leq |\beta-g_1'(z)\alpha|^2+|\beta-g_2'(z)\alpha|^2,$$

for all $w\in\mathbb{C}$ and (α,β) fixed in $\mathbb{C}^2$ with $\alpha\neq 0$. Now if $(g_1''(z)+g_2''(z))\neq 0$, then the subset $\mathbb{C}$ is bounded. We get a contradiction. Consequently, $g_1''+g_2''=0$ over $\mathbb{C}$. We have

$$|g_1''(z)\overline{g_1}(z)+g_2''(z)\overline{g_2}(z)||\alpha|^2\leq|\beta-g_1'(z)\alpha|^2+|\beta-g_2'(z)\alpha|^2,$$

$\forall z\in\mathbb{C}$, $\forall(\alpha,\beta)\in\mathbb{C}^2$. Thus, we have

$$|g_1''(\overline{g_1}-\overline{g_2})||\alpha|^2\leq 2|\beta|^2+|g_1'\alpha|^2+|g_2'\alpha|^2-2\mathrm{Re}\left(g_1'\alpha\overline{\beta}+g_2'\alpha\overline{\beta}\right)$$

over $\mathbb{C}$ and for all $(\alpha,\beta)\in\mathbb{C}^2$. It follows that

$$2|\beta|^2+[|g_1'|^2+|g_2'|^2-|g_1''(\overline{g_1}-\overline{g_2})|]|\alpha|^2-2\mathrm{Re}\left((g_1'+g_2')\alpha\overline{\beta}\right)\geq 0\quad\text{over }\mathbb{C},$$

for all $(\alpha,\beta)\in\mathbb{C}^2$. By the above Lemma 2.7, we have

$$|g_1'+g_2'|^2\leq 2[|g_1'|^2+|g_2'|^2-|g_1''(g_1-g_2)|],\quad\text{over }\mathbb{C}.$$

Thus,

$$|g_1'|^2+|g_2'|^2+2\mathrm{Re}[g_1'\overline{g_2'}]\leq 2[|g_1'|^2+|g_2'|^2]-2|g_1''(g_1-g_2)|,\quad\text{over }\mathbb{C}.$$

Then,

$$|g_1'|^2+|g_2'|^2-2\mathrm{Re}[g_1'\overline{g_2'}]-2|g_1''(g_1-g_2)|\geq 0,\quad\text{over }\mathbb{C}.$$

Therefore, $|g_1'-g_2'|^2-2|g_1''(g_1-g_2)|\geq 0$, on $\mathbb{C}$. Then, we have

$$(2)\quad |g_1''(g_1-g_2)|\leq\frac{1}{2}|g_1'-g_2'|^2,\quad\text{over }\mathbb{C}.$$

Now we replace g_1'' by $(-g_2'')$, we have $|g_2''(g_1-g_2)|\leq\frac{1}{2}|g_1'-g_2'|^2$, over $\mathbb{C}$. Then,

$$(3)\quad |-g_2''(g_1-g_2)|\leq\frac{1}{2}|g_1'-g_2'|^2,\quad\text{over }\mathbb{C}.$$

The sum of (2) and (3) implies that $|g_1''(g_1-g_2)|+|-g_2''(g_1-g_2)|\leq|g_1'-g_2'|^2$, over $\mathbb{C}$. By the triangle inequality, we have $|(g_1''-g_2'')(g_1-g_2)|\leq|g_1'-g_2'|^2$, over $\mathbb{C}$. Now put $k=g_1-g_2$. Observe that the function k is holomorphic on $\mathbb{C}$ and satisfies the holomorphic differential inequality $|k''k|\leq|k'|^2$ over $\mathbb{C}$. Therefore, $k''(z)k(z)=\gamma(k'(z))^2$, for each $z\in\mathbb{C}$, where $\gamma\in\mathbb{C}$ and $|\gamma|\leq 1$. By [2, Théorème 21], we have $k(z)=(cz+d)^s$, for every $z\in\mathbb{C}$, where $c,d\in\mathbb{C}$ and $s\in\mathbb{N}$, or $k(z)=e^{(cz+d)}$, for each $z\in\mathbb{C}$, where $c,d\in\mathbb{C}$.

Case 1. $k(z) = (cz+d)^s$, for each $z \in \mathbb{C}$. Then, $g_1(z) - g_2(z) = (cz+d)^s$, for all $z \in \mathbb{C}$. Now recall that we have $g_1'' + g_2'' = 0$ on $\mathbb{C}$. Thus, $g_1(z) + g_2(z) = az + b$, for all $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$. Now we have

$$\begin{cases} g_1(z) - g_2(z) = (cz+d)^s \\ g_1(z) + g_2(z) = az + b \end{cases}$$

for each $z \in \mathbb{C}$. Then,

$$\begin{cases} g_1(z) = a_1 z + b_1 + (c_1 z + d_1)^s \\ g_2(z) = a_1 z + b_1 - (c_1 z + d_1)^s \end{cases}$$

for each $z \in \mathbb{C}$, with a_1, b_1, c_1 and $d_1 \in \mathbb{C}$.

Case 2. $k(z) = e^{(cz+d)}$, for all $z \in \mathbb{C}$. Then, $g_1(z) - g_2(z) = e^{(cz+d)}$, for $z \in \mathbb{C}$. Now recall that we have $g_1'' + g_2'' = 0$ on $\mathbb{C}$. It follows that $g_1(z) + g_2(z) = az + b$ for all $z \in \mathbb{C}$ where $a, b \in \mathbb{C}$. We have

$$\begin{cases} g_1(z) - g_2(z) = e^{(cz+d)} \\ g_1(z) + g_2(z) = az + b \end{cases}$$

Thus,

$$\begin{cases} g_1(z) = a_2 z + b_2 + e^{(c_2 z + d_2)} \\ g_2(z) = a_2 z + b_2 - e^{(c_2 z + d_2)} \end{cases}$$

for each $z \in \mathbb{C}$, with a_2, b_2, c_2 and $d_2 \in \mathbb{C}$.

(II) implies (I) is evident. $\square$

Now a number of fundamental properties can be deduced from the above Theorem 2.10. We have

Corollary 2.11. *Given $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ and two holomorphic functions $g_1, g_2 : \mathbb{C} \to \mathbb{C}$, let $v(z,w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^2$. The following assertions are equivalent*

(I) *v is convex on $\mathbb{C}^2$;*

(II) *$g_1(z) = A_1(az+b) + \overline{A_2}(cz+d)^m$ and $g_2(z) = A_2(az+b) - \overline{A_1}(cz+d)^m$, for each $z \in \mathbb{C}$, with $a, b, c, d \in \mathbb{C}$ and $m \in \mathbb{N}$, or $g_1(z) = A_1(a_1 z + b_1) + \overline{A_2} e^{(c_1 z + d_1)}$ and $g_2(z) = A_2(a_1 z + b_1) - \overline{A_1} e^{(c_1 z + d_1)}$, for every $z \in \mathbb{C}$, where $a_1, b_1, c_1, d_1 \in \mathbb{C}$.*

Proof. This is a consequence of the proof of Theorem 2.3 and the proof of Theorem 2.10. $\square$

Proposition 2.12. *Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions, $n \geq 1$. Put $u(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Suppose that u is convex on $\mathbb{C}^n \times \mathbb{C}$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_n = m$. Then, v is convex on $\mathbb{C}^n \times \mathbb{C}$,*

$$v(z,w) = \left| w - \frac{\partial^m g_1}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}(z) \right|^2 + \left| w - \frac{\partial^m g_2}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}}(z) \right|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. $\Delta(u)$ is convex on $\mathbb{C}^n \times \mathbb{C}$, for all $s \in \mathbb{N} \setminus \{0\}$.

$$\Delta = 4 \left(\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \cdots + \frac{\partial^2}{\partial z_n \partial \overline{z_n}} \right)$$

is the Laplace operator acting on $\mathbb{C}^n$.

