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HIGHER-DIMENSIONAL GENERAL JACOBI IDENTITIES I

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Abstract. It was shown by the author [International Journal of Theoretical Physics **36** (1997), 1099–1131] that what is called the general Jacobi identity, obtaining in microcubes, underlies the Jacobi identity of vector fields. It is well known in the theory of Lie algebras that a plethora of higher-dimensional generalizations of the Jacobi identity hold, though they are usually established not as a derivation on the nose from the axioms of Lie algebras but by making an appeal to the so-called Poincaré–Birkhoff–Witt theorem and the like. The general Jacobi identity was rediscovered by Kirill Mackenzie in the second decade of this century [Geometric Methods in Physics, Birkhäuser/Springer, 2013, 357–366]. The principal objective of this paper is to investigate a four-dimensional generalization of the general Jacobi identity in detail. In a subsequent paper we will propose a uniform method for establishing a bevy of higher-dimensional generalizations of the general Jacobi identity under a single umbrella.

1. Introduction

It is known in synthetic differential geometry (cf. [2] and [4]) that vector fields on a microlinear space M form a Lie algebra, for which the following antisymmetry holds:

$$[X_1, X_2] + [X_2, X_1] = 0.$$

It was shown in [3] that a somewhat deeper theorem below underlies the above identity.

Theorem 1.1. *Let M be a microlinear space. Given microsquares $\gamma_{12}, \gamma_{21} : D^2 \to M$ with $\gamma_{12} \mid D(2) = \gamma_{21} \mid D(2)$, we have*

$$\left(\gamma_{12} \dot{-} \gamma_{21}\right) + \left(\gamma_{21} \dot{-} \gamma_{12}\right) = 0.$$

Now we consider the famous Jacobi identity

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

It claims that the sum of $[X_1, [X_2, X_3]]$'s with the three cyclic permutations of $\{1, 2, 3\}$ applied vanishes. We note in passing that the three cyclic permutations of $\{1, 2, 3\}$ are no other than the three even permutations of $\{1, 2, 3\}$. It has been demonstrated in [6], [7], [8] and [10] that the following deeper theorem underlies the above identity.

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Theorem 1.2. (General Jacobi identity) *Let M be a microlinear space. Given microcubes $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} : D^3 \to M$ with*

$$\begin{aligned}
\gamma_{123} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\right\} &= \gamma_{132} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\right\}, \\
\gamma_{231} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\right\} &= \gamma_{321} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_2 d_3 = 0\right\}, \\
\gamma_{231} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\right\} &= \gamma_{213} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\right\}, \\
\gamma_{312} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\right\} &= \gamma_{132} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_3 = 0\right\}, \\
\gamma_{312} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\right\} &= \gamma_{321} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\right\}, \\
\gamma_{123} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\right\} &= \gamma_{213} \mid \left\{(d_1, d_2, d_3) \in D^3 \mid d_1 d_2 = 0\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\left(\left(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}\right) \dot{-} \left(\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}\right)\right) + \left(\left(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}\right) \dot{-} \left(\gamma_{312} \underset{2}{\dot{-}} \gamma_{132}\right)\right) \\
&\qquad + \left(\left(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}\right) \dot{-} \left(\gamma_{123} \underset{3}{\dot{-}} \gamma_{213}\right)\right) = 0.
\end{aligned}$$

The general Jacobi identity was rediscovered by Kirill Mackenzie [5] in a somewhat different context. We add that the general Jacobi identity plays a fundamental role in a combinatorial or geometric proof of Jacobi-like identities in the so-called Frölicher–Nijenhuis calculus (cf. [9]).

Now we consider the following four-dimensional analogue of the Jacobi identity.

$$\begin{aligned}
&[X_1, [X_2, [X_3, X_4]]] + [X_1, [X_3, [X_4, X_2]]] + [X_1, [X_4, [X_2, X_3]]] + \\
&[X_2, [X_1, [X_4, X_3]]] + [X_2, [X_3, [X_1, X_4]]] + [X_2, [X_4, [X_3, X_1]]] + \\
&[X_3, [X_1, [X_2, X_4]]] + [X_3, [X_2, [X_4, X_1]]] + [X_3, [X_4, [X_1, X_2]]] + \\
&[X_4, [X_1, [X_3, X_2]]] + [X_4, [X_2, [X_1, X_3]]] + [X_4, [X_3, [X_2, X_1]]] = 0.
\end{aligned} \tag{1.1}$$

It claims that the sum of $[X_1, [X_2, [X_3, X_4]]]$'s with the twelve even permutations of $\{1, 2, 3, 4\}$ applied vanishes.

The principal objective in this paper is to establish a four-dimensional version of the general Jacobi identity underpinning the above identity (1.1). Below we will discuss a slew of higher-dimensional general Jacobi identities underlying the higher-dimensional Jacobi identities discussed in [1] and [12] (the former called them *generalized Jacobi identities*) from a coherent standpoint. For a good introduction to generalized Jacobi identities, the reader is referred to Chapter 8 of [11]. We know well that various higher-dimensional Jacobi identities are logical consequences of the three-dimensional Jacobi identity, but we guess that higher-dimensional general Jacobi identities are by no means logical consequences of the three-dimensional general Jacobi identity. We assume the reader to be familiar with [4] up to Chapter 3.

2. Strong differences

First we introduce the notion of a simplicial small object after [6], though in a somewhat generalized form.

Notation 2.1. (*Simplicial small objects*) Let n be a natural number. Given a subset $\mathfrak{p}$ of

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}$$

and a subset ξ of

$$\{i \in \mathbb{N} \mid 1 \leq i \leq n\},$$

$D^n \{\mathfrak{p},\xi\}$ denotes the set

$$\{(d_1, ..., d_n) : D^n \mid d_i d_j = 0 \text{ for any } (i,j) \in \mathfrak{p},\ d_i = 0 \text{ for any } i \in \xi\},$$

which is surely a small object. By way of an example, we have

$$D(2) = D^2 \{(1,2)\},$$
$$D(3) = D^3 \{(1,2), (1,3), (2,3)\}$$

and $D^3 \{1,3\}$ can be identified with D via the canonical isomorphism

$$d \in D \mapsto (0,d,0) \in D^3 \{1,3\}.$$

The notion of strong difference in synthetic differential geometry is based upon the following lemma.

Lemma 2.2. (cf. the first Lemma in §3.4 of [4]) *The diagram*

$$\begin{array}{ccccc} & & D^3 \{(1,3), (2,3)\} & & \\ & \nearrow & & \nwarrow & \\ D^2 & & & & D^2 \\ & \nwarrow & & \nearrow & \\ & & D^2 \{(1,2)\} & & \end{array}$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$j_1^2 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, d_1 d_2) \in D^3 \{(1,3), (2,3)\},$$
$$j_2^2 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3 \{(1,3), (2,3)\}$$

from left to right is a quasi-colimit diagram.

Corollary 2.3. *Let M be a microlinear space with two microsquares $\gamma_1, \gamma_2 : D^2 \to M$ abiding by*

$$\gamma_1 \mid D^2 \{(1,2)\} = \gamma_2 \mid D^2 \{(1,2)\}.$$

Then, there exists a unique mapping

$$\mathfrak{n}^2_{(\gamma_1,\gamma_2)} : D^3 \{(1,3), (2,3)\} \to M$$

such that $\mathfrak{n}^2_{(\gamma_1,\gamma_2)} \circ j_1^2 = \gamma_1$ and $\mathfrak{n}^2_{(\gamma_1,\gamma_2)} \circ j_2^2 = \gamma_2$.

Notation 2.4. In the above notation, in Corollary 2.3 we write $\gamma_1 \overset{\cdot}{-} \gamma_2$ for the mapping

$$d \in D \mapsto \mathfrak{n}^2_{(\gamma_1,\gamma_2)} (0,0,d).$$

The notion of strong difference can easily be relativized.

Lemma 2.5. *Let n be a natural number. The diagram*

$$\begin{array}{ccccc} & & D^{n+3}\left\{(n+1,n+3),(n+2,n+3)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^{n+2} & & & & D^{n+2} \\ & \nwarrow & & \nearrow & \\ & & D^{n+2}\left\{(n+1,n+2)\right\} & & \end{array} \quad (2.1)$$

with the lower two arrows being the canonical injections and the upper two arrows being

$$j_1^{n+2}:(d_1,...,d_n,d_{n+1},d_{n+2})\in D^{n+2}\mapsto(d_1,...,d_n,d_{n+1},d_{n+2},d_{n+1}d_{n+2})\in$$
$$\in D^{n+3}\left\{(n+1,n+3),(n+2,n+3)\right\},$$

$$j_2^{n+2}:(d_1,...,d_n,d_{n+1},d_{n+2})\in D^{n+2}\mapsto(d_1,...,d_n,d_{n+1},d_{n+2},0)\in$$
$$\in D^{n+3}\left\{(n+1,n+3),(n+2,n+3)\right\}$$

from left to right is a quasi-colimit diagram.

Corollary 2.6. *Let n be a natural number. Let M be a microlinear space with two mappings $\gamma_1,\gamma_2:D^{n+2}\to M$ abiding by*

$$\gamma_1\mid D^{n+2}\left\{(n+1,n+2)\right\}=\gamma_2\mid D^{n+2}\left\{(n+1,n+2)\right\}.$$

Then, there exists a unique mapping

$$\mathfrak{n}_{(\gamma_1,\gamma_2)}^{n+2}:D^{n+3}\left\{(n+1,n+3),(n+2,n+3)\right\}\to M$$

such that $\mathfrak{n}_{(\gamma_1,\gamma_2)}^{n+2}\circ j_1^{n+2}=\gamma_1$ and $\mathfrak{n}_{(\gamma_1,\gamma_2)}^{n+2}\circ j_2^{3}=\gamma_2$.

Notation 2.7. Let n be a natural number. Let M be a microlinear space. Given $\gamma:D^n\to M$ and a permutation σ of $\{1,...,n\}$, we write γ^σ for the mapping

$$(d_1,...,d_n)\in D^n\mapsto\gamma\left(d_{\sigma^{-1}(1)},...,d_{\sigma^{-1}(n)}\right)\in M.$$

Notation 2.8. Let n be a natural number. Let M be a microlinear space.

(1) Given $\gamma_1,\gamma_2:D^{n+2}\to M$ with

$$\gamma_1\mid D^{n+2}\left\{(n+1,n+2)\right\}=\gamma_2\mid D^{n+2}\left\{(n+1,n+2)\right\}$$

we write $\gamma_1\underset{1...n}{\dot{-}}\gamma_2$ for the mapping

$$(d_1,...,d_n,d_{n+1})\in D^{n+1}\mapsto\mathfrak{n}_{(\gamma_1,\gamma_2)}^{n+2}(d_1,...,d_n,0,0,d_{n+1})\in M.$$

(2) Given a permutation σ of $\{1,...,n,n+1,n+2\}$ and $\gamma_1,\gamma_2:D^{n+2}\to M$ with

$$\gamma_1\mid D^{n+2}\left\{(\sigma(n+1),\sigma(n+2))\right\}=\gamma_2\mid D^{n+2}\left\{(\sigma(n+1),\sigma(n+2))\right\}$$

we write $\gamma_1\underset{\sigma(1)...\sigma(n)}{\dot{-}}\gamma_2$ for $(\gamma_1)^\sigma\underset{1...n}{\dot{-}}(\gamma_2)^\sigma$.

The following result is well known.

Lemma 2.9. (cf. Proposition 6 in §2.2 of [4]) *The diagram*

$$\begin{array}{ccccc} & & D^2\{(1,2)\} & & \\ & \nearrow & & \nwarrow & \\ D^2\{1\} & & & & D^2\{2\} \\ & \nwarrow & & \nearrow & \\ & & 1 & & \end{array}$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 2.10. *Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^2 \to M$,*

$$\gamma_1 \mid D^2\{(1,2)\} = \gamma_2 \mid D^2\{(1,2)\}$$

is obtained iff both

$$\gamma_1 \mid D^2\{1\} = \gamma_2 \mid D^2\{1\}$$

and

$$\gamma_1 \mid D^2\{2\} = \gamma_2 \mid D^2\{2\}$$

obtain.