Remark 2.13. For 3 functions, the above Theorem 2.3 does not hold. In fact, we have the following fundamental 3 examples. Define 9 holomorphic functions over $\mathbb{C}$ by $g_1(z) = z+z^2$, $g_2(z) = (z-z^2)$, $g_3(z) = 4(z+1)$, $k_1(z) = z^2$, $k_2(z) = z^2$, $k_3(z) = -2z^2$, $f_1(z) = z - e^z$, $f_2(z) = -(z + e^z)$, $f_3(z) = 2e^z$, for $z \in \mathbb{C}$. Put $u_1(z,w) = |w-g_1(z)|^2 + |w-g_2(z)|^2 + |w-g_3(z)|^2$, $u_2(z,w) = |w-k_1(z)|^2 + |w-k_2(z)|^2 + |w-k_3(z)|^2$ and $u_3(z,w) = |w-f_1(z)|^2 + |w-f_2(z)|^2 + |w-f_3(z)|^2$, $(z,w) \in \mathbb{C}^2$. We have the following 2 properties.

(I) g_1, g_2, k_j and f_j are not affine functions on $\mathbb{C}$, for all $j \in \{1,2,3\}$.

(II) u_1 and u_3 are strictly convex on $\mathbb{C}^2$. u_2 is convex on $\mathbb{C}^2$ and strictly convex on $\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}$ but u_2 is not strictly convex on $\mathbb{C}^2$.

Theorem 2.14. *Given two numbers $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ and two holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, $n \geq 2$, lG aet $v(z,w) = |A_1 w - g_1(z)|^2 + |A_2 w - g_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following assertions are equivalent*

(I) *v is convex on $\mathbb{C}^n \times \mathbb{C}$;*

(II) *$g_1(z) = A_1(< z/a > +b) + \overline{A_2}(< z/c > +d)^m$ and $g_2(z) = A_2(< z/a > +b) - \overline{A_1}(< z/c > +d)^m$, (for each $z \in \mathbb{C}^n$, with $a, c \in \mathbb{C}^n$ and $b, d \in \mathbb{C}$), or $g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2} e^{(<z/c_1>+d_1)}$ and $g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1} e^{(<z/c_1>+d_1)}$ for every $z \in \mathbb{C}^n$, where $a_1, c_1 \in \mathbb{C}^n$ and $b_1, d_1 \in \mathbb{C}$.*

Proof. v is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Put $w = z_{n+1}$. We have

$$\Big|\sum_{j,k=1}^{n+1} \frac{\partial^2 v}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big| \leq \sum_{j,k=1}^{n+1} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $(\alpha_1 \ldots, \alpha_n, \alpha_{n+1}) = (\alpha_1, \ldots, \alpha_n, \beta) \in \mathbb{C}^{n+1}$. Therefore,

$$\text{(E)} \quad \sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\overline{g_1}(z) + \sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\overline{g_2}(z)$$
$$- \overline{w}\Big(\overline{A_1}\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k - \overline{A_2}\sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big)$$
$$\leq \Big|A_1\beta - \sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j\Big|^2 + \Big|A_2\beta - \sum_{j=1}^{n}\frac{\partial g_2}{\partial z_j}(z)\alpha_j\Big|^2,$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and $(\alpha_1, \ldots, \alpha_n, \beta) \in \mathbb{C}^{n+1}$. Now fix $z \in \mathbb{C}^n$ and $(\alpha_1, \ldots, \alpha_n, \beta) \in \mathbb{C}^{n+1}$. It follows that, for each $w \in \mathbb{C}$, we obtain inequality (E). If

$$\Big(\overline{A_1}\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k + \overline{A_2}\sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big) \neq 0,$$

then the subset $\mathbb{C}$ is bounded. We get a contradiction. Consequently,

$$\overline{A_1}\sum_{j,k=1}^{n} \frac{\partial^2 g_1}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k + \overline{A_2}\sum_{j,k=1}^{n} \frac{\partial^2 g_2}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k = 0,$$

for each $z=(z_1,\dots z_n)\in\mathbb{C}^n$, and $(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n$. Thus, $(\overline{A_1}g_1+\overline{A_2}g_2)$ is an affine function on $\mathbb{C}^n$. Now we have

$$\text{(E)}\quad \Big|\sum_{j,k=1}^{n}\frac{\partial^2 g_1}{\partial z_j\partial z_k}(z)\alpha_j\alpha_k\overline{g_1}(z)+\sum_{j,k=1}^{n}\frac{\partial^2 g_2}{\partial z_j\partial z_k}(z)\alpha_j\alpha_k\overline{g_2}(z)\Big| \\ \leq\Big|A_1\beta-\sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j\Big|^2+\Big|A_2\beta-\sum_{j=1}^{n}\frac{\partial g_2}{\partial z_j}(z)\alpha_j\Big|^2,$$

for each $(z,w)\in\mathbb{C}^n\times\mathbb{C}$ and $(\alpha_1,\dots,\alpha_n,\beta)\in\mathbb{C}^{n+1}$. Now fix $(\gamma_1,\dots,\gamma_n)\in\mathbb{C}^n$ and define $(\alpha_1,\dots,\alpha_n)=\delta(\gamma_1,\dots,\gamma_n)$, where $\delta\in\mathbb{C}$. Then,

$$\Big|\sum_{j,k=1}^{n}\frac{\partial^2 g_1}{\partial z_j\partial z_k}(z)\gamma_j\gamma_k\overline{g_1}(z)+\sum_{j,k=1}^{n}\frac{\partial^2 g_2}{\partial z_j\partial z_k}(z)\gamma_j\gamma_k\overline{g_2}(z)\Big||\delta|^2 \\ \leq\Big|A_1\beta-\Big(\sum_{j=1}^{n}\frac{\partial g_1}{\partial z_j}(z)\alpha_j\Big)\delta\Big|^2+\Big|A_2\beta-\Big(\sum_{j=1}^{n}\frac{\partial g_2}{\partial z_j}(z)\alpha_j\Big)\delta\Big|^2,$$

for each $(\beta,\delta)\in\mathbb{C}^2$. This condition implies that the holomorphic function $\varphi=(A_2g_1-A_1g_2)$ that satisfies $|\varphi|^2$ is convex on $\mathbb{C}^n$. Therefore, we have the following 2 possible cases.

Case 1. $|\varphi|>0$ on $\mathbb{C}^n$. Therefore, $\varphi=e^K$, where $K:\mathbb{C}^n\to\mathbb{C}$ is a holomorphic function. Now we use the following result proved in [3, Theorem 9]: If $g:\mathbb{C}^n\to\mathbb{C}$ is an analytic function, $n\geq 1$, such that $|g|$ is convex and satisfying $|g|>0$ on $\mathbb{C}^n$, then $g(z)=e^{F(z)}$, for all $z\in\mathbb{C}^n$, where $F:\mathbb{C}^n\to\mathbb{C}$ is analytic and affine on $\mathbb{C}^n$. We conclude that K is an affine function on $\mathbb{C}^n$. Consequently, for each $z\in\mathbb{C}^n$, we have $(\overline{A_1}g_1(z)+\overline{A_2}g_2(z))=<z/a>+b$, where $(a,b)\in\mathbb{C}^n\times\mathbb{C}$ and $A_2g_1(z)-A_1g_2(z)=e^{K(z)}$. Since $(|A_1|^2+|A_2|^2)>0$, we calculate the expressions of g_1 and g_2 (thus having two representations of the functions g_1 and g_2).