It can readily be relativized and generalized.

Lemma 2.11. *The diagram*

$$\begin{array}{ccccc} & & D^{n+m_1+m_2}\left\{\begin{array}{c}(n+i,n+m_1+j)\\ 1\leq i\leq m_1,\\ 1\leq j\leq m_2\end{array}\right\} & & \\ & \nearrow & & \nwarrow & \\ D^{n+m_1+m_2}\left\{\begin{array}{c}n+1,...,\\ n+m_1\end{array}\right\} & & & & D^{n+m_1+m_2}\left\{\begin{array}{c}n+m_1+1,...,\\ n+m_1+m_2\end{array}\right\} \\ & \nwarrow & & \nearrow & \\ & & D^n & & \end{array}$$

with the four arrows being the canonical injections is a quasi-colimit diagram.

Corollary 2.12. *Let M be a microlinear space. Given $\gamma_1, \gamma_2 : D^{n+m_1+m_2} \to M$,*

$$\begin{aligned} &\gamma_1 \mid D^{n+m_1+m_2}\{(n+i,n+m_1+j) \mid 1\leq i\leq m_1, 1\leq j\leq m_2\} \\ &= \gamma_2 \mid D^{n+m_1+m_2}\{(n+i,n+m_1+j) \mid 1\leq i\leq m_1, 1\leq j\leq m_2\} \end{aligned}$$

is obtained iff both

$$\gamma_1 \mid D^{n+m_1+m_2}\{n+1,...,n+m_1\} = \gamma_2 \mid D^{n+m_1+m_2}\{n+1,...,n+m_1\}$$

and

$$\begin{aligned} &\gamma_1 \mid D^{n+m_1+m_2}\{n+m_1+1,...,n+m_1+m_2\} \\ &= \gamma_2 \mid D^{n+m_1+m_2}\{n+m_1+1,...,n+m_1+m_2\} \end{aligned}$$

are obtained.

Proposition 2.13. *Let M be a microlinear space. Then, we have the following two statements:*

(1) *Given*

$$\gamma_1 : D^4 \to M, \gamma_2 : D^4 \to M, \gamma_3 : D^4 \to M, \gamma_4 : D^4 \to M$$

if it holds that

$$\gamma_1 \mid D^4\{(3,4)\} = \gamma_2 \mid D^4\{(3,4)\},$$
$$\gamma_3 \mid D^4\{(3,4)\} = \gamma_4 \mid D^4\{(3,4)\},$$
$$\gamma_1 \mid D^4\{(2,3),(2,4)\} = \gamma_3 \mid D^4\{(2,3),(2,4)\}, \tag{2.2}$$
$$\gamma_2 \mid D^4\{(2,3),(2,4)\} = \gamma_4 \mid D^4\{(2,3),(2,4)\}, \tag{2.3}$$

Then, all of

$$\gamma_1 \underset{12}{\dot{-}} \gamma_2, \qquad \gamma_3 \underset{12}{\dot{-}} \gamma_4, \qquad \left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)$$

are well defined.

(2) *Given*

$$\gamma_1 : D^4 \to M, \gamma_2 : D^4 \to M, \gamma_3 : D^4 \to M, \gamma_4 : D^4 \to M,$$
$$\gamma_5 : D^4 \to M, \gamma_6 : D^4 \to M, \gamma_7 : D^4 \to M, \gamma_8 : D^4 \to M$$

if it holds that

$$\gamma_1 \mid D^4\{(3,4)\} = \gamma_2 \mid D^4\{(3,4)\},$$
$$\gamma_3 \mid D^4\{(3,4)\} = \gamma_4 \mid D^4\{(3,4)\},$$
$$\gamma_5 \mid D^4\{(3,4)\} = \gamma_6 \mid D^4\{(3,4)\},$$
$$\gamma_7 \mid D^4\{(3,4)\} = \gamma_8 \mid D^4\{(3,4)\},$$
$$\gamma_1 \mid D^4\{(2,3),(2,4)\} = \gamma_3 \mid D^4\{(2,3),(2,4)\},$$
$$\gamma_2 \mid D^4\{(2,3),(2,4)\} = \gamma_4 \mid D^4\{(2,3),(2,4)\},$$
$$\gamma_5 \mid D^4\{(2,3),(2,4)\} = \gamma_7 \mid D^4\{(2,3),(2,4)\},$$
$$\gamma_6 \mid D^4\{(2,3),(2,4)\} = \gamma_8 \mid D^4\{(2,3),(2,4)\},$$
$$\gamma_1 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_5 \mid D^4\{(1,2),(1,3),(1,4)\}, \tag{2.4}$$
$$\gamma_2 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_6 \mid D^4\{(1,2),(1,3),(1,4)\}, \tag{2.5}$$
$$\gamma_3 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_7 \mid D^4\{(1,2),(1,3),(1,4)\}, \tag{2.6}$$
$$\gamma_4 \mid D^4\{(1,2),(1,3),(1,4)\} = \gamma_8 \mid D^4\{(1,2),(1,3),(1,4)\}, \tag{2.7}$$

Then, all of

$$\gamma_1 \underset{12}{\dot{-}} \gamma_2, \quad \gamma_3 \underset{12}{\dot{-}} \gamma_4, \quad \gamma_5 \underset{12}{\dot{-}} \gamma_6, \quad \gamma_7 \underset{12}{\dot{-}} \gamma_8$$
$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right), \quad \left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right),$$
$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \dot{-} \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right)$$

are well defined.

Proof. We deal with the above two statements in order.

(1) For the first statement, we have to show that

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\left\{(2,3)\right\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\left\{(2,3)\right\}$$

which is, by dint of Corollary 2.12, equivalent to showing that

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\left\{2\right\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\left\{2\right\}, \tag{2.8}$$

$$\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\left\{3\right\} = \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\left\{3\right\} \tag{2.9}$$

because of the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^3\left\{(2,3)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^3\left\{2\right\} & & & & D^3\left\{3\right\} \\ & \nwarrow & & \nearrow & \\ & & D^3\left\{2,3\right\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 2.11 with $n=m_1=m_2=1$). Due to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\left\{(2,3),(2,4)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^4\left\{2\right\} & & & & D^4\left\{3,4\right\} \\ & \nwarrow & & \nearrow & \\ & & D^4\left\{2,3,4\right\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 2.11 with $n=m_1=1$ and $m_2=2$), the condition (2.2) is equivalent to the conditions

$$\gamma_1 \mid D^4\left\{2\right\} = \gamma_3 \mid D^4\left\{2\right\}, \tag{2.10}$$

$$\gamma_1 \mid D^4\left\{3,4\right\} = \gamma_3 \mid D^4\left\{3,4\right\} \tag{2.11}$$

while the condition (2.3) is equivalent to the conditions

$$\gamma_2 \mid D^4\left\{2\right\} = \gamma_4 \mid D^4\left\{2\right\}, \tag{2.12}$$

$$\gamma_2 \mid D^4\left\{3,4\right\} = \gamma_4 \mid D^4\left\{3,4\right\}. \tag{2.13}$$

In order to show that (2.8) is obtained, we note that the quasi-colimit diagram in (2.1) with $n=2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^5\left\{2,(3,5),(4,5)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^4\left\{2\right\} & & & & D^4\left\{2\right\} \\ & \nwarrow & & \nearrow & \\ & & D^4\left\{2,(3,4)\right\} & & \end{array}$$

so that the conditions (2.10) and (2.12) imply (2.8). It is easy to see that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \mid D^3\left\{3\right\}\right) \circ i_2^3 = \left(\gamma_1 \mid D^4\left\{3,4\right\}\right) \circ i_2^4 = \left(\gamma_2 \mid D^4\left\{3,4\right\}\right) \circ i_2^4,$$

$$\left(\left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right) \mid D^3\left\{3\right\}\right) \circ i_2^3 = \left(\gamma_3 \mid D^4\left\{3,4\right\}\right) \circ i_2^4 = \left(\gamma_4 \mid D^4\left\{3,4\right\}\right) \circ i_2^4$$

are obtained with

$$i_2^3 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0) \in D^3,$$
$$i_2^4 : (d_1, d_2) \in D^2 \mapsto (d_1, d_2, 0, 0) \in D^4$$

so that (2.11) and (2.13) imply (2.9).

(2) For the second statement, we have to show that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\left\{(1,2)\right\} = \\ = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\left\{(1,2)\right\}$$

which is equivalent to showing that

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\left\{1\right\} = \\ = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\left\{1\right\}, \quad (2.14)$$

$$\left(\left(\gamma_1 \underset{12}{\dot{-}} \gamma_2\right) \underset{1}{\dot{-}} \left(\gamma_3 \underset{12}{\dot{-}} \gamma_4\right)\right) \mid D^2\left\{2\right\} = \\ = \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\left\{2\right\} \quad (2.15)$$

because of the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^2\left\{(1,2)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^2\left\{1\right\} & & & & D^2\left\{2\right\} \\ & \nwarrow & & \nearrow & \\ & & D^2\left\{1,2\right\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 2.9). Due to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\left\{(1,2),(1,3),(1,4)\right\} & & \\ & \nearrow & & \nwarrow & \\ D^4\left\{1\right\} & & & & D^4\left\{2,3,4\right\} \\ & \nwarrow & & \nearrow & \\ & & D^4\left\{1,2,3,4\right\} & & \end{array}$$

with the four arrows being the canonical injections (Lemma 2.11 with $n = 0$, $m_1 = 1$ and $m_2 = 3$), we have

- the condition (2.4) is equivalent to the conditions

$$\gamma_1 \mid D^4\left\{1\right\} = \gamma_5 \mid D^4\left\{1\right\}, \quad (2.16)$$

$$\gamma_1 \mid D^4\left\{2,3,4\right\} = \gamma_5 \mid D^4\left\{2,3,4\right\}, \quad (2.17)$$

- the condition (2.5) is equivalent to the conditions

$$\gamma_2 \mid D^4\left\{1\right\} = \gamma_6 \mid D^4\left\{1\right\}, \quad (2.18)$$

$$\gamma_2 \mid D^4\{2,3,4\} = \gamma_6 \mid D^4\{2,3,4\}, \tag{2.19}$$

- the condition (2.6) is equivalent to the conditions

$$\gamma_3 \mid D^4\{1\} = \gamma_7 \mid D^4\{1\}, \tag{2.20}$$
$$\gamma_3 \mid D^4\{2,3,4\} = \gamma_7 \mid D^4\{2,3,4\}, \tag{2.21}$$

- the condition (2.7) is equivalent to the conditions

$$\gamma_4 \mid D^4\{1\} = \gamma_8 \mid D^4\{1\}, \tag{2.22}$$
$$\gamma_4 \mid D^4\{2,3,4\} = \gamma_8 \mid D^4\{2,3,4\}. \tag{2.23}$$

In order to show that (2.14) is obtained, we note first that the quasi-colimit diagram in 2.1 with $n=2$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^5\{1,(3,5),(4,5)\} & & \\ & \nearrow & & \nwarrow & \\ D^4\{1\} & & & & D^4\{1\} \\ & \nwarrow & & \nearrow & \\ & & D^4\{1,(3,4)\} & & \end{array}$$

so that we have

- the conditions (2.16) and (2.18) imply

$$\left(\gamma_1 \underset{12}{\overset{\cdot}{-}} \gamma_2\right) \mid D^3\{1\} = \left(\gamma_5 \underset{12}{\overset{\cdot}{-}} \gamma_6\right) \mid D^3\{1\}, \tag{2.24}$$

- the conditions (2.20) and (2.22) imply

$$\left(\gamma_3 \underset{12}{\overset{\cdot}{-}} \gamma_4\right) \mid D^3\{1\} = \left(\gamma_7 \underset{12}{\overset{\cdot}{-}} \gamma_8\right) \mid D^3\{1\}. \tag{2.25}$$