Case 2. There exists $z_0\in\mathbb{C}^n$, such that $\varphi(z_0)=0$. It was proved in [4, Theorem 10] that, if $g:\mathbb{C}^n\to\mathbb{C}$ is a holomorphic function, $n\geq 1$, such that $|g|$ is convex on $\mathbb{C}^n$ and $g(z^0)=0$, where $z^0\in\mathbb{C}^n$, then $g(z)=(<z/\lambda>+\mu)^m$, for all $z\in\mathbb{C}^n$, where $\lambda\in\mathbb{C}^n$, $\mu\in\mathbb{C}$ and $m\in\mathbb{N}$. Therefore, we have $\varphi(z)=(<z/\lambda>+\mu)^m$, for each $z\in\mathbb{C}^n$, where $(\lambda,\mu)\in\mathbb{C}^n\times\mathbb{C}$. Thus, $(\overline{A_1}g_1(z)+\overline{A_2}g_2(z))=<z/a>+b$ and $A_2g_1(z)-A_1g_2(z)=\varphi(z)=(<z/\lambda>+\mu)^m$, for each $z\in\mathbb{C}^n$. Since $(|A_1|^2+|A_2|^2)>0$, we can calculate the two representations of g_1 and g_2 on $\mathbb{C}^n$. □

Theorem 2.15. *Let $g_1,g_2:\mathbb{C}^n\to\mathbb{C}$ be two analytic functions, $n\geq 1$ and $A_1,A_2\in\mathbb{C}\setminus\{0\}$. Put $u(z,w)=|A_1w-g_1(z)|^2+|A_2w-g_2(z)|^2$, $(z,w)\in\mathbb{C}^n\times\mathbb{C}$. The following are equivalent*

(I) *u is convex and strictly psh on $\mathbb{C}^n\times\mathbb{C}$;*
(II) *$n=1$ and*

$$\begin{cases} g_1(z)=A_1(az+b)+\overline{A_2}(cz+d)\\ g_2(z)=A_2(az+b)-\overline{A_1}(cz+d)\end{cases}$$

for each $z \in \mathbb{C}$, *where* $a, b, c, d \in \mathbb{C}$, $c \neq 0$), *or*

$$\begin{cases} g_1(z) = A_1(a_1 z + b_1) + \overline{A_2} e^{(c_1 z + d_1)} \\ g_2(z) = A_2(a_1 z + b_1) - \overline{A_1} e^{(c_1 z + d_1)} \end{cases}$$

for every $z \in \mathbb{C}$, *with* $a_1, b_1, c_1, d_1 \in \mathbb{C}$, $c_1 \neq 0$).

Proof. (I) implies (II). We have

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}(< z/c > +d)^m \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}(< z/c > +d)^m \end{cases}$$

for each $z \in \mathbb{C}^n$, with ($a, c \in \mathbb{C}^n$ and $b, d \in \mathbb{C}, m \in \mathbb{N}$), or

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2} e^{(< z/c_1 > +d_1)} \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1} e^{(< z/c_1 > +d_1)} \end{cases}$$

for every $z \in \mathbb{C}^n$, where ($a_1, c_1 \in \mathbb{C}^n$ and $b_1, d_1 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) = A_1(< z/a > +b) + \overline{A_2}(< z/c > +d)^m \\ g_2(z) = A_2(< z/a > +b) - \overline{A_1}(< z/c > +d)^m \end{cases}$$

for each $z \in \mathbb{C}^n$. For $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, $u(z, w) = (|A_1|^2 + |A_2|^2)[|w- < z/a > -b|^2 + | < z/c > +d|^{2m}]$. Define $v(z, w) = |w- < z/a > -b|^2 + | < z/c > +d|^{2m}$. u and v are C^∞ functions on $\mathbb{C}^n \times \mathbb{C}$. For $z \in \mathbb{C}^n$, we denote $z = (z_1, \dots, z_n)$. Note that u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Now define T by $T(z, w) = (z, w+ < z/a >) \in \mathbb{C}^n \times \mathbb{C}$, for all $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. T is a $\mathbb{C}$ linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Let $v_1(z, w) = voT(z, w) = |w - b|^2 + | < z/c > +d|^{2m}$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. v_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Let $(\alpha_1, \dots, \alpha_n, \beta) \in \mathbb{C}^{n+1} \setminus \{0\}$; $\alpha = (\alpha_1, \dots, \alpha_n)$. The Hermitian Levi form of v_1 is $L(v_1)(z, w)(\alpha_1, \dots, \alpha_n, \beta) = |\beta|^2 + m^2| < \alpha/c > |^2| < z/c > +d|^{2m-2} > 0$, $\forall (\alpha, \beta) \in (\mathbb{C}^n \times \mathbb{C}) \setminus \{0\}$. But the above strict inequality is true if and only if $v_2(\alpha) = m^2| < \alpha/c > |^2| < z/c > +d|^{2m-2} > 0$, $\forall \alpha \in \mathbb{C}^n \setminus \{0\}$, $\forall z \in \mathbb{C}^n$. Thus, $m = 1$, $n = 1$ and $c \in \mathbb{C} \setminus \{0\}$.

Case 2.

$$\begin{cases} g_1(z) = A_1(< z/a_1 > +b_1) + \overline{A_2} e^{(< z/c_1 > +d_1)} \\ g_2(z) = A_2(< z/a_1 > +b_1) - \overline{A_1} e^{(< z/c_1 > +d_1)} \end{cases}$$

for all $z \in \mathbb{C}^n$.

$$u(z, w) = (|A_1|^2 + |A_2|^2)\Big[|w- < z/a_1 > -b_1|^2 + |e^{(< z/c_1 > +d_1)}|^2\Big],$$

$(z, w) \in \mathbb{C}^n \times \mathbb{C}$. Let $\varphi(z, w) = |w- < z/a_1 > -b_1|^2 + |e^{(< z/c_1 > +d_1)}|^2$, $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. φ and u are C^∞ functions on $\mathbb{C}^n \times \mathbb{C}$. Now u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. For $(z, w) \in \mathbb{C}^n \times \mathbb{C}$, let $T(z, w) = (z, w+ < z/a_1 >) \in \mathbb{C}^n \times \mathbb{C}$. T is a $\mathbb{C}$ linear bijective transformation over $\mathbb{C}^n \times \mathbb{C}$. Let $\varphi_1(z, w) = \varphi o T(z, w) = |w - b_1|^2 + |e^{(< z/c_1 > +d_1)}|^2$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. φ_1 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$, so that φ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Let $(\alpha_1, \dots, \alpha_n, \beta) \in \mathbb{C}^{n+1} \setminus \{0\}$, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. The Hermitian Levi form of φ_1 is $L(\varphi_1)(z, w)(\alpha, \beta) = |\beta|^2 + | < \alpha/c_1 > |^2|e^{(< z/c_1 > +d_1)}|^2 > 0$. But this last strict inequality is true for any

$(\alpha, \beta) \in \mathbb{C}^{n+1} \setminus \{0\}$ if and only if $\varphi_2(\alpha) = | < \alpha/c_1 > |^2 > 0$, for each $\alpha \in \mathbb{C}^n \setminus \{0\}$. But now $\varphi_2(\alpha) > 0$, for every $\alpha \in \mathbb{C}^n \setminus \{0\}$ if and only if $n = 1$ and $c_1 \in \mathbb{C} \setminus \{0\}$.

(II) implies (I) is a classical case and the proof is finished. $\square$

An original question in complex analysis is

Question 2.16. Let $n \geq 1$ and $N \geq 2$. Find all the analytic functions

$$g_1, g_2, \ldots, g_N : \mathbb{C}^n \to \mathbb{C}$$

such that the function v is psh on $\mathbb{C}^n \times \mathbb{C}^n$ (or strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$, psh and not strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$, psh and not strictly psh on any not empty open ball subset of $\mathbb{C}^n \times \mathbb{C}^n$), where $v(z, w) = |g_1(w - \overline{z})|^2 + |g_2(w - \overline{z})|^2 + \cdots + |g_N(w - \overline{z})|^2$ for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$.

For example the case of one holomorphic function was studied by Abidi in the earlier paper [2]. In this case we find all the real numbers $\alpha > 0$ and all the families of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ such that u is psh (or strictly psh) on $\mathbb{C}^n \times \mathbb{C}^n$; $u(z, w) = |f(w - \overline{z})|^\alpha$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$. In this situation, we have proved that u is psh over $\mathbb{C}^n \times \mathbb{C}^n$ if and only if $f(z) = (< z/a > +b)^m$ for each $z \in \mathbb{C}^n$, with ($m \in \mathbb{N}, m\alpha \geq 1, a \in \mathbb{C}^n,$ and $b \in \mathbb{C}$); or $f(z) = e^{(<z/c>+d)}$ for every $z \in \mathbb{C}^n$, where ($c \in \mathbb{C}^n$ and $d \in \mathbb{C}$).

On the other hand, we can study the above problem for $g_1, \ldots, g_N : \mathbb{C}^n \to \mathbb{C}$ are N pluriharmonic (prh) functions. We can also study the case of $\mathrm{Re}(g_1), \ldots, \mathrm{Re}(g_N)$ or ($\mathrm{Im}(g_1), \ldots, \mathrm{Im}(g_N)$). The same question for the case of $g_1, \ldots, g_N$ being n-harmonic functions.

On the other hand, fix $T_1, \tau_1, \ldots, T_N, \tau_N : \mathbb{C}^n \to \mathbb{C}^n$ are $(2N)-$ complex linear bijective transformations. Find all the subharmonic functions $u_1, \ldots, u_N \in \mathrm{sh}(\mathbb{C}^n)$ (or $\mathrm{sh}(\mathbb{C}^n) \cap C(\mathbb{C}^n)$) such that v is psh (or strictly psh) over $\mathbb{C}^n \times \mathbb{C}^n$, where $v(z, w) = u_1(T_1(w) - \overline{\tau_1}(z)) + \cdots + u_N(T_N(w) - \overline{\tau_N}(z))$, for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$, $n, N \geq 1$.

Proposition 2.17. *Let p be an analytic polynomial over $\mathbb{C}$ with deg(p) ≤ 3. Then, it has the following 2 properties.*

1. *Assume that deg(p) ≤ 2. Then, there exists an analytic polynomial q over $\mathbb{C}$, deg(q) ≤ 1 and $u = (|p|^2 + |q|^2)$ is strictly convex on $\mathbb{C}$.*
2. *Suppose that deg(p) $= 3$. Then, there exists a polynomial q over $\mathbb{C}$, deg(q) $=$ 3 and $u = (|p|^2 + |q|^2)$ is convex on $\mathbb{C}$.*

Proof. (1) If $\deg(p) \in \{0, 1\}$, we choose $q(z) = z$, for all $z \in \mathbb{C}$. Suppose that $\deg(p) = 2$. Write $p(z) = az^2 + bz + c$, $a \in \mathbb{C} \setminus \{0\}$, $b, c \in \mathbb{C}$. We prove that there exists an analytic polynomial q, $\deg(q) = 1$ and $(|p|^2 + |q|^2) = u$ is strictly convex on $\mathbb{C}$. Let $B \in \mathbb{R}_+ \setminus \{0\}$. We study the strict convexity of u, $u(z) = |az^2 + bz + c|^2 + B|z|^2$, for $z \in \mathbb{C}$. u is strictly convex on $\mathbb{C}$ if

$$\left| 2a^2 \Big[\Big(z + \frac{b}{2a} \Big)^2 - \Big(\frac{b^2 - 4ac}{4a^2} \Big) \Big] \right| < |2az + b|^2 + B$$

for each $z \in \mathbb{C}$. Take $z_0 = \frac{-b}{2a}$. We choose B satisfying the condition

$$2B > |b^2 - 4ac|.$$

In this situation, we observe that, by the triangle inequality,

$$\Big|\frac{\partial^2 u}{\partial z^2}(z)\Big| = \Big|2a(az^2+bz+c)\Big| = \Big|2a^2\Big[\Big(z+\frac{b}{2a}\Big)^2 - \Big(\frac{b^2-4ac}{4a^2}\Big)\Big]\Big|$$
$$\le \Big|2a^2\Big(z+\frac{b}{2a}\Big)^2\Big| + 2|a|^2\Big|\frac{b^2-4ac}{4a^2}\Big|$$
$$< \left|4a^2\left(z+\frac{b}{2a}\right)^2\right| + 2|a|^2\Big|\frac{2B}{4a^2}\Big| = |2az+b|^2 + B = \frac{\partial^2 u}{\partial z\partial\overline{z}}(z),$$

for each $z \in \mathbb{C}$. Therefore, u is strictly convex on $\mathbb{C}$.

(2) Write $p(z) = az^3 + bz^2 + cz + d$, $a \in \mathbb{C}\setminus\{0\}$, $b, c, d \in \mathbb{C}$. In this case, $p(z) = a(z+\frac{b}{3a})^3 + (\gamma z+\delta)$, where $\gamma, \delta \in \mathbb{C}$. Let $q(z) = a(z+\frac{b}{3a})^3 - (\gamma z+\delta)$, for $z \in \mathbb{C}$. Then, $u(z) = 2|a|^2|z+\frac{b}{3a}|^6 + 2|\gamma z+\delta|^2$. Consequently, u is convex on $\mathbb{C}$. □

Lemma 2.18. (A) *There does not exist an $n \ge 1$, a convex domain D of $\mathbb{C}^n$ and a holomorphic function $g : D \to \mathbb{C}$ such that u (or v) is strictly convex on $D\times\mathbb{C}$, where $u(z,w) = |w-g(z)|^2 + |w-\overline{g}(z)|^2$ and $v(z,w) = |w-g(z)|^2 + |w+\overline{g}(z)|^2$, for $(z,w) \in D\times\mathbb{C}$. But u is convex on $D\times\mathbb{C}$ if and only if v is convex on $D\times\mathbb{C}$ (if and only if g is an affine function).*

(B) *There does not exist an $n \ge 2$, a convex domain G of $\mathbb{C}^n$ and a holomorphic function $k : G \to \mathbb{C}$ such that v is strictly psh (or strictly psh and convex) on $G\times\mathbb{C}$, where $v(z,w) = |w-k(z)|^2 + |w-\overline{k}(z)|^2$, for $(z,w) \in G\times\mathbb{C}$.*

Proof. (A) Suppose that u is strictly convex on $D\times\mathbb{C}$, where D is a non-empty convex domain of $\mathbb{C}^n$, $n \ge 1$ and $g : D \to \mathbb{C}$ be an holomorphic function. Note that u is a function of class C^∞ over $D\times\mathbb{C}$. By the problem of fibration, we assume that $n = 1$. Then, we have

$$\Big|\frac{\partial^2 u}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u}{\partial z\partial w}(z,w)\alpha\beta\Big|$$
$$< \frac{\partial^2 u}{\partial z\partial\overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w\partial\overline{w}}(z,w)\beta\overline{\beta} + 2\mathrm{Re}\left(\frac{\partial^2 u}{\partial z\partial\overline{w}}(z,w)\alpha\overline{\beta}\right),$$

$\forall (z,w) \in D\times\mathbb{C}$, $\forall(\alpha,\beta) \in \mathbb{C}^2\setminus\{0\}$. Let $(z,w) \in D\times\mathbb{C}$. Then, we have

$$\frac{\partial^2 u}{\partial w^2}(z,w) = 0, \quad \frac{\partial^2 u}{\partial z^2}(z,w) = g''(z)\overline{g}(z) - (w+\overline{w})g''(z)$$

and

$$\frac{\partial^2 u}{\partial z\partial w}(z,w) = -g'(z).$$

$$\frac{\partial^2 u}{\partial w\partial\overline{w}}(z,w) = 2, \quad \frac{\partial^2 u}{\partial z\partial\overline{z}}(z,w) = 2|g'(z)|^2 \quad \text{and} \quad \frac{\partial^2 u}{\partial z\partial\overline{w}}(z,w) = -g'(z).$$

Fix $z \in D$ and $(\alpha,\beta) \in \mathbb{C}^2\setminus\{0\}$, with the condition $\alpha \ne 0$. It follows that $|g''(z)\overline{g}(z)\alpha^2 - (w+\overline{w})g''(z)\alpha^2 - 2g'(z)\alpha\beta| < |\beta - g'(z)\alpha|^2 + |\beta|^2 + |g'(z)\alpha|^2$, $\forall w \in \mathbb{C}$. Therefore, $|g''(z)\overline{g}(z)\alpha^2 - 2wg''(z)\alpha^2 - 2g'(z)\alpha\beta| < |\beta-g'(z)\alpha|^2 + |\beta|^2 + |g'(z)\alpha|^2$, $\forall w \in \mathbb{R}$. Suppose that $g''(z) \ne 0$. Then, the subset $\mathbb{R}$ is bounded. We get a contradiction.