We note also that the quasi-colimit diagram in (2.1) with $n=1$ is to be restricted to the quasi-colimit diagram

$$\begin{array}{ccccc} & & D^4\{1,(2,4),(3,4)\} & & \\ & \nearrow & & \nwarrow & \\ D^3\{1\} & & & & D^3\{1\} \\ & \nwarrow & & \nearrow & \\ & & D^3\{1,(2,3)\} & & \end{array}$$

so that the conditions (2.24) and (2.25) imply the condition (2.14). It is easy to see that

$$\begin{aligned}
&\left(\left(\left(\gamma_1 \underset{12}{\overset{\cdot}{-}} \gamma_2\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_3 \underset{12}{\overset{\cdot}{-}} \gamma_4\right)\right) \mid D^2\{2\}\right) \circ i^2 \\
&= \left(\left(\gamma_1 \underset{12}{\overset{\cdot}{-}} \gamma_2\right) \mid D^3\{2,3\}\right) \circ i^3 = \left(\left(\gamma_3 \underset{12}{\overset{\cdot}{-}} \gamma_4\right) \mid D^3\{2,3\}\right) \circ i^3 \\
&= \left(\gamma_1 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_2 \mid D^4\{2,3,4\}\right) \circ i^4 \\
&= \left(\gamma_3 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_4 \mid D^4\{2,3,4\}\right) \circ i^4
\end{aligned}$$

and

$$\left(\left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \underset{1}{\dot{-}} \left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right)\right) \mid D^2\{2\}\right) \circ i^2$$
$$= \left(\left(\gamma_5 \underset{12}{\dot{-}} \gamma_6\right) \mid D^3\{2,3\}\right) \circ i^3 = \left(\left(\gamma_7 \underset{12}{\dot{-}} \gamma_8\right) \mid D^3\{2,3\}\right) \circ i^3$$
$$= \left(\gamma_5 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_6 \mid D^4\{2,3,4\}\right) \circ i^4$$
$$= \left(\gamma_7 \mid D^4\{2,3,4\}\right) \circ i^4 = \left(\gamma_8 \mid D^4\{2,3,4\}\right) \circ i^4$$

are obtained with

$$i^2 : d \in D \mapsto (d,0) \in D^2\{2\},$$
$$i^3 : d \in D \mapsto (d,0,0) \in D^3\{2,3\},$$
$$i^4 : d \in D \mapsto (d,0,0,0) \in D^4\{2,3,4\}$$

so that (2.17), (2.19), (2.21) and (2.23) imply (2.15).

□

Notation 2.14. Let M be a microlinear space.

(1) We denote by $\mathfrak{X}(M)$ the totality of vector fields on M. It forms a Lie algebra. We take the third viewpoint of a vector field in the essentially equivalent three discussed in §3.2 of [4]. Namely, a vector field X on M is a mapping $d \in D \mapsto X_d \in M^M$ with $X_0 = \mathrm{id}_M$.

(2) Given $X, ..., X_n \in \mathfrak{X}(M)$, we denote by $X_n * ... * X_1$ the mapping $(d_1, ..., d_n) \in D^n \mapsto (X_n)_{d_n} \circ ... \circ (X_1)_{d_1} \in M^M$.

(3) We recall (cf. Proposition 8 in §3.4 of [4]) that

$$[X_1, X_2] = X_2 * X_1 \dot{-} (X_1 * X_2)^{\sigma_{21}}$$

with

$$\sigma_{21} = \begin{pmatrix} 12 \\ 21 \end{pmatrix}.$$

We recall (cf. Proposition 2.7 of [6]) that

$$[X_1, [X_2, X_3]] = \left(X_3 * X_2 * X_1 \underset{1}{\dot{-}} (X_2 * X_3 * X_1)^{\sigma_{132}}\right) \dot{-} \left((X_1 * X_3 * X_2)^{\sigma_{231}} \underset{1}{\dot{-}} (X_1 * X_2 * X_3)^{\sigma_{321}}\right)$$

with

$$\sigma_{132} = \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \quad \sigma_{231} = \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \quad \sigma_{321} = \begin{pmatrix} 123 \\ 321 \end{pmatrix}.$$

We note that

$$[X_1, [X_2, [X_3, X_4]]]$$
$$= \begin{pmatrix} \left(X_4 * X_3 * X_2 * X_1 \underset{12}{\dot{-}} (X_3 * X_4 * X_2 * X_1)^{\sigma_{1243}}\right) \underset{1}{\dot{-}} \\ \left((X_2 * X_4 * X_3 * X_1)^{\sigma_{1342}} \underset{12}{\dot{-}} (X_2 * X_3 * X_4 * X_1)^{\sigma_{1432}}\right) \end{pmatrix} \dot{-}$$

$$\left(\begin{array}{c} \left((X_1 * X_4 * X_3 * X_2)^{\sigma_{2341}} \underset{12}{\dot{-}} (X_1 * X_3 * X_4 * X_2)^{\sigma_{2431}} \right) \underset{1}{\dot{-}} \\ \left((X_1 * X_2 * X_4 * X_3)^{\sigma_{3421}} \underset{12}{\dot{-}} (X_1 * X_2 * X_3 * X_4)^{\sigma_{4321}} \right) \end{array} \right)$$

with

$$\sigma_{1243} = \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, \ \sigma_{1342} = \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, \ \sigma_{1432} = \begin{pmatrix} 1234 \\ 1432 \end{pmatrix}, \ \sigma_{2341} = \begin{pmatrix} 1234 \\ 4123 \end{pmatrix},$$
$$\sigma_{2431} = \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, \ \sigma_{3421} = \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, \ \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}.$$

3. A four-dimensional general Jacobi identity

Theorem 3.1. *The diagram whose underlying directed graph consists of vertices*

$P, Q^{1234}, Q^{1243}, Q^{1324}, Q^{1342}, Q^{1423}, Q^{1432}, Q^{2134}, Q^{2143}, Q^{2314}, Q^{2341}, Q^{2413}, Q^{2431},$
$Q^{3124}, Q^{3142}, Q^{3214}, Q^{3241}, Q^{3412}, Q^{3421}, Q^{4123}, Q^{4132}, Q^{4213}, Q^{4231}, Q^{4312}, Q^{4321},$
$R_{12}^{1234,1243}, R_{12}^{1342,1432}, R_{12}^{2341,2431}, R_{12}^{3421,4321}, R_{12}^{2134,2143}, R_{12}^{3412,4312}, R_{13}^{1324,1342},$
$R_{13}^{1243,1423}, R_{13}^{3241,3421}, R_{13}^{2431,4231}, R_{13}^{3124,3142}, R_{13}^{2413,4213}, R_{14}^{1423,1432}, R_{14}^{1234,1324},$
$R_{14}^{4231,4321}, R_{14}^{2341,3241}, R_{14}^{4123,4132}, R_{14}^{2314,3214}, R_{23}^{2314,2341}, R_{23}^{2143,2413}, R_{23}^{3142,3412},$
$R_{23}^{1432,4132}, R_{23}^{3214,3241}, R_{23}^{1423,4123}, R_{24}^{2413,2431}, R_{24}^{2134,2314}, R_{24}^{4132,4312}, R_{24}^{1342,3142},$
$R_{24}^{4213,4231}, R_{24}^{1324,3124}, R_{34}^{3412,3421}, R_{34}^{3124,3214}, R_{34}^{4123,4213}, R_{34}^{1243,2143}, R_{34}^{4312,4321},$
$R_{34}^{1234,2134}$

and edges

$$f_{1234} : Q^{1234} \to P, f_{1243} : Q^{1243} \to P, f_{1324} : Q^{1324} \to P, f_{1342} : Q^{1342} \to P,$$
$$f_{1423} : Q^{1423} \to P, f_{1432} : Q^{1432} \to P, f_{2134} : Q^{2134} \to P, f_{2143} : Q^{2143} \to P,$$
$$f_{2314} : Q^{2314} \to P, f_{2341} : Q^{2341} \to P, f_{2413} : Q^{2413} \to P, f_{2431} : Q^{2431} \to P,$$
$$f_{3124} : Q^{3124} \to P, f_{3142} : Q^{3142} \to P, f_{3214} : Q^{3214} \to P, f_{3241} : Q^{3241} \to P,$$
$$f_{3412} : Q^{3412} \to P, f_{3421} : Q^{3421} \to P, f_{4123} : Q^{4123} \to P, f_{4132} : Q^{4132} \to P,$$
$$f_{4213} : Q^{4213} \to P, f_{4231} : Q^{4231} \to P, f_{4312} : Q^{4312} \to P, f_{4321} : Q^{4321} \to P,$$

$$g_{12}^{1234,1243} : R_{12}^{1234,1243} \to Q^{1234}, h_{12}^{1234,1243} : R_{12}^{1234,1243} \to Q^{1243},$$
$$g_{12}^{1342,1432} : R_{12}^{1342,1432} \to Q^{1342}, h_{12}^{1342,1432} : R_{12}^{1342,1432} \to Q^{1432},$$
$$g_{12}^{2341,2431} : R_{12}^{2341,2431} \to Q^{2341}, h_{12}^{2341,2431} : R_{12}^{2341,2431} \to Q^{2431},$$
$$g_{12}^{3421,4321} : R_{12}^{3421,4321} \to Q^{3421}, h_{12}^{3421,4321} : R_{12}^{3421,4321} \to Q^{4321},$$
$$g_{12}^{2134,2143} : R_{12}^{2134,2143} \to Q^{2134}, h_{12}^{2134,2143} : R_{12}^{2134,2143} \to Q^{2143},$$
$$g_{12}^{3412,4312} : R_{12}^{3412,4312} \to Q^{3412}, h_{12}^{3412,4312} : R_{12}^{3412,4312} \to Q^{4312},$$

$$g_{13}^{1324,1342} : R_{13}^{1324,1342} \to Q^{1324}, h_{13}^{1324,1342} : R_{13}^{1324,1342} \to Q^{1342},$$

$$g_{13}^{1243,1423}: R_{13}^{1243,1423} \to Q^{1243}, h_{13}^{1243,1423}: R_{13}^{1243,1423} \to Q^{1423},$$
$$g_{13}^{3241,3421}: R_{13}^{3241,3421} \to Q^{3241}, h_{13}^{3241,3421}: R_{13}^{3241,3421} \to Q^{3421},$$
$$g_{13}^{2431,4231}: R_{13}^{2431,4231} \to Q^{2431}, h_{13}^{2431,4231}: R_{13}^{2431,4231} \to Q^{4231},$$
$$g_{13}^{3124,3142}: R_{13}^{3124,3142} \to Q^{3124}, h_{13}^{3124,3142}: R_{13}^{3124,3142} \to Q^{3142},$$
$$g_{13}^{2413,4213}: R_{13}^{2413,4213} \to Q^{2413}, h_{13}^{2413,4213}: R_{13}^{2413,4213} \to Q^{4213},$$

$$g_{14}^{1423,1432}: R_{14}^{1423,1432} \to Q^{1423}, h_{14}^{1423,1432}: R_{14}^{1423,1432} \to Q^{1432},$$
$$g_{14}^{1234,1324}: R_{14}^{1234,1324} \to Q^{1234}, h_{14}^{1234,1324}: R_{14}^{1234,1324} \to Q^{1324},$$
$$g_{14}^{4231,4321}: R_{14}^{4231,4321} \to Q^{4231}, h_{14}^{4231,4321}: R_{14}^{4231,4321} \to Q^{4321},$$
$$g_{14}^{2341,3241}: R_{14}^{2341,3241} \to Q^{2341}, h_{14}^{2341,3241}: R_{14}^{2341,3241} \to Q^{3241},$$
$$g_{14}^{4123,4132}: R_{14}^{4123,4132} \to Q^{4123}, h_{14}^{4123,4132}: R_{14}^{4123,4132} \to Q^{4132},$$
$$g_{14}^{2314,3214}: R_{14}^{2314,3214} \to Q^{2314}, h_{14}^{2314,3214}: R_{14}^{2314,3214} \to Q^{3214},$$