Consequently, $g''(z) = 0$, for all $z \in D$. Thus, $|-2g'(z)\alpha\beta| < |\beta - g'(z)\alpha|^2 + |\beta|^2 + |g'(z)\alpha|^2$, for each $z \in D$ and any $(\alpha,\beta) \in \mathbb{C}^2\setminus\{0\}$. Then, we have

$$|\beta - g'(z)\alpha|^2 + (|\beta| - |g'(z)\alpha|)^2 > 0 \quad \text{for all} \quad (\alpha,\beta) \in \mathbb{C}^2\setminus\{0\}.$$

Now take $\beta = g'(z)\alpha$, where $\alpha \in \mathbb{C}\setminus\{0\}$. We conclude that the last strict inequality above is not possible. $\square$

Now,

Corollary 2.19. *For each $n \geq 1$, for each D, a convex domain of $\mathbb{C}^n$, and for each holomorphic function $g : D \to \mathbb{C}$, we have the following 2 properties.*

(a) *φ is not strictly convex on $D\times\mathbb{C}$, where $\varphi(z,w) = |w-g(z)|^2+|w+\overline{g}(z)|^2$, for $(z,w) \in D \times \mathbb{C}$.*

(b) *φ_1 is not strictly convex on $D \times \mathbb{C}$, $\varphi_1(z,w) = |w - \overline{g}(z)|^2$, for $(z,w) \in D\times\mathbb{C}$. But there exist several possible cases for $n = 1$ where φ_1 is strictly psh and convex on $D \times \mathbb{C}$.*

Proof. Let $v : D \to \mathbb{R}$ be a function of class C^2, D is a convex domain of $\mathbb{C}^n$. Recall that v is convex on D if and only if

$$\Big|\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k\Big| \leq \sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z = (z_1,\dots,z_n) \in D$ and all $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{C}^n$.
v is strictly convex on D if and only if

$$|\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k| < \sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z \in D$ and every $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{C}^n \setminus \{0\}$. $\square$

Question 2.20. Let $g_1,\dots,g_N : \mathbb{C}^n \to \mathbb{C}$ be N analytic functions, $n, N \geq 1$. Put

$$u(z,w) = \int_{B(0,1)} (|w - g_1(z+\xi)|^2 + \cdots + |w - g_N(z+\xi)|^2)dm_{2n}(\xi),$$

for $(z,w) \in \mathbb{C}^n\times\mathbb{C}$. Find the condition satisfied by $g_1,\dots,g_N$ such that u is convex on $\mathbb{C}^n \times \mathbb{C}$.

3. Some extensions

Now the following theorems plays a classical role in many problems of complex analysis.

Theorem 3.1. *Let $A_1, A_2 \in \mathbb{C}\setminus\{0\}$. Put $k(w) = e^{(aw+b)}$, for $w \in \mathbb{C}$, where $a \in \mathbb{C}\setminus\{0\}$, $b \in \mathbb{C}$ and $n \geq 1$.*
Let $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ be two analytic functions. Define

$$u(z,w) = |A_1k(w) - g_1(z)|^2 + |A_2k(w) - g_2(z)|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. The following are equivalent

(A) *u is convex on $\mathbb{C}^n \times \mathbb{C}$;*

(B) *We have 2 representations*

$$\begin{cases} g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -\overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ *and* $s \in \mathbb{N}$ *or*

$$\begin{cases} g_1(z) = \overline{A_2} e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -\overline{A_1} e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$ *and* $\mu_1 \in \mathbb{C}$.

Proof. (A) implies (B). After effectuating a holomorphic $\mathbb{C}$ linear change of variable, we assume that $a = 1$. Since convex functions are invariant by a translation, we assume that $b = 0$. Therefore, $k(w) = e^w$, for each $w \in \mathbb{C}$. Now u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Observe that, if $n = 1$, we have

$$\begin{aligned} \Big|\frac{\partial^2 u}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u}{\partial z \partial w}(z,w)\alpha\beta\Big| \\ \leq \frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w)\beta\overline{\beta} 2\mathrm{Re}\Big(\frac{\partial^2 u}{\partial z \partial \overline{w}}(z,w)\alpha\overline{\beta}\Big), \end{aligned}$$

$\forall (z,w) \in \mathbb{C}^2$, $\forall (\alpha,\beta) \in \mathbb{C}^2$. It follows that

$$\begin{aligned} |(g_1''(z)\overline{g_1}(z) + g_2''(z)\overline{g_2}(z))\alpha^2 - e^{\overline{w}}(\overline{A_1}g_1''(z) + \overline{A_2}g_2''(z))\alpha^2 \\ + (|A_1|^2 + |A_2|^2)|e^w|^2\beta^2 - (A_1\overline{g_1}(z) + A_2\overline{g_2}(z))e^w\beta^2| \\ \leq |A_1\beta e^w - g_1'(z)\alpha|^2 + |A_2\beta e^w - g_2'(z)\alpha|^2. \end{aligned}$$

If $\alpha_0 = 0$ and $\beta_0 = 1$, then

$$|(|A_1|^2 + |A_2|^2)e^{2x_1} - (A_1\overline{g_1}(z) + A_2\overline{g_2}(z))e^{x_1}| \leq (|A_1|^2 + |A_2|^2)e^{2x_1}, \quad \forall x_1 \in \mathbb{R}.$$

Then,

$$|(|A_1|^2 + |A_2|^2)e^{x_1} - (A_1\overline{g_1}(z) + A_2\overline{g_2}(z))| \leq (|A_1|^2 + |A_2|^2)e^{x_1}$$

for each $x_1 \in \mathbb{R}$. Therefore,

$$\begin{aligned} 0 \leq |A_1\overline{g_1}(z) + A_2\overline{g_2}(z)| = \lim_{x_1 \to -\infty} |(|A_1|^2 + |A_2|^2)e^{x_1} - (A_1\overline{g_1}(z) + A_2\overline{g_2}(z))| \\ \leq (|A_1|^2 + |A_2|^2) \lim_{x_1 \to -\infty} e^{x_1} = 0. \end{aligned}$$

Thus, $A_1\overline{g_1}(z) + A_2\overline{g_2}(z) = 0$ for each $z \in \mathbb{C}$. The case with $n \geq 2$, is identical to the one above. Now since $g_2 = -\frac{\overline{A_1}}{\overline{A_2}} g_1$,

$$\begin{aligned} u(z,w) = |A_1 e^w - g_1(z)|^2 + \Big|A_2 e^w + \frac{\overline{A_1}}{\overline{A_2}} g_1(z)\Big|^2 \\ = (|A_1|^2 + |A_2|^2)|e^w|^2 + \Big(1 + \frac{|A_1|^2}{|A_2|^2}\Big)|g_1(z)|^2, \end{aligned}$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. $u(.,0)$ is convex on $\mathbb{C}^n$, thus $|g_1|^2$ is convex on $\mathbb{C}^n$. Therefore, $g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s$, for each $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $s \in \mathbb{N}$, or $g_1(z) = \overline{A_2} e^{(<z/\lambda_1>+\mu_1)}$, for each $z \in \mathbb{C}^n$, with $\lambda_1 \in \mathbb{C}^n$ and $\mu_1 \in \mathbb{C}$. Consequently, we get the 2 representations. □

Theorem 3.2. *Let* $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ *and* $n \geq 1$. *Put* $k(w) = (aw+b)^m$, *for* $w \in \mathbb{C}$, *where* $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ *and* $m \in \mathbb{N}$, $m \geq 2$. *Given two holomorphic functions* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, *define* $u(z,w) = |A_1(aw+b)^m - g_1(z)|^2 + |A_2(aw+b)^m - g_2(z)|^2$, *for* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. *The following conditions are equivalent*

(A) *u is convex on* $\mathbb{C}^n \times \mathbb{C}$;
(B) *We have the* 2 *representations.*

$$\begin{cases} g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -\overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

for each $z \in \mathbb{C}^n$, *where* $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ *and* $s \in \mathbb{N}$, *or*

$$\begin{cases} g_1(z) = \overline{A_2}e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -\overline{A_1}e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

for every $z \in \mathbb{C}^n$, *where* $\lambda_1 \in \mathbb{C}^n$ *and* $\mu_1 \in \mathbb{C}$).

Proof. This is similar to the proof of Theorem 4.3. □

The following is true:

Theorem 3.3. *Let* $k : \mathbb{C} \to \mathbb{C}$ *be analytic non-constant,* $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ *and* $n \geq 1$. *Suppose that* $|k|$ *is convex on* $\mathbb{C}$. *Given* $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ *are two analytic functions. Define* $u(z,w) = |A_1k(w) - g_1(z)|^2 + |A_2k(w) - g_2(z)|^2$, *for* $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. *Then the following conditions are equivalent*

(A) *u is convex and strictly psh on* $\mathbb{C}^n \times \mathbb{C}$;
(B) $n = 1$ *and we have one of the following cases:*
  (I) $k(w) = aw + b$ *for* $w \in \mathbb{C}$, $a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}$ *(the representation for* g_1, g_2 *is given in Theorem 2.15).*
  (II) $k(w) = e^{(aw+b)}$ *for each* $w \in \mathbb{C}$, *with* $(a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C})$, *in which case there are the following two representations:*

$$\begin{cases} g_1(z) = \overline{A_2}(\lambda z + \mu) \\ g_2(z) = -\overline{A_1}(\lambda z + \mu) \end{cases}$$

for each $z \in \mathbb{C}$ *with* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $\mu \in \mathbb{C}$, *or*

$$\begin{cases} g_1(z) = \overline{A_2}e^{(\lambda_1 z+\mu_1)} \\ g_2(z) = -\overline{A_1}e^{(\lambda_1 z+\mu_1)} \end{cases}$$

for each $z \in \mathbb{C}$ *with* $\lambda_1 \in \mathbb{C} \setminus \{0\}$ *and* $\mu_1 \in \mathbb{C}$.

Proof. We will prove that (A) implies (B). The proof of the opposite implication is a standard exercise. From the assumptions of the theorem, it follows that k is of one of the following two forms: $k(w) = e^{(aw+b)}$, $(a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C})$, or $k(w) = (aw + b)^m$, $m \in \mathbb{N}$, $a \in \mathbb{C} \setminus \{0\}, b \in \mathbb{C}$. First we consider the former case $k(w) = e^{(aw+b)}$. Since u is convex on $\mathbb{C}^n \times \mathbb{C}$,

$$\begin{cases} g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -\overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

for each $z \in \mathbb{C}^n$ where $\lambda \in \mathbb{C}^n$, $\mu \in \mathbb{C}$ and $s \in \mathbb{N}$, or

$$\begin{cases} g_1(z) = \overline{A_2}e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -\overline{A_1}e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

for each $z \in \mathbb{C}^n$, where $\lambda_1 \in \mathbb{C}^n$ and $\mu_1 \in \mathbb{C}$).

Case 1.

$$\begin{cases} g_1(z) = \overline{A_2}(< z/\lambda > +\mu)^s \\ g_2(z) = -\overline{A_1}(< z/\lambda > +\mu)^s \end{cases}$$

for each $z \in \mathbb{C}^n$. Therefore, $u(z,w) = (|A_1|^2 + |A_2|^2)[|e^{(aw+b)}|^2 + | < z/\lambda > + \mu|^{2s}]$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Put $u_1 = \frac{u}{|A_1|^2+|A_2|^2}$. Note that u and u_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$ and, therefore, u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if u_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. The Levi Hermitian form of u_1 is $L(u_1)(z,w)(\alpha,\beta) = |a|^2|\beta|^2|e^{(aw+b)}|^2 + s^2| < \alpha/\lambda > |^2| < z/\lambda > + \mu|^{2s-2} > 0$, for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$ and any $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C} \setminus \{0\}$. Now observe that

$$L(u_1(z,w)(\alpha,\beta) > 0, \quad \forall (z,w) \in \mathbb{C}^n \times \mathbb{C},\ \forall (\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C} \setminus \{0\}$$

if and only if

$$\varphi(z,\alpha) = s^2| < \alpha/\lambda > |^2| < z/\lambda > + \mu|^{2s-2} > 0, \quad \forall z \in \mathbb{C}^n,\ \forall \alpha \in \mathbb{C}^n \setminus \{0\}.$$

Therefore, $s = 1, n = 1$ and $\lambda \in \mathbb{C} \setminus \{0\}$.

Case 2.

$$\begin{cases} g_1(z) = \overline{A_2} e^{(<z/\lambda_1>+\mu_1)} \\ g_2(z) = -\overline{A_1} e^{(<z/\lambda_1>+\mu_1)} \end{cases}$$

for each $z \in \mathbb{C}^n$. Then,

$$u(z,w) = (|A_1|^2 + |A_2|^2)\Big[|e^{(aw+b)}|^2 + |e^{(<z/+\lambda_1>+\mu_1)}|^2\Big],$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Define $v_1 = \frac{u}{|A_1|^2+|A_2|^2}$. Note that u and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. u is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. The Hermitian Levi form of v_1 is

$$L(v_1)(z,w)(\alpha,\beta) = |a|^2|\beta|^2|e^{(aw+b)}|^2 + | < \alpha/\lambda_1 > |^2|e^{(<z/\lambda_1>+\mu_1)}|^2 > 0,$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, for every $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C} \setminus \{0\}$. Therefore,

$$| < \alpha/\lambda_1 > |^2|e^{(<z/\lambda_1>+\mu_1)}|^2 > 0,$$

for each $z \in \mathbb{C}^n$ and every $\alpha \in \mathbb{C}^n \setminus \{0\}$. Then, $n = 1$ and $\lambda_1 \in \mathbb{C} \setminus \{0\}$.

Now we assume that $k(w) = (aw+b)^m$. This case is, in fact, treated in Theorem 2.15 and its proof. Note that, in particular, we get $m = 1$. □

4. A complete characterization

We have the following theorem

Theorem 4.1. *Let $k : \mathbb{C} \to \mathbb{C}$ be an analytic non-constant function, $A_1, A_2 \in \mathbb{C} \setminus \{0\}$ and $n \geq 1$. Suppose that $|k|$ is convex on $\mathbb{C}$. Given two analytic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$, define $u(z,w) = |A_1k(w) - g_1(z)|^2 + |A_2k(w) - g_2(z)|^2$, $v(z,w) = u(z,w) + |A_1k(w) - \overline{g_1}(z)|^2 + |A_2k(w) - \overline{g_2}(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Then, the following are equivalent*

(A) *u is convex on $\mathbb{C}^n \times \mathbb{C}$ and v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$;*
(B) *$n \in \{1,2\}$ and we have the following 2 cases.*
(I) *$k(w) = aw + b$, for each $w \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Then, we have the 2 representations.*

$$\begin{cases} g_1(z) = A_1(< z/\lambda > + \mu) + \overline{A_2}(< z/\lambda_1 > + \mu_1)^m, \\ g_2(z) = A_2(< z/\lambda > + \mu) - \overline{A_1}(< z/\lambda_1 > \mu_1)^m. \end{cases}$$

For each $z \in \mathbb{C}^n$ *with* $n = 2, m = 1, \lambda, \lambda_1 \in \mathbb{C}^2$ *and* (λ, λ_1) *is a basis of the complex vector space* $\mathbb{C}^2$, *or* $n = 1, m = 1,\ \lambda, \lambda_1 \in \mathbb{C}$ *and* $\lambda\lambda_1 \neq 0$, *or*

$$\begin{cases} g_1(z) = A_1(< z/\lambda_2 > +\mu_2) + \overline{A_2} e^{(<z/\lambda_3>+\mu_3)}, \\ g_2(z) = A_2(< z/\lambda_2 > +\mu_2) - \overline{A_1} e^{(<z/\lambda_3>+\mu_3)}. \end{cases}$$

For each $z \in \mathbb{C}$ *with* $n = 2, \lambda_2, \lambda_3 \in \mathbb{C}^2$ *and* (λ_2, λ_3) *is a basis of the complex vector space* $\mathbb{C}^2$), *or* $n = 1, \lambda_2, \lambda_3 \in \mathbb{C}, \lambda_2 \neq 0$ *or* $\lambda_3 \neq 0$.

(II) $k(w) = e^{(aw+b)}$, *for each* $w \in \mathbb{C}$, *with* $a \in \mathbb{C} \setminus \{0\}$ *and* $b \in \mathbb{C}$.
Then, $n = 1$ *and*

$$\begin{cases} g_1(z) = \overline{A_2}(\lambda z + \mu), \\ g_2(z) = -\overline{A_1}(\lambda z + \mu). \end{cases}$$

For each $z \in \mathbb{C}$ *with* $\lambda \in \mathbb{C} \setminus \{0\}$ *and* $\mu \in \mathbb{C}$ *or*

$$\begin{cases} g_1(z) = \overline{A_2} e^{(\lambda_1 z + \mu_1)}, \\ g_2(z) = -\overline{A_1} e^{(\lambda_1 z + \mu_1)}. \end{cases}$$

For each $z \in \mathbb{C}$ *with* $\lambda_1 \in \mathbb{C} \setminus \{0\}$ *and* $\mu_1 \in \mathbb{C}$.

For the proof of this theorem we use the following lemma which is fundamental in the theory of psh and strictly psh functions.