$$g_{23}^{2314,2341}: R_{23}^{2314,2341} \to Q^{2314}, h_{23}^{2314,2341}: R_{23}^{2314,2341} \to Q^{2341},$$
$$g_{23}^{2143,2413}: R_{23}^{2143,2413} \to Q^{2143}, h_{23}^{2143,2413}: R_{23}^{2143,2413} \to Q^{2413},$$
$$g_{23}^{3142,3412}: R_{23}^{3142,3412} \to Q^{3142}, h_{23}^{3142,3412}: R_{23}^{3142,3412} \to Q^{3412},$$
$$g_{23}^{1432,4132}: R_{23}^{1432,4132} \to Q^{1432}, h_{23}^{1432,4132}: R_{23}^{1432,4132} \to Q^{4132},$$
$$g_{23}^{3214,3241}: R_{23}^{3214,3241} \to Q^{3214}, h_{23}^{3214,3241}: R_{23}^{3214,3241} \to Q^{3241},$$
$$g_{23}^{1423,4123}: R_{23}^{1423,4123} \to Q^{1423}, h_{23}^{1423,4123}: R_{23}^{1423,4123} \to Q^{4123},$$

$$g_{24}^{2413,2431}: R_{24}^{2413,2431} \to Q^{2413}, h_{24}^{2413,2431}: R_{24}^{2413,2431} \to Q^{2431},$$
$$g_{24}^{2134,2314}: R_{24}^{2134,2314} \to Q^{2134}, h_{24}^{2134,2314}: R_{24}^{2134,2314} \to Q^{2314},$$
$$g_{24}^{4132,4312}: R_{24}^{4132,4312} \to Q^{4132}, h_{24}^{4132,4312}: R_{24}^{4132,4312} \to Q^{4312},$$
$$g_{24}^{1342,3142}: R_{24}^{1342,3142} \to Q^{1342}, h_{24}^{1342,3142}: R_{24}^{1342,3142} \to Q^{3142},$$
$$g_{24}^{4213,4231}: R_{24}^{4213,4231} \to Q^{4213}, h_{24}^{4213,4231}: R_{24}^{4213,4231} \to Q^{4231},$$
$$g_{24}^{1324,3124}: R_{24}^{1324,3124} \to Q^{1324}, h_{24}^{1324,3124}: R_{24}^{1324,3124} \to Q^{3124},$$

$$g_{34}^{3412,3421}: R_{34}^{3412,3421} \to Q^{3412}, h_{34}^{3412,3421}: R_{34}^{3412,3421} \to Q^{3421},$$
$$g_{34}^{3124,3214}: R_{34}^{3124,3214} \to Q^{3124}, h_{34}^{3124,3214}: R_{34}^{3124,3214} \to Q^{3214},$$
$$g_{34}^{4123,4213}: R_{34}^{4123,4213} \to Q^{4123}, h_{34}^{4123,4213}: R_{34}^{4123,4213} \to Q^{4213},$$
$$g_{34}^{1243,2143}: R_{34}^{1243,2143} \to Q^{1243}, h_{34}^{1243,2143}: R_{34}^{1243,2143} \to Q^{2143},$$
$$g_{34}^{4312,4321}: R_{34}^{4312,4321} \to Q^{4312}, h_{34}^{4312,4321}: R_{34}^{4312,4321} \to Q^{4321},$$
$$g_{34}^{1234,2134}: R_{34}^{1234,2134} \to Q^{1234}, h_{34}^{1234,2134}: R_{34}^{1234,2134} \to Q^{2134}$$

with P being labelled

$$D^{53}\left\{\begin{array}{c}(1,5),(2,5),(1,6),(3,6),(1,7),(4,7),(2,8),(3,8),(2,9),(4,9),\\(3,10),(4,10),(5,6),(5,7),(5,8),(5,9),(6,7),(6,8),(6,10),(7,9),\\(7,10),(8,9),(8,10),(9,10),(1,i_{11,15}),(2,i_{11,15}),(3,i_{11,15}),\\(5,i_{11,15}),(6,i_{11,15}),(7,i_{11,15}),(8,i_{11,15}),(9,i_{11,15}),(10,i_{11,15}),\\(1,i_{16,20}),(2,i_{16,20}),(4,i_{16,20}),(5,i_{16,20}),(6,i_{16,20}),(7,i_{16,20}),\\(8,i_{16,20}),(9,i_{16,20}),(10,i_{16,20}),(1,i_{21,25})(3,i_{21,25}),(4,i_{21,25}),\\(5,i_{21,25}),(6,i_{21,25}),(7,i_{21,25}),(8,i_{21,25}),(9,i_{21,25}),(10,i_{21,25}),\\(2,i_{26,30}),(3,i_{26,30}),(4,i_{26,30}),(5,i_{26,30}),(6,i_{26,30}),(7,i_{26,30}),\\(8,i_{26,30}),(9,i_{26,30}),(10,i_{26,30}),\left(i_{11,15},i'_{11,15}\right),(i_{11,15},i_{16,20}),\\(i_{11,15},i_{21,25}),(i_{11,15},i_{26,30}),\left(i_{16,20},i'_{16,20}\right),(i_{16,20},i_{21,25}),\\(i_{16,20},i_{26,30}),\left(i_{21,25},i'_{21,25}\right),(i_{21,25},i_{26,30}),\left(i_{26,30},i'_{26,30}\right),\\(1,i_{31,53}),(2,i_{31,53}),(3,i_{31,53}),(4,i_{31,53}),(5,i_{31,53}),(6,i_{31,53}),\\(7,i_{31,53}),(8,i_{31,53}),(9,i_{31,53}),(10,i_{31,53}),(i_{11,15},i_{31,53}),\\(i_{16,20},i_{31,53}),(i_{21,25},i_{31,53}),(i_{26,30},i_{31,53}),\left(i_{31,53},i'_{31,53}\right)\\ \mid 1i_{11,15},i'_{11,15},i_{16,20},i'_{16,20},i_{21,25},i'_{21,25},i_{26,30},i'_{26,30},i_{31,53},i'_{31,53}\in\mathbb{N},\\1\le i_{11,15}\le 15,11\le i'_{11,15}\le 15,16\le i_{16,20}\le 20,\\16\le i'_{16,20}\le 20,21\le i_{21,25}\le 25,21\le i'_{21,25}\le 25,\\26\le i_{26,30}\le 30,26\le i'_{26,30}\le 30,31\le i_{31,53}\le 53,\\31\le i'_{31,53}\le 53\end{array}\right\}$$

all of

$$Q^{1234},Q^{1243},Q^{1324},Q^{1342},Q^{1423},Q^{1432},Q^{2134},Q^{2143},Q^{2314},Q^{2341},Q^{2413},Q^{2431},$$
$$Q^{3124},Q^{3142},Q^{3214},Q^{3241},Q^{3412},Q^{3421},Q^{4123},Q^{4132},Q^{4213},Q^{4231},Q^{4312},Q^{4321}$$

being laballed D^4, all of

$$R_{12}^{1234,1243},R_{12}^{1342,1432},R_{12}^{2341,2431},R_{12}^{3421,4321},R_{12}^{2134,2143},R_{12}^{3412,4312}$$

being labelled $D^4\{(3,4)\}$, all of

$$R_{13}^{1324,1342},R_{13}^{1243,1423},R_{13}^{3241,3421},R_{13}^{2431,4231},R_{13}^{3124,3142},R_{13}^{2413,4213}$$

being labelled $D^4\{(2,4)\}$, all of

$$R_{14}^{1423,1432},R_{14}^{1234,1324},R_{14}^{4231,4321},R_{14}^{2341,3241},R_{14}^{4123,4132},R_{14}^{2314,3214}$$

being labelled $D^4\{(2,3)\}$, all of

$$R_{23}^{2314,2341},R_{23}^{2143,2413},R_{23}^{3142,3412},R_{23}^{1432,4132},R_{23}^{3214,3241},R_{23}^{1423,4123}$$

being labelled $D^4\{(1,4)\}$, all of

$$R_{24}^{2413,2431},R_{24}^{2134,2314},R_{24}^{4132,4312},R_{24}^{1342,3142},R_{24}^{4213,4231},R_{24}^{1324,3124}$$

being labelled $D^4\{(1,3)\}$

$$R_{34}^{3412,3421},R_{34}^{3124,3214},R_{34}^{4123,4213},R_{34}^{1243,2143},R_{34}^{4312,4321},R_{34}^{1234,2134}$$

being labelled $D^4\{(1,2)\}$, the edges

$$f_{1234},f_{1243},f_{1324},f_{1342},f_{1423},f_{1432},f_{2134},f_{2143},f_{2314},f_{2341},f_{2413},f_{2431},$$
$$f_{3124},f_{3142},f_{3214},f_{3241},f_{3412},f_{3421},f_{4123},f_{4132},f_{4213},f_{4231},f_{4312},f_{4321}$$

standing for mappings

$$f_{1234}(d_1,d_2,d_3,d_4) = \left(d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{53}{0}\right),$$

$$f_{1243}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},\dots,\underset{20}{0},d_1d_3d_4,\underset{22}{0},\dots,\underset{25}{0},d_2d_3d_4,\underset{27}{0},\dots,\underset{30}{0}, \\ d_1d_2d_3d_4,\underset{32}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{1324}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{7}{0},d_2d_3,\underset{9}{0},\underset{10}{0},d_1d_2d_3,\underset{12}{0},\dots,\underset{26}{0},d_2d_3d_4,\underset{28}{0},\dots,\underset{31}{0}, \\ d_1d_2d_3d_4,\underset{33}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{1342}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},\dots,\underset{15}{0},d_1d_2d_4,\underset{17}{0},\dots,\underset{27}{0}, \\ d_2d_3d_4,\underset{29}{0},\dots,\underset{32}{0},d_1d_2d_3d_4,\underset{34}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{1423}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\dots,\underset{15}{0},d_1d_2d_4,\underset{17}{0},\dots,\underset{20}{0},d_1d_3d_4, \\ \underset{22}{0},\dots,\underset{28}{0},d_2d_3d_4,\underset{30}{0},\dots,\underset{33}{0},d_1d_2d_3d_4,\underset{35}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{1432}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\dots,\underset{7}{0},d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},\dots,\underset{15}{0},d_1d_2d_4,\underset{16}{0},\dots,\underset{20}{0}, \\ d_1d_3d_4,\underset{22}{0},\dots,\underset{29}{0},d_2d_3d_4,\underset{31}{0},\dots,\underset{34}{0},d_1d_2d_3d_4,\underset{36}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{2134}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},\dots,\underset{11}{0},d_1d_2d_3,\underset{13}{0},\dots,\underset{16}{0},d_1d_2d_4, \\ \underset{18}{0},\dots,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{2143}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},\dots,\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},\dots,\underset{16}{0},d_1d_2d_4,\underset{18}{0},\dots,\underset{20}{0}, \\ d_1d_3d_4,\underset{22}{0},\dots,\underset{25}{0},d_2d_3d_4,\underset{27}{0},\dots,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{2314}(d_1,d_2,d_3,d_4) = \left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},\dots,\underset{12}{0},d_1d_2d_3,\underset{14}{0},\dots,\underset{16}{0},d_1d_2d_4,\underset{18}{0},\dots,\underset{21}{0},d_1d_3d_4, \\ \underset{23}{0},\dots,\underset{37}{0},d_1d_2d_3d_4,\underset{39}{0},\dots,\underset{53}{0} \end{array}\right),$$

$$f_{2341}(d_1,d_2,d_3,d_4)$$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, ..., \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{38}{0}, d_1d_2d_3d_4, \underset{40}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{2413}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{23}{0}, \\ d_1d_3d_4, \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, ..., \underset{39}{0}, d_1d_2d_3d_4, \underset{41}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{2431}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{24}{0}, d_1d_3d_4, d_2d_3d_4, \underset{27}{0}, ..., \underset{40}{0}, d_1d_2d_3d_4, \underset{42}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3124}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{21}{0}, d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, \\ d_2d_3d_4, \underset{28}{0}, ..., \underset{41}{0}, d_1d_2d_3d_4, \underset{43}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3142}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{42}{0}, d_1d_2d_3d_4, \underset{44}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3214}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{43}{0}, d_1d_2d_3d_4, \underset{45}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3241}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{44}{0}, d_1d_2d_3d_4, \underset{46}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3412}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{3421}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, d_1d_2d_4, \\ \underset{21}{0}, \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{46}{0}, d_1d_2d_3d_4, \underset{48}{0}, ..., \underset{53}{0} \end{array} \right),$$

$f_{4123}\,(d_1, d_2, d_3, d_4)$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1d_4, \underset{7}{0}, \underset{7}{0}, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{18}{0}, d_1d_2d_4, \underset{20}{0}, ..., \underset{23}{0}, d_1d_3d_4, \\ \underset{25}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{47}{0}, d_1d_2d_3d_4, \underset{49}{0}, ..., \underset{53}{0} \end{array} \right),$$

$$f_{4132}(d_1,d_2,d_3,d_4)$$
$$=\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0}, \\ d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0} \end{pmatrix},$$

$$f_{4213}(d_1,d_2,d_3,d_4)$$
$$=\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{pmatrix},$$

$$f_{4231}(d_1,d_2,d_3,d_4)$$
$$=\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,\underset{52}{0},\underset{53}{0} \end{pmatrix},$$

$$f_{4312}(d_1,d_2,d_3,d_4)$$
$$=\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0}, \\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0} \end{pmatrix},$$

$$f_{4321}(d_1,d_2,d_3,d_4)$$
$$=\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{52}{0},d_1d_2d_3d_4 \end{pmatrix},$$

respectively, and all the other edges standing for the identity mappings (e.g., both $g_{12}^{1234,1243}$ *and* $h_{12}^{1234,1243}$ *standing for the mapping* $(d_1,d_2,d_3,d_4)\in D^4\{(3,4)\}\mapsto(d_1,d_2,d_3,d_4)\in D^4$ *while both* $g_{13}^{1324,1342}$ *and* $h_{13}^{1324,1342}$ *standing for the mapping* $(d_1,d_2,d_3,d_4)\in D^4\{(2,4)\}\mapsto(d_1,d_2,d_3,d_4)\in D^4$*) is a quasi-colimit diagram.*

Proof. Let $\theta^{1234}:Q^{1234}\to\mathbb{R}$, $\theta^{1243}:Q^{1243}\to\mathbb{R}$, $\theta^{1324}:Q^{1324}\to\mathbb{R}$, $\theta^{1342}:Q^{1342}\to\mathbb{R}$, $\theta^{1423}:Q^{1423}\to\mathbb{R}$, $\theta^{1432}:Q^{1432}\to\mathbb{R}$, $\theta^{2134}:Q^{2134}\to\mathbb{R}$, $\theta^{2143}:Q^{2143}\to\mathbb{R}$, $\theta^{2314}:Q^{2314}\to\mathbb{R}$, $\theta^{2341}:Q^{2341}\to\mathbb{R}$, $\theta^{2413}:Q^{2413}\to\mathbb{R}$, $\theta^{2431}:Q^{2431}\to\mathbb{R}$, $\theta^{3124}:Q^{3124}\to\mathbb{R}$, $\theta^{3142}:Q^{3142}\to\mathbb{R}$, $\theta^{3214}:Q^{3214}\to\mathbb{R}$, $\theta^{3241}:Q^{3241}\to\mathbb{R}$, $\theta^{3412}:Q^{3412}\to\mathbb{R}$, $\theta^{3421}:Q^{3421}\to\mathbb{R}$, $\theta^{4123}:Q^{4123}\to\mathbb{R}$, $\theta^{4132}:Q^{4132}\to\mathbb{R}$, $\theta^{4213}:Q^{4213}\to\mathbb{R}$, $\theta^{4231}:Q^{4231}\to\mathbb{R}$, $\theta^{4312}:Q^{4312}\to\mathbb{R}$ and $\theta^{4321}:Q^{4321}\to\mathbb{R}$ be mappings, which are to be of the following forms by dint of the general Kock–Lawvere axiom (cf. §2.1.3 of [4]):

$$\theta^{1234}(d_1,d_2,d_3,d_4)$$
$$=a^{1234}+a_1^{1234}d_1+a_2^{1234}d_2+a_3^{1234}d_3+a_4^{1234}d_1+a_{12}^{1234}d_1d_2+a_{13}^{1234}d_1d_3+$$
$$a_{14}^{1234}d_1d_4+a_{23}^{1234}d_2d_3+a_{24}^{1234}d_2d_4+a_{34}^{1234}d_3d_4+a_{123}^{1234}d_1d_2d_3+a_{124}^{1234}d_1d_2d_4+$$
$$a_{134}^{1234}d_1d_3d_4+a_{234}^{1234}d_2d_3d_4+a_{1234}^{1234}d_1d_2d_3d_4,$$

$$\theta^{1243}(d_1,d_2,d_3,d_4)$$
$$=a^{1243}+a_1^{1243}d_1+a_2^{1243}d_2+a_3^{1243}d_3+a_4^{1243}d_1+a_{12}^{1243}d_1d_2+a_{13}^{1243}d_1d_3+$$

$a_{14}^{1243}d_1d_4 + a_{23}^{1243}d_2d_3 + a_{24}^{1243}d_2d_4 + a_{34}^{1243}d_3d_4 + a_{123}^{1243}d_1d_2d_3 + a_{124}^{1243}d_1d_2d_4 +$
$a_{134}^{1243}d_1d_3d_4 + a_{234}^{1243}d_2d_3d_4 + a_{1234}^{1243}d_1d_2d_3d_4,$

$\theta^{1324}(d_1, d_2, d_3, d_4)$
$= a^{1324} + a_1^{1324}d_1 + a_2^{1324}d_2 + a_3^{1324}d_3 + a_4^{1324}d_1 + a_{12}^{1324}d_1d_2 + a_{13}^{1324}d_1d_3 +$
$a_{14}^{1324}d_1d_4 + a_{23}^{1324}d_2d_3 + a_{24}^{1324}d_2d_4 + a_{34}^{1324}d_3d_4 + a_{123}^{1324}d_1d_2d_3 + a_{124}^{1324}d_1d_2d_4 +$
$a_{134}^{1324}d_1d_3d_4 + a_{234}^{1324}d_2d_3d_4 + a_{1234}^{1324}d_1d_2d_3d_4,$

$\theta^{1342}(d_1, d_2, d_3, d_4)$
$= a^{1342} + a_1^{1342}d_1 + a_2^{1342}d_2 + a_3^{1342}d_3 + a_4^{1342}d_1 + a_{12}^{1342}d_1d_2 + a_{13}^{1342}d_1d_3 +$
$a_{14}^{1342}d_1d_4 + a_{23}^{1342}d_2d_3 + a_{24}^{1342}d_2d_4 + a_{34}^{1342}d_3d_4 + a_{123}^{1342}d_1d_2d_3 + a_{124}^{1342}d_1d_2d_4 +$
$a_{134}^{1342}d_1d_3d_4 + a_{234}^{1342}d_2d_3d_4 + a_{1234}^{1342}d_1d_2d_3d_4,$

$\theta^{1423}(d_1, d_2, d_3, d_4)$
$= a^{1423} + a_1^{1423}d_1 + a_2^{1423}d_2 + a_3^{1423}d_3 + a_4^{1423}d_1 + a_{12}^{1423}d_1d_2 + a_{13}^{1423}d_1d_3 +$
$a_{14}^{1423}d_1d_4 + a_{23}^{1423}d_2d_3 + a_{24}^{1423}d_2d_4 + a_{34}^{1423}d_3d_4 + a_{123}^{1423}d_1d_2d_3 + a_{124}^{1423}d_1d_2d_4 +$
$a_{134}^{1423}d_1d_3d_4 + a_{234}^{1423}d_2d_3d_4 + a_{1234}^{1423}d_1d_2d_3d_4,$

$\theta^{1432}(d_1, d_2, d_3, d_4)$
$= a^{1432} + a_1^{1432}d_1 + a_2^{1432}d_2 + a_3^{1432}d_3 + a_4^{1432}d_1 + a_{12}^{1432}d_1d_2 + a_{13}^{1432}d_1d_3 +$
$a_{14}^{1432}d_1d_4 + a_{23}^{1432}d_2d_3 + a_{24}^{1432}d_2d_4 + a_{34}^{1432}d_3d_4 + a_{123}^{1432}d_1d_2d_3 + a_{124}^{1432}d_1d_2d_4 +$
$a_{134}^{1432}d_1d_3d_4 + a_{234}^{1432}d_2d_3d_4 + a_{1234}^{1432}d_1d_2d_3d_4,$

$\theta^{2134}(d_1, d_2, d_3, d_4)$
$= a^{2134} + a_1^{2134}d_1 + a_2^{2134}d_2 + a_3^{2134}d_3 + a_4^{2134}d_1 + a_{12}^{2134}d_1d_2 + a_{13}^{2134}d_1d_3 +$
$a_{14}^{2134}d_1d_4 + a_{23}^{2134}d_2d_3 + a_{24}^{2134}d_2d_4 + a_{34}^{2134}d_3d_4 + a_{123}^{2134}d_1d_2d_3 + a_{124}^{2134}d_1d_2d_4 +$
$a_{134}^{2134}d_1d_3d_4 + a_{234}^{2134}d_2d_3d_4 + a_{1234}^{2134}d_1d_2d_3d_4,$

$\theta^{2143}(d_1, d_2, d_3, d_4)$
$= a^{2143} + a_1^{2143}d_1 + a_2^{2143}d_2 + a_3^{2143}d_3 + a_4^{2143}d_1 + a_{12}^{2143}d_1d_2 + a_{13}^{2143}d_1d_3 +$
$a_{14}^{2143}d_1d_4 + a_{23}^{2143}d_2d_3 + a_{24}^{2143}d_2d_4 + a_{34}^{2143}d_3d_4 + a_{123}^{2143}d_1d_2d_3 + a_{124}^{2143}d_1d_2d_4 +$
$a_{134}^{2143}d_1d_3d_4 + a_{234}^{2143}d_2d_3d_4 + a_{1234}^{2143}d_1d_2d_3d_4,$

$\theta^{2314}(d_1, d_2, d_3, d_4)$
$= a^{2314} + a_1^{2314}d_1 + a_2^{2314}d_2 + a_3^{2314}d_3 + a_4^{2314}d_1 + a_{12}^{2314}d_1d_2 + a_{13}^{2314}d_1d_3 +$
$a_{14}^{2314}d_1d_4 + a_{23}^{2314}d_2d_3 + a_{24}^{2314}d_2d_4 + a_{34}^{2314}d_3d_4 + a_{123}^{2314}d_1d_2d_3 + a_{124}^{2314}d_1d_2d_4 +$
$a_{134}^{2314}d_1d_3d_4 + a_{234}^{2314}d_2d_3d_4 + a_{1234}^{2314}d_1d_2d_3d_4,$

$\theta^{2341}(d_1, d_2, d_3, d_4)$

$= a^{2341} + a_1^{2341} d_1 + a_2^{2341} d_2 + a_3^{2341} d_3 + a_4^{2341} d_1 + a_{12}^{2341} d_1 d_2 + a_{13}^{2341} d_1 d_3 +$
$a_{14}^{2341} d_1 d_4 + a_{23}^{2341} d_2 d_3 + a_{24}^{2341} d_2 d_4 + a_{34}^{2341} d_3 d_4 + a_{123}^{2341} d_1 d_2 d_3 + a_{124}^{2341} d_1 d_2 d_4 +$
$a_{134}^{2341} d_1 d_3 d_4 + a_{234}^{2341} d_2 d_3 d_4 + a_{1234}^{2341} d_1 d_2 d_3 d_4,$