Lemma 4.2. *Let* $f, g : D \to \mathbb{C}^N$, $n, N \geq 1$. $f = (f_1, \ldots, f_N)$, $g = (g_1, \ldots, g_N)$. *Suppose that* f *and* g *are holomorphic on* D. *Then,* $\|f + \overline{g}\|^2$ *and* $\|f\|^2 + \|g\|^2$ *have the same Hermitian Levi form on* D. *Moreover, let* $u : D \to \mathbb{R}$ *be a function of class* C^2 *on* D. *Denote* $u_1 = u + \|f + \overline{g}\|^2$ *and* $u_2 = u + (\|f\|^2 + \|g\|^2)$. *Then,* u_1 *is strictly psh on* D *if and only if* u_2 *is strictly psh on* D.

Proof. We have

$$\|f + \overline{g}\|^2 = |f_1 + \overline{g_1}|^2 + \cdots + |f_N + \overline{g_N}|^2 =$$
$$|f_1|^2 + |g_1|^2 + \cdots + |f_N|^2 + |g_N|^2$$
$$+ (f_1 g_1 + \overline{f_1 g_1} + \cdots + f_N g_N + \overline{f_N g_N})$$

on D. Since $(f_1 g_1 + \overline{f_1 g_1} + \cdots + f_N g_N + \overline{f_N g_N})$ is pluriharmonic (prh) and real valued, the functions $\|f + \overline{g}\|^2$ and

$$|f_1|^2 + |g_1|^2 + \cdots + |f_N|^2 + |g_N|^2 = \|f\|^2 + \|g\|^2$$

have the same Hermitian Levi form on D. □

This lemma is fundamental in complex analysis and plays a classical role in many problems in several questions of pluripotential theory.

Proof of Theorem 4.1. (A) implies (B). We study the case where $k(w) = e^{(aw+b)}$ for each $w \in \mathbb{C}$, where $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. Define

$$v_1(z, w) = |A_1 k(w) - \overline{g_1}(z)|^2 + |A_2 k(w) - \overline{g_2}(z)|^2,$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}$. u, v and v_1 are functions of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ by Lemma 4.2. Since u

is convex on $\mathbb{C}^n \times \mathbb{C}$, then $g_2 = -\frac{\overline{A_1}}{\overline{A_2}} g_1$ on $\mathbb{C}^n$. We have

$$v_1(z,w) = (|A_1|^2 + |A_2|^2)|k(w)|^2 + \Big(1 + \frac{|A_1|^2}{|A_2|^2}\Big)|g_1(z)|^2 + \psi(z,w),$$

for each $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, where ψ is a pluriharmonic (prh) function on $\mathbb{C}^n \times \mathbb{C}$. By Lemma 4.2, v is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. Let $v_2(z,w) = |k(w)|^2 + |g_1(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. v_2 is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. v_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. The Levi Hermitian form of v_2 is

$$L(v_2)(z,w)(\alpha,\beta) = |k'(w)\beta|^2 + \Big|\sum_{j=1}^n \frac{\partial g_1}{\partial z_j}(z)\alpha_j\Big|^2,$$

for each $(z,w) = (z_1,\dots,z_n,w) \in \mathbb{C}^n \times \mathbb{C}$, for every $(\alpha,\beta) \in \mathbb{C}^n \times \mathbb{C}$, $\alpha = (\alpha_1,\dots,\alpha_n)$. $|k'(w)\beta|^2 = |a|^2|\beta|^2|e^{(aw+b)}|^2 > 0$, for each $\beta \in \mathbb{C} \setminus \{0\}$. Therefore, v_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $|g_1|^2$ is strictly psh on $\mathbb{C}^n$. Since g_1 is holomorphic on $\mathbb{C}^n$, $n = 1$ (observe that $|g_1|^2$ is not strictly psh at any point of $\mathbb{C}^n$ for each $n \geq 2$). It follows that $|g_1'| > 0$ on $\mathbb{C}$. In fact

$$\begin{cases} g_1(z) = \overline{A_2}(\lambda z + \mu) \\ g_2(z) = -\overline{A_1}(\lambda z + \mu) \end{cases}$$

For each $z \in \mathbb{C}$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\mu \in \mathbb{C}$, or

$$\begin{cases} g_1(z) = \overline{A_2} e^{(\lambda_1 z + \mu_1)} \\ g_2(z) = -\overline{A_1} e^{(\lambda_1 z + \mu_1)} \end{cases}$$

For each $z \in \mathbb{C}$, with $\lambda_1 \in \mathbb{C} \setminus \{0\}$ and $\mu_1 \in \mathbb{C}$. □

Theorem 4.3. *Let $A_1, A_2 \in \mathbb{C} \setminus \{0\}$. Put $p(w) = w^3 + w^2$ and $k(w) = e^{w^2}$, for $w \in \mathbb{C}$ ($|p|^2$ and $|k|^2$ are not convex functions on $\mathbb{C}$). We have the following*

(A) *There does not exist $n \geq 1$ and 2 holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex on $\mathbb{C}^n \times \mathbb{C}$;*

$$u(z,w) = |A_1 p(w) - g_1(z)|^2 + |A_2 p(w) - g_2(z)|^2,$$

$(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

(B) *There does not exist $n \geq 1$ and 2 holomorphic functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex on $\mathbb{C}^n \times \mathbb{C}$;*

$$v(z,w) = |A_1 k(w) - f_1(z)|^2 + |A_2 k(w) - f_2(z)|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

Proof. (A) Suppose that there exists $n \geq 1$ and 2 holomorphic functions $g_1, g_2 : \mathbb{C}^n \to \mathbb{C}$ such that u is convex on $\mathbb{C}^n \times \mathbb{C}$. u is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Then, for all fixed $z \in \mathbb{C}^n$, the function $u(z,.)$ is convex on $\mathbb{C}$. Therefore,

$$\Big|\frac{\partial^2 u}{\partial w^2}(z,w)\Big| \leq \frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w) \quad \text{for each } w \in \mathbb{C}.$$

Then,

$$|A_1 p''(w)(\overline{A_1 p(w) - g_1(z)}) + A_2 p''(w)(\overline{A_2 p(w) - g_2(z)})| \leq (|A_1|^2 + |A_2|^2)|p'(w)|^2,$$

for each $(w,z) \in \mathbb{C} \times \mathbb{C}^n$. Since $p'(0) = 0$, $|-2A_1\overline{g_1}(z) - 2A_2\overline{g_2}(z)| \le 0$. Thus, $A_1\overline{g_1}(z) + A_2\overline{g_2}(z) = 0$, for each $z \in \mathbb{C}^n$. Therefore,

$$u(z,w) = |A_1p(w) - g_1(z)|^2 + \left|A_2p(w) + \frac{\overline{A_1}}{\overline{A_2}}g_1(z)\right|^2$$
$$= (|A_1|^2 + |A_2|^2)|p(w)|^2 + \Big(1 + \frac{|A_1|^2}{|A_2|^2}\Big)|g_1(z)|^2,$$

for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Since $u(0,.)$ is convex on $\mathbb{C}$, $|p|^2$ is convex on $\mathbb{C}$ and, consequently, $|p''(w)\overline{p}(w)| \le |p'(w)|^2$, for each $w \in \mathbb{C}$. Let $w_0 = -\frac{2}{3}$. We have $p''(-\frac{2}{3})p(-\frac{2}{3}) \neq 0$, but $p'(-\frac{2}{3}) = 0$. This is a contradiction.

(B) Suppose that there exists $n \ge 1$ and 2 holomorphic functions $f_1, f_2 : \mathbb{C}^n \to \mathbb{C}$ such that v is convex on $\mathbb{C}^n \times \mathbb{C}$. v is a function of class C^∞ on $\mathbb{C}^n \times \mathbb{C}$. Let $z \in \mathbb{C}^n$. Then, $v(z,.)$ is convex on $\mathbb{C}$. Thus, $|\frac{\partial^2 v}{\partial w^2}(z,w)| \le \frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w)$, for each $w \in \mathbb{C}$. Then,

$$\text{(S)} \quad |A_1(2+4w^2)(\overline{A_1k(w) - f_1(z)}) + A_2(2+4w^2)(\overline{A_2k(w) - f_2(z)})|$$
$$\le 4(|A_1|^2 + |A_2|^2)|w|^2,$$

for every $w \in \mathbb{C}$. Put $w_0 = 0$. Then, we have

$$|2|A_1|^2 + 2|A_2|^2 - 2A_1\overline{f_1}(z) - 2A_2\overline{f_2}(z)| \le 0,$$

for every $z \in \mathbb{C}^n$. Then $A_1\overline{f_1}(z) + A_2\overline{f_2}(z) = c = |A_1|^2 + |A_2|^2$, for each $z \in \mathbb{C}^n$. Note that $c > 0$. Therefore, the inequality (S) implies that $|ck''(w)(\overline{k}(w) - 1)| \le |k'(w)|^2$, for each $w \in \mathbb{C}$. Then, $c|k''(w)(k(w) - 1)| \le |k'(w)|^2$, for every $w \in \mathbb{C}$. Put $k_1 = k - 1$. k_1 is an analytic function on $\mathbb{C}$. k_1 satisfies the holomorphic differential inequality $|k_1''k_1| \le \frac{1}{c}|k_1'|^2$ on $\mathbb{C}$. Consequently, $k_1''(w)k_1(w) = \gamma(k_1'(w))^2$, for each $w \in \mathbb{C}$, where $\gamma \in \mathbb{C}$. By [2], $|k_1|^2 = |k-1|^2$ is convex on $\mathbb{C}$. We get a contradiction. □

Note that the function k defined by the above theorem has the following property. For every holomorphic functions $\varphi_1, \varphi_2 : \mathbb{C}^n \to \mathbb{C}$, for every $B_1, B_2 \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{N} \setminus \{0\}$, if we define $\psi(z,w) = |B_1k^{(m)}(w) - \varphi_1(z)|^2 + |B_2k^{(m)}(w) - \varphi_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, we get that the function ψ is not convex on $\mathbb{C}^n \times \mathbb{C}$. But p satisfies for every $C_1, C_2 \in \mathbb{C} \setminus \{0\}$, there exist $p_1, p_2 : \mathbb{C}^n \to \mathbb{C}$ 2 holomorphic functions such that ψ_1 is convex on $\mathbb{C}^n \times \mathbb{C}$; $\psi_1(z,w) = |C_1p(w) - p_1(z)|^2 + |C_2p(w) - p_2(z)|^2$, for $(z,w) \in \mathbb{C}^n \times \mathbb{C}$.

5. The convexity and the complex structure

In this section we study some fundamental properties concerning plurisubharmonic, convex, strictly plurisubharmonic (or convex and strictly psh) and strictly convex functions. For analytic functions g_1, k_1, g_2, $k_2 : \mathbb{C}^n \to \mathbb{C}$ $(n \ge 1)$, put $\varphi_1(z,w) = (|w - g_1(z) - \overline{k_1}(z)|^2 + |w - g_2(z) - \overline{k_2}(z)|^2)$, $\varphi_2(z,w) = (|w - g_1(z)|^2 + |k_1(z)|^2 + |w - g_2(z)|^2 + |k_2(z)|^2)$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. Recall that φ_1 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if φ_2 is strictly psh on $\mathbb{C}^n \times \mathbb{C}$. This is not true for strictly convex functions. Indeed, we have

Theorem 5.1. *Let $g_1(z) = k_1(z) = z$, $g_2(z) = k_2(z) = 2z$, for $z \in \mathbb{C}$. Define*

$$u(z,w) = |w - g_1(z) - \overline{k_1}(z)|^2 + |w - g_2(z) - \overline{k_2}(z)|^2,$$

$$v(z,w) = |w - g_1(z)|^2 + |k_1(z)|^2 + |w - g_2(z)|^2 + |k_2(z)|^2$$

for $(z,w) \in \mathbb{C}^2$. *Then, we get that* u *and* v *are* C^∞ *and strictly psh functions over* $\mathbb{C}^2$. v *is strictly convex on* $\mathbb{C}^2$. *But* u *is not strictly convex at all points of* $\mathbb{C}^2$ *(then,* u *is not strictly convex on all not empty euclidean open ball subsets of* $\mathbb{C}^2$*).*

Proof. $u(z,w) = |w - 2x_1|^2 + |w - 4x_1|^2$, $z = (x_1 + ix_2)$, $w = (x_3 + ix_4) \in \mathbb{C}$, $(x_1, x_2, x_3, x_4 \in \mathbb{R})$. u is a function of class C^∞ on $\mathbb{C}^2$. u is independent of the variable x_2. It follows that u is not strictly convex at all points of $\mathbb{C}^2$. But u is convex and strictly psh on $\mathbb{C}^2$. □

Claim 5.2. *Let* $v(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2 + |g_1(z)|^2 + |g_2(z)|^2$, *where* $g_1(z) = z - z^2$, $g_2(z) = z + z^2$, *for* $(z,w) \in \mathbb{C}^2$. *Then,* v *is strictly convex on* $\mathbb{C}^2$, *but* g_1 *and* g_2 *are not affine functions.*

Remark 5.3. Put $g_1(z) = g_2(z) = z$, $k_1(z) = k_2(z) = z^2$, for $z \in \mathbb{C}$.

$$u(z,w) = |w - g_1(z) - \overline{k_1}(z)|^2 + |w - g_2(z) - \overline{k_2}(z)|^2,$$
$$v(z,w) = |w - g_1(z)|^2 + |k_1(z)|^2 + |w - g_2(z)|^2 + |k_2(z)|^2,$$

$(z,w) \in \mathbb{C}^2$. Then, v is convex on $\mathbb{C}^2$, but u is not convex on any non-empty euclidean open ball subset of $\mathbb{C}^2$. Put $f_1(z) = -1$, $f_2(z) = 1$, $k_1(z) = -k_2(z) = z^2 + 1$, for $z \in \mathbb{C}$. For $(z,w) \in \mathbb{C}^2$, let

$$u_1(z,w) = |w - f_1(z) - \overline{k_1}(z)|^2 + |w - f_2(z) - \overline{k_2}(z)|^2 = 2|w|^2 + 2|z^2|^2.$$

Therefore, u_1 is convex on $\mathbb{C}^2$. Let

$$v_1(z,w) = |w - f_1(z)|^2 + |k_1(z)|^2 + |w - f_2(z)|^2 + |k_2(z)|^2.$$

v_1 is not convex on $\mathbb{C}^2$. Consequently, we cannot compare the convexity of φ_1 and φ_2 (φ_1 and φ_2 as defined above).

Theorem 5.4. *Let* $g_1(z) = a(z^2 - 1)$, $g_2(z) = 2az$, $z \in \mathbb{C}$, $a \in \mathbb{C} \setminus \{0\}$. *For* $(z,w) \in \mathbb{C}^2$, *define*

$$v_1(z,w) = |g_1(w - z)|^2 + |g_2(w - z)|^2,$$
$$v_2(z,w) = |g_1(w - \overline{z})|^2 + |g_2(w - \overline{z})|^2,$$
$$v_3(z,w) = |g_1(w - z)|^2 + |g_2(w - \overline{z})|^2,$$
$$v_4(z,w) = |g_1(w - \overline{z})|^2 + |g_2(w - z)|^2.$$

Then, we have the following 5 *properties.*

(1) $|g_1|^2 + |g_2|^2$ *is strictly convex on* $\mathbb{C}$.
(2) v_1 *is convex on* $\mathbb{C}^2$, *but* v_1 *is not strictly psh at all point of* $\mathbb{C}^2$.
(3) v_2 *is convex and strictly psh on* $\mathbb{C}^2$, *but* v_2 *is not strictly convex on any non-empty open ball of* $\mathbb{C}^2$.
(4) v_3 *is strictly psh on* $\mathbb{C}^2$, v_3 *is not convex on* $\mathbb{C}^2$, *but* v_3 *is not strictly convex at any point of* $\mathbb{C}^2$.
(5) v_4 *is strictly psh on* $\mathbb{C}^2$, v_4 *is not convex on* $\mathbb{C}^2$, *but* v_4 *is not strictly convex on any non-empty open ball of* $\mathbb{C}^2$.

Proof. Obvious. □

Observe that there is a great differences between the 2 families of functions (convex and strictly psh) and (strictly convex) in $\mathbb{C}^2$. However, there is also a great difference between the classes of convex function and strictly psh functions in general in convex domains of $\mathbb{C}^n$, $n \geq 1$.

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