$\theta^{2413}(d_1, d_2, d_3, d_4)$
$= a^{2413} + a_1^{2413} d_1 + a_2^{2413} d_2 + a_3^{2413} d_3 + a_4^{2413} d_1 + a_{12}^{2413} d_1 d_2 + a_{13}^{2413} d_1 d_3 +$
$a_{14}^{2413} d_1 d_4 + a_{23}^{2413} d_2 d_3 + a_{24}^{2413} d_2 d_4 + a_{34}^{2413} d_3 d_4 + a_{123}^{2413} d_1 d_2 d_3 + a_{124}^{2413} d_1 d_2 d_4 +$
$a_{134}^{2413} d_1 d_3 d_4 + a_{234}^{2413} d_2 d_3 d_4 + a_{1234}^{2413} d_1 d_2 d_3 d_4,$

$\theta^{2431}(d_1, d_2, d_3, d_4)$
$= a^{2431} + a_1^{2431} d_1 + a_2^{2431} d_2 + a_3^{2431} d_3 + a_4^{2431} d_1 + a_{12}^{2431} d_1 d_2 + a_{13}^{2431} d_1 d_3 +$
$a_{14}^{2431} d_1 d_4 + a_{23}^{2431} d_2 d_3 + a_{24}^{2431} d_2 d_4 + a_{34}^{2431} d_3 d_4 + a_{123}^{2431} d_1 d_2 d_3 + a_{124}^{2431} d_1 d_2 d_4 +$
$a_{134}^{2431} d_1 d_3 d_4 + a_{234}^{2431} d_2 d_3 d_4 + a_{1234}^{2431} d_1 d_2 d_3 d_4,$

$\theta^{3124}(d_1, d_2, d_3, d_4)$
$= a^{3124} + a_1^{3124} d_1 + a_2^{3124} d_2 + a_3^{3124} d_3 + a_4^{3124} d_1 + a_{12}^{3124} d_1 d_2 + a_{13}^{3124} d_1 d_3 +$
$a_{14}^{3124} d_1 d_4 + a_{23}^{3124} d_2 d_3 + a_{24}^{3124} d_2 d_4 + a_{34}^{3124} d_3 d_4 + a_{123}^{3124} d_1 d_2 d_3 + a_{124}^{3124} d_1 d_2 d_4 +$
$a_{134}^{3124} d_1 d_3 d_4 + a_{234}^{3124} d_2 d_3 d_4 + a_{1234}^{3124} d_1 d_2 d_3 d_4,$

$\theta^{3142}(d_1, d_2, d_3, d_4)$
$= a^{3142} + a_1^{3142} d_1 + a_2^{3142} d_2 + a_3^{3142} d_3 + a_4^{3142} d_1 + a_{12}^{3142} d_1 d_2 + a_{13}^{3142} d_1 d_3 +$
$a_{14}^{3142} d_1 d_4 + a_{23}^{3142} d_2 d_3 + a_{24}^{3142} d_2 d_4 + a_{34}^{3142} d_3 d_4 + a_{123}^{3142} d_1 d_2 d_3 + a_{124}^{3142} d_1 d_2 d_4 +$
$a_{134}^{3142} d_1 d_3 d_4 + a_{234}^{3142} d_2 d_3 d_4 + a_{1234}^{3142} d_1 d_2 d_3 d_4,$

$\theta^{3214}(d_1, d_2, d_3, d_4)$
$= a^{3214} + a_1^{3214} d_1 + a_2^{3214} d_2 + a_3^{3214} d_3 + a_4^{3214} d_1 + a_{12}^{3214} d_1 d_2 + a_{13}^{3214} d_1 d_3 +$
$a_{14}^{3214} d_1 d_4 + a_{23}^{3214} d_2 d_3 + a_{24}^{3214} d_2 d_4 + a_{34}^{3214} d_3 d_4 + a_{123}^{3214} d_1 d_2 d_3 + a_{124}^{3214} d_1 d_2 d_4 +$
$a_{134}^{3214} d_1 d_3 d_4 + a_{234}^{3214} d_2 d_3 d_4 + a_{1234}^{3214} d_1 d_2 d_3 d_4,$

$\theta^{3241}(d_1, d_2, d_3, d_4)$
$= a^{3241} + a_1^{3241} d_1 + a_2^{3241} d_2 + a_3^{3241} d_3 + a_4^{3241} d_1 + a_{12}^{3241} d_1 d_2 + a_{13}^{3241} d_1 d_3 +$
$a_{14}^{3241} d_1 d_4 + a_{23}^{3241} d_2 d_3 + a_{24}^{3241} d_2 d_4 + a_{34}^{3241} d_3 d_4 + a_{123}^{3241} d_1 d_2 d_3 + a_{124}^{3241} d_1 d_2 d_4 +$
$a_{134}^{3241} d_1 d_3 d_4 + a_{234}^{3241} d_2 d_3 d_4 + a_{1234}^{3241} d_1 d_2 d_3 d_4,$

$\theta^{3412}(d_1, d_2, d_3, d_4)$
$= a^{3412} + a_1^{3412} d_1 + a_2^{3412} d_2 + a_3^{3412} d_3 + a_4^{3412} d_1 + a_{12}^{3412} d_1 d_2 + a_{13}^{3412} d_1 d_3 +$
$a_{14}^{3412} d_1 d_4 + a_{23}^{3412} d_2 d_3 + a_{24}^{3412} d_2 d_4 + a_{34}^{3412} d_3 d_4 + a_{123}^{3412} d_1 d_2 d_3 + a_{124}^{3412} d_1 d_2 d_4 +$
$a_{134}^{3412} d_1 d_3 d_4 + a_{234}^{3412} d_2 d_3 d_4 + a_{1234}^{3412} d_1 d_2 d_3 d_4,$

$\theta^{3421}(d_1,d_2,d_3,d_4)$
$= a^{3421} + a_1^{3421}d_1 + a_2^{3421}d_2 + a_3^{3421}d_3 + a_4^{3421}d_1 + a_{12}^{3421}d_1d_2 + a_{13}^{3421}d_1d_3 +$
$a_{14}^{3421}d_1d_4 + a_{23}^{3421}d_2d_3 + a_{24}^{3421}d_2d_4 + a_{34}^{3421}d_3d_4 + a_{123}^{3421}d_1d_2d_3 + a_{124}^{3421}d_1d_2d_4 +$
$a_{134}^{3421}d_1d_3d_4 + a_{234}^{3421}d_2d_3d_4 + a_{1234}^{3421}d_1d_2d_3d_4,$

$\theta^{4123}(d_1,d_2,d_3,d_4)$
$= a^{4123} + a_1^{4123}d_1 + a_2^{4123}d_2 + a_3^{4123}d_3 + a_4^{4123}d_1 + a_{12}^{4123}d_1d_2 + a_{13}^{4123}d_1d_3 +$
$a_{14}^{4123}d_1d_4 + a_{23}^{4123}d_2d_3 + a_{24}^{4123}d_2d_4 + a_{34}^{4123}d_3d_4 + a_{123}^{4123}d_1d_2d_3 + a_{124}^{4123}d_1d_2d_4 +$
$a_{134}^{4123}d_1d_3d_4 + a_{234}^{4123}d_2d_3d_4 + a_{1234}^{4123}d_1d_2d_3d_4,$

$\theta^{4132}(d_1,d_2,d_3,d_4)$
$= a^{4132} + a_1^{4132}d_1 + a_2^{4132}d_2 + a_3^{4132}d_3 + a_4^{4132}d_1 + a_{12}^{4132}d_1d_2 + a_{13}^{4132}d_1d_3 +$
$a_{14}^{4132}d_1d_4 + a_{23}^{4132}d_2d_3 + a_{24}^{4132}d_2d_4 + a_{34}^{4132}d_3d_4 + a_{123}^{4132}d_1d_2d_3 + a_{124}^{4132}d_1d_2d_4 +$
$a_{134}^{4132}d_1d_3d_4 + a_{234}^{4132}d_2d_3d_4 + a_{1234}^{4132}d_1d_2d_3d_4,$

$\theta^{4213}(d_1,d_2,d_3,d_4)$
$= a^{4213} + a_1^{4213}d_1 + a_2^{4213}d_2 + a_3^{4213}d_3 + a_4^{4213}d_1 + a_{12}^{4213}d_1d_2 + a_{13}^{4213}d_1d_3 +$
$a_{14}^{4213}d_1d_4 + a_{23}^{4213}d_2d_3 + a_{24}^{4213}d_2d_4 + a_{34}^{4213}d_3d_4 + a_{123}^{4213}d_1d_2d_3 + a_{124}^{4213}d_1d_2d_4 +$
$a_{134}^{4213}d_1d_3d_4 + a_{234}^{4213}d_2d_3d_4 + a_{1234}^{4213}d_1d_2d_3d_4,$

$\theta^{4231}(d_1,d_2,d_3,d_4)$
$= a^{4231} + a_1^{4231}d_1 + a_2^{4231}d_2 + a_3^{4231}d_3 + a_4^{4231}d_1 + a_{12}^{4231}d_1d_2 + a_{13}^{4231}d_1d_3 +$
$a_{14}^{4231}d_1d_4 + a_{23}^{4231}d_2d_3 + a_{24}^{4231}d_2d_4 + a_{34}^{4231}d_3d_4 + a_{123}^{4231}d_1d_2d_3 + a_{124}^{4231}d_1d_2d_4 +$
$a_{134}^{4231}d_1d_3d_4 + a_{234}^{4231}d_2d_3d_4 + a_{1234}^{4231}d_1d_2d_3d_4,$

$\theta^{4312}(d_1,d_2,d_3,d_4)$
$= a^{4312} + a_1^{4312}d_1 + a_2^{4312}d_2 + a_3^{4312}d_3 + a_4^{4312}d_1 + a_{12}^{4312}d_1d_2 + a_{13}^{4312}d_1d_3 +$
$a_{14}^{4312}d_1d_4 + a_{23}^{4312}d_2d_3 + a_{24}^{4312}d_2d_4 + a_{34}^{4312}d_3d_4 + a_{123}^{4312}d_1d_2d_3 + a_{124}^{4312}d_1d_2d_4 +$
$a_{134}^{4312}d_1d_3d_4 + a_{234}^{4312}d_2d_3d_4 + a_{1234}^{4312}d_1d_2d_3d_4,$

$\theta^{4321}(d_1,d_2,d_3,d_4)$
$= a^{4321} + a_1^{4321}d_1 + a_2^{4321}d_2 + a_3^{4321}d_3 + a_4^{4321}d_1 + a_{12}^{4321}d_1d_2 + a_{13}^{4321}d_1d_3 +$
$a_{14}^{4321}d_1d_4 + a_{23}^{4321}d_2d_3 + a_{24}^{4321}d_2d_4 + a_{34}^{4321}d_3d_4 + a_{123}^{4321}d_1d_2d_3 + a_{124}^{4321}d_1d_2d_4 +$
$a_{134}^{4321}d_1d_3d_4 + a_{234}^{4321}d_2d_3d_4 + a_{1234}^{4321}d_1d_2d_3d_4.$

The conditions

$$\theta^{1234} \circ g_{12}^{1234,1243} = \theta^{1243} \circ h_{12}^{1234,1243}, \theta^{1342} \circ g_{12}^{1342,1432} = \theta^{1432} \circ h_{12}^{1342,1432},$$
$$\theta^{2341} \circ g_{12}^{2341,2431} = \theta^{2431} \circ h_{12}^{2341,2431}, \theta^{3421} \circ g_{12}^{3421,4321} = \theta^{4321} \circ h_{12}^{3421,4321},$$

$$\theta^{2134}\circ g_{12}^{2134,2143}=\theta^{2143}\circ h_{12}^{2134,2143},\theta^{3412}\circ g_{12}^{3412,4312}=\theta^{4312}\circ h_{12}^{3412,4312},$$
$$\theta^{1324}\circ g_{13}^{1324,1342}=\theta^{1342}\circ h_{13}^{1324,1342},\theta^{1243}\circ g_{13}^{1243,1423}=\theta^{1423}\circ h_{13}^{1243,1423},$$
$$\theta^{3241}\circ g_{13}^{3241,3421}=\theta^{3421}\circ h_{13}^{3241,3421},\theta^{2431}\circ g_{13}^{2431,4231}=\theta^{4231}\circ h_{13}^{2431,4231},$$
$$\theta^{3124}\circ g_{13}^{3124,3142}=\theta^{3142}\circ h_{13}^{3124,3142},\theta^{2413}\circ g_{13}^{2413,4213}=\theta^{4213}\circ h_{13}^{2413,4213},$$
$$\theta^{1423}\circ g_{14}^{1423,1432}=\theta^{1432}\circ h_{14}^{1423,1432},\theta^{1234}\circ g_{14}^{1234,1324}=\theta^{1324}\circ h_{14}^{1234,1324},$$
$$\theta^{4231}\circ g_{14}^{4231,4321}=\theta^{4321}\circ h_{14}^{4231,4321},\theta^{2341}\circ g_{14}^{2341,3241}=\theta^{3241}\circ h_{14}^{2341,3241},$$
$$\theta^{4123}\circ g_{14}^{4123,4132}=\theta^{4132}\circ h_{14}^{4123,4132},\theta^{2314}\circ g_{14}^{2314,3214}=\theta^{3214}\circ h_{14}^{2314,3214},$$
$$\theta^{2314}\circ g_{23}^{2314,2341}=\theta^{2341}\circ h_{23}^{2314,2341},\theta^{2143}\circ g_{23}^{2143,2413}=\theta^{2413}\circ h_{23}^{2143,2413},$$
$$\theta^{3142}\circ g_{23}^{3142,3412}=\theta^{3412}\circ h_{23}^{3142,3412},\theta^{1432}\circ g_{23}^{1432,4132}=\theta^{4132}\circ h_{23}^{1432,4132},$$
$$\theta^{3214}\circ g_{23}^{3214,3241}=\theta^{3241}\circ h_{23}^{3214,3241},\theta^{1423}\circ g_{23}^{1423,4123}=\theta^{4123}\circ h_{23}^{1423,4123},$$
$$\theta^{2413}\circ g_{24}^{2413,2431}=\theta^{2431}\circ h_{24}^{2413,2431},\theta^{2134}\circ g_{24}^{2134,2314}=\theta^{2314}\circ h_{24}^{2134,2314},$$
$$\theta^{4132}\circ g_{24}^{4132,4312}=\theta^{4312}\circ h_{24}^{4132,4312},\theta^{1342}\circ g_{24}^{1342,3142}=\theta^{3142}\circ h_{24}^{1342,3142},$$
$$\theta^{4213}\circ g_{24}^{4213,4231}=\theta^{4231}\circ h_{24}^{4213,4231},\theta^{1324}\circ g_{24}^{1324,3124}=\theta^{3124}\circ h_{24}^{1324,3124},$$
$$\theta^{3412}\circ g_{34}^{3412,3421}=\theta^{3421}\circ h_{34}^{3412,3421},\theta^{3124}\circ g_{34}^{3124,3214}=\theta^{3214}\circ h_{34}^{3124,3214},$$
$$\theta^{4123}\circ g_{34}^{4123,4213}=\theta^{4213}\circ h_{34}^{4123,4213},\theta^{1243}\circ g_{34}^{1243,2143}=\theta^{2143}\circ h_{34}^{1243,2143},$$
$$\theta^{4312}\circ g_{34}^{4312,4321}=\theta^{4321}\circ h_{34}^{4312,4321},\theta^{1234}\circ g_{34}^{1234,2134}=\theta^{2134}\circ h_{34}^{1234,2134}$$

are equivalent to the following conditions in terms of coefficients of the polynomials

$$a^{1342}=a^{1432},a_1^{1342}=a_1^{1432},a_2^{1342}=a_2^{1432},a_3^{1342}=a_3^{1432},a_4^{1342}=a_4^{1432},$$
$$a_{12}^{1342}=a_{12}^{1432},a_{13}^{1342}=a_{13}^{1432},a_{14}^{1342}=a_{14}^{1432},a_{23}^{1342}=a_{23}^{1432},a_{24}^{1342}=a_{24}^{1432},$$
$$a_{123}^{1342}=a_{123}^{1432},a_{124}^{1342}=a_{124}^{1432},$$

$$a^{2341}=a^{2431},a_1^{2341}=a_1^{2431},a_2^{2341}=a_2^{2431},a_3^{2341}=a_3^{2431},a_4^{2341}=a_4^{2431},$$
$$a_{12}^{2341}=a_{12}^{2431},a_{13}^{2341}=a_{13}^{2431},a_{14}^{2341}=a_{14}^{2431},a_{23}^{2341}=a_{23}^{2431},a_{24}^{2341}=a_{24}^{2431},$$
$$a_{123}^{2341}=a_{123}^{2431},a_{124}^{2341}=a_{124}^{2431},$$

$$a^{3421}=a^{4321},a_1^{3421}=a_1^{4321},a_2^{3421}=a_2^{4321},a_3^{3421}=a_3^{4321},a_4^{3421}=a_4^{4321},$$
$$a_{12}^{3421}=a_{12}^{4321},a_{13}^{3421}=a_{13}^{4321},a_{14}^{3421}=a_{14}^{4321},a_{23}^{3421}=a_{23}^{4321},a_{24}^{3421}=a_{24}^{4321},$$
$$a_{123}^{3421}=a_{123}^{4321},a_{124}^{3421}=a_{124}^{4321},$$

$$a^{2341}=a^{2314},a_1^{2341}=a_1^{2314},a_2^{2341}=a_2^{2314},a_3^{2341}=a_3^{2314},a_4^{2341}=a_4^{2314},$$
$$a_{12}^{2341}=a_{12}^{2314},a_{13}^{2341}=a_{13}^{2314},a_{14}^{2341}=a_{14}^{2314},a_{24}^{2341}=a_{24}^{2314},a_{34}^{2341}=a_{34}^{2314},$$
$$a_{124}^{2341}=a_{124}^{2314},a_{134}^{2341}=a_{134}^{2314},$$

$$a^{2413}=a^{2143},a_1^{2413}=a_1^{2143},a_2^{2413}=a_2^{2143},a_3^{2413}=a_3^{2143},a_4^{2413}=a_4^{2143},$$
$$a_{12}^{2413}=a_{12}^{2143},a_{13}^{2413}=a_{13}^{2143},a_{14}^{2413}=a_{14}^{2143},a_{24}^{2413}=a_{24}^{2143},a_{34}^{2413}=a_{34}^{2143},$$
$$a_{124}^{2413}=a_{124}^{2143},a_{134}^{2413}=a_{134}^{2143},$$

$a^{3412} = a^{3142}, a_1^{3412} = a_1^{3142}, a_2^{3412} = a_2^{3142}, a_3^{3412} = a_3^{3142}, a_4^{3412} = a_4^{3142},$
$a_{12}^{3412} = a_{12}^{3142}, a_{13}^{3412} = a_{13}^{3142}, a_{14}^{3412} = a_{14}^{3142}, a_{24}^{3412} = a_{24}^{3142}, a_{34}^{3412} = a_{34}^{3142},$
$a_{124}^{3412} = a_{124}^{3142}, a_{134}^{3412} = a_{134}^{3142},$

$a^{4132} = a^{1432}, a_1^{4132} = a_1^{1432}, a_2^{4132} = a_2^{1432}, a_3^{4132} = a_3^{1432}, a_4^{4132} = a_4^{1432},$
$a_{12}^{4132} = a_{12}^{1432}, a_{13}^{4132} = a_{13}^{1432}, a_{14}^{4132} = a_{14}^{1432}, a_{24}^{4132} = a_{24}^{1432}, a_{34}^{4132} = a_{34}^{1432},$
$a_{124}^{4132} = a_{124}^{1432}, a_{134}^{4132} = a_{134}^{1432},$

$a^{3412} = a^{3421}, a_1^{3412} = a_1^{3421}, a_2^{3412} = a_2^{3421}, a_3^{3412} = a_3^{3421}, a_4^{3412} = a_4^{3421},$
$a_{12}^{3412} = a_{12}^{3421}, a_{13}^{3412} = a_{13}^{3421}, a_{14}^{3412} = a_{14}^{3421}, a_{23}^{3412} = a_{23}^{3421}, a_{24}^{3412} = a_{24}^{3421},$
$a_{123}^{3412} = a_{123}^{3421}, a_{124}^{3412} = a_{124}^{3421},$

$a^{3124} = a^{3214}, a_1^{3124} = a_1^{3214}, a_2^{3124} = a_2^{3214}, a_3^{3124} = a_3^{3214}, a_4^{3124} = a_4^{3214},$
$a_{12}^{3124} = a_{12}^{3214}, a_{13}^{3124} = a_{13}^{3214}, a_{14}^{3124} = a_{14}^{3214}, a_{23}^{3124} = a_{23}^{3214}, a_{24}^{3124} = a_{24}^{3214},$
$a_{123}^{3124} = a_{123}^{3214}, a_{124}^{3124} = a_{124}^{3214},$

$a^{4123} = a^{4213}, a_1^{4123} = a_1^{4213}, a_2^{4123} = a_2^{4213}, a_3^{4123} = a_3^{4213}, a_4^{4123} = a_4^{4213},$
$a_{12}^{4123} = a_{12}^{4213}, a_{13}^{4123} = a_{13}^{4213}, a_{14}^{4123} = a_{14}^{4213}, a_{23}^{4123} = a_{23}^{4213}, a_{24}^{4123} = a_{24}^{4213},$
$a_{123}^{4123} = a_{123}^{4213}, a_{124}^{4123} = a_{124}^{4213},$

$a^{1243} = a^{2143}, a_1^{1243} = a_1^{2143}, a_2^{1243} = a_2^{2143}, a_3^{1243} = a_3^{2143}, a_4^{1243} = a_4^{2143},$
$a_{12}^{1243} = a_{12}^{2143}, a_{13}^{1243} = a_{13}^{2143}, a_{14}^{1243} = a_{14}^{2143}, a_{23}^{1243} = a_{23}^{2143}, a_{24}^{1243} = a_{24}^{2143},$
$a_{123}^{1243} = a_{123}^{2143}, a_{124}^{1243} = a_{124}^{2143},$

$a^{4123} = a^{4132}, a_1^{4123} = a_1^{4132}, a_2^{4123} = a_2^{4132}, a_3^{4123} = a_3^{4132}, a_4^{4123} = a_4^{4132},$
$a_{12}^{4123} = a_{12}^{4132}, a_{13}^{4123} = a_{13}^{4132}, a_{23}^{4123} = a_{23}^{4132}, a_{24}^{4123} = a_{24}^{4132}, a_{34}^{4123} = a_{34}^{4132},$
$a_{123}^{4123} = a_{123}^{4132}, a_{234}^{4123} = a_{234}^{4132},$

$a^{4231} = a^{4321}, a_1^{4231} = a_1^{4321}, a_2^{4231} = a_2^{4321}, a_3^{4231} = a_3^{4321}, a_4^{4231} = a_4^{4321},$
$a_{12}^{4231} = a_{12}^{4321}, a_{13}^{4231} = a_{13}^{4321}, a_{23}^{4231} = a_{23}^{4321}, a_{24}^{4231} = a_{24}^{4321}, a_{34}^{4231} = a_{34}^{4321},$
$a_{123}^{4231} = a_{123}^{4321}, a_{234}^{4231} = a_{234}^{4321},$

$a^{1234} = a^{1324}, a_1^{1234} = a_1^{1324}, a_2^{1234} = a_2^{1324}, a_3^{1234} = a_3^{1324}, a_4^{1234} = a_4^{1324},$
$a_{12}^{1234} = a_{12}^{1324}, a_{13}^{1234} = a_{13}^{1324}, a_{23}^{1234} = a_{23}^{1324}, a_{24}^{1234} = a_{24}^{1324}, a_{34}^{1234} = a_{34}^{1324},$
$a_{123}^{1234} = a_{123}^{1324}, a_{234}^{1234} = a_{234}^{1324},$

$a^{2314} = a^{3214}, a_1^{2314} = a_1^{3214}, a_2^{2314} = a_2^{3214}, a_3^{2314} = a_3^{3214}, a_4^{2314} = a_4^{3214},$
$a_{12}^{2314} = a_{12}^{3214}, a_{13}^{2314} = a_{13}^{3214}, a_{23}^{2314} = a_{23}^{3214}, a_{24}^{2314} = a_{24}^{3214}, a_{34}^{2314} = a_{34}^{3214},$
$a_{123}^{2314} = a_{123}^{3214}, a_{234}^{2314} = a_{234}^{3214}$

which can succinctly be summarized as

$$a^{1234}=a^{1243}=a^{1324}=a^{1342}=a^{1423}=a^{1432}=$$
$$a^{2134}=a^{2143}=a^{2314}=a^{2341}=a^{2413}=a^{2431}=$$
$$a^{3124}=a^{3142}=a^{3214}=a^{3241}=a^{3412}=a^{3421}=$$
$$a^{4123}=a^{4132}=a^{4213}=a^{4231}=a^{4312}=a^{4321},$$

$$a_1^{1234}=a_1^{1243}=a_1^{1324}=a_1^{1342}=a_1^{1423}=a_1^{1432}=$$
$$a_1^{2134}=a_1^{2143}=a_1^{2314}=a_1^{2341}=a_1^{2413}=a_1^{2431}=$$
$$a_1^{3124}=a_1^{3142}=a_1^{3214}=a_1^{3241}=a_1^{3412}=a_1^{3421}=$$
$$a_1^{4123}=a_1^{4132}=a_1^{4213}=a_1^{4231}=a_1^{4312}=a_1^{4321},$$

$$a_2^{1234}=a_2^{1243}=a_2^{1324}=a_2^{1342}=a_2^{1423}=a_2^{1432}=$$
$$a_2^{2134}=a_2^{2143}=a_2^{2314}=a_2^{2341}=a_2^{2413}=a_2^{2431}=$$
$$a_2^{3124}=a_2^{3142}=a_2^{3214}=a_2^{3241}=a_2^{3412}=a_2^{3421}=$$
$$a_2^{4123}=a_2^{4132}=a_2^{4213}=a_2^{4231}=a_2^{4312}=a_2^{4321},$$

$$a_3^{1234}=a_3^{1243}=a_3^{1324}=a_3^{1342}=a_3^{1423}=a_3^{1432}=$$
$$a_3^{2134}=a_3^{2143}=a_3^{2314}=a_3^{2341}=a_3^{2413}=a_3^{2431}=$$
$$a_3^{3124}=a_3^{3142}=a_3^{3214}=a_3^{3241}=a_3^{3412}=a_3^{3421}=$$
$$a_3^{4123}=a_3^{4132}=a_3^{4213}=a_3^{4231}=a_3^{4312}=a_3^{4321},$$

$$a_4^{1234}=a_4^{1243}=a_4^{1324}=a_4^{1342}=a_4^{1423}=a_4^{1432}=$$
$$a_4^{2134}=a_4^{2143}=a_4^{2314}=a_4^{2341}=a_4^{2413}=a_4^{2431}=$$
$$a_4^{3124}=a_4^{3142}=a_4^{3214}=a_4^{3241}=a_4^{3412}=a_4^{3421}=$$
$$a_4^{4123}=a_4^{4132}=a_4^{4213}=a_4^{4231}=a_4^{4312}=a_4^{4321},$$

$$a_{12}^{1234}=a_{12}^{1243}=a_{12}^{1324}=a_{12}^{1342}=a_{12}^{1423}=a_{12}^{1432}=$$
$$a_{12}^{3124}=a_{12}^{3142}=a_{12}^{3412}=a_{12}^{4123}=a_{12}^{4132}=a_{12}^{4312},$$

$$a_{12}^{2134}=a_{12}^{2143}=a_{12}^{2314}=a_{12}^{2341}=a_{12}^{2413}=a_{12}^{2431}=$$
$$a_{12}^{3214}=a_{12}^{3241}=a_{12}^{3421}=a_{12}^{4213}=a_{12}^{4231}=a_{12}^{4321},$$

$$a_{13}^{1342}=a_{13}^{1324}=a_{13}^{1432}=a_{13}^{1423}=a_{13}^{1234}=a_{13}^{1243}=$$
$$a_{13}^{4132}=a_{13}^{4123}=a_{13}^{4213}=a_{13}^{2134}=a_{13}^{2143}=a_{13}^{2413},$$

$$a_{13}^{3142}=a_{13}^{3124}=a_{13}^{3412}=a_{13}^{3421}=a_{13}^{3214}=a_{13}^{3241}=$$
$$a_{13}^{4312}=a_{13}^{4321}=a_{13}^{4231}=a_{13}^{2314}=a_{13}^{2341}=a_{13}^{2431},$$

$$a_{14}^{1423}=a_{14}^{1432}=a_{14}^{1243}=a_{14}^{1234}=a_{14}^{1342}=a_{14}^{1324}=$$
$$a_{14}^{2143}=a_{14}^{2134}=a_{14}^{2314}=a_{14}^{3142}=a_{14}^{3124}=a_{14}^{3214},$$

$$a_{14}^{4123} = a_{14}^{4132} = a_{14}^{4213} = a_{14}^{4231} = a_{14}^{4312} = a_{14}^{4321} =$$
$$a_{14}^{2413} = a_{14}^{2431} = a_{14}^{2341} = a_{14}^{3412} = a_{14}^{3421} = a_{14}^{3241},$$

$$a_{23}^{2314} = a_{23}^{2341} = a_{23}^{2134} = a_{23}^{2143} = a_{23}^{2431} = a_{23}^{2413} =$$
$$a_{23}^{1234} = a_{23}^{1243} = a_{23}^{1423} = a_{23}^{4231} = a_{23}^{4213} = a_{23}^{4123},$$

$$a_{23}^{3214} = a_{23}^{3241} = a_{23}^{3124} = a_{23}^{3142} = a_{23}^{3421} = a_{23}^{3412} =$$
$$a_{23}^{1324} = a_{23}^{1342} = a_{23}^{1432} = a_{23}^{4321} = a_{23}^{4312} = a_{23}^{4132},$$

$$a_{24}^{2413} = a_{24}^{2431} = a_{24}^{2143} = a_{24}^{2134} = a_{24}^{2341} = a_{24}^{2314} =$$
$$a_{24}^{1243} = a_{24}^{1234} = a_{24}^{1324} = a_{24}^{3241} = a_{24}^{3214} = a_{24}^{3124},$$

$$a_{24}^{4213} = a_{24}^{4231} = a_{24}^{4123} = a_{24}^{4132} = a_{24}^{4321} = a_{24}^{4312} =$$
$$a_{24}^{1423} = a_{24}^{1432} = a_{24}^{1342} = a_{24}^{3421} = a_{24}^{3412} = a_{24}^{3142},$$

$$a_{34}^{3412} = a_{34}^{3421} = a_{34}^{3142} = a_{34}^{3124} = a_{34}^{3241} = a_{34}^{3214} =$$
$$a_{34}^{1342} = a_{34}^{1324} = a_{34}^{1234} = a_{34}^{2341} = a_{34}^{2314} = a_{34}^{2134},$$

$$a_{34}^{4312} = a_{34}^{4321} = a_{34}^{4132} = a_{34}^{4123} = a_{34}^{4231} = a_{34}^{4213} =$$
$$a_{34}^{1432} = a_{34}^{1423} = a_{34}^{1243} = a_{34}^{2431} = a_{34}^{2413} = a_{34}^{2143},$$

$$a_{123}^{1234} = a_{123}^{1243} = a_{123}^{1423} = a_{123}^{4123},$$
$$a_{123}^{1324} = a_{123}^{1342} = a_{123}^{1432} = a_{123}^{4132},$$
$$a_{123}^{2134} = a_{123}^{2143} = a_{123}^{2413} = a_{123}^{4213},$$
$$a_{123}^{2314} = a_{123}^{2341} = a_{123}^{2431} = a_{123}^{4231},$$
$$a_{123}^{3124} = a_{123}^{3142} = a_{123}^{3412} = a_{123}^{4312},$$
$$a_{123}^{3214} = a_{123}^{3241} = a_{123}^{3421} = a_{123}^{4321},$$
$$a_{124}^{1243} = a_{124}^{1234} = a_{124}^{1324} = a_{124}^{3124},$$
$$a_{124}^{1423} = a_{124}^{1432} = a_{124}^{1342} = a_{124}^{3142},$$
$$a_{124}^{2143} = a_{124}^{2134} = a_{124}^{2314} = a_{124}^{3214},$$
$$a_{124}^{2413} = a_{124}^{2431} = a_{124}^{2341} = a_{124}^{3241},$$
$$a_{124}^{4123} = a_{124}^{4132} = a_{124}^{4312} = a_{124}^{3412},$$
$$a_{124}^{4213} = a_{124}^{4231} = a_{124}^{4321} = a_{124}^{3421},$$
$$a_{134}^{1342} = a_{134}^{1324} = a_{134}^{1234} = a_{134}^{2134},$$
$$a_{134}^{1432} = a_{134}^{1423} = a_{134}^{1243} = a_{134}^{2143},$$
$$a_{134}^{3142} = a_{134}^{3124} = a_{134}^{3214} = a_{134}^{2314},$$
$$a_{134}^{3412} = a_{134}^{3421} = a_{134}^{3241} = a_{134}^{2341},$$
$$a_{134}^{4132} = a_{134}^{4123} = a_{134}^{4213} = a_{134}^{2413},$$

$$a_{134}^{4312} = a_{134}^{4321} = a_{134}^{4231} = a_{134}^{2431},$$
$$a_{234}^{2341} = a_{234}^{2314} = a_{234}^{2134} = a_{234}^{1234},$$
$$a_{234}^{2431} = a_{234}^{2413} = a_{234}^{2143} = a_{234}^{1243},$$
$$a_{234}^{3241} = a_{234}^{3214} = a_{234}^{3124} = a_{234}^{1324},$$
$$a_{234}^{3421} = a_{234}^{3412} = a_{234}^{3142} = a_{234}^{1342},$$
$$a_{234}^{4231} = a_{234}^{4213} = a_{234}^{4123} = a_{234}^{1423},$$
$$a_{234}^{4321} = a_{234}^{4312} = a_{234}^{4132} = a_{234}^{1432}.$$

Therefore, there exists a unique mapping $\theta : P \to \mathbb{R}$ such that

$$\theta^{1234} = \theta \circ f_{1234}, \theta^{1243} = \theta \circ f_{1243}, \theta^{1324} = \theta \circ f_{1324},$$
$$\theta^{1342} = \theta \circ f_{1342}, \theta^{1423} = \theta \circ f_{1423}, \theta^{1432} = \theta \circ f_{1432},$$
$$\theta^{2134} = \theta \circ f_{2134}, \theta^{2143} = \theta \circ f_{2143}, \theta^{2314} = \theta \circ f_{2314},$$
$$\theta^{2341} = \theta \circ f_{2341}, \theta^{2413} = \theta \circ f_{2413}, \theta^{2431} = \theta \circ f_{2431},$$
$$\theta^{3124} = \theta \circ f_{3124}, \theta^{3142} = \theta \circ f_{3142}, \theta^{3214} = \theta \circ f_{3214},$$
$$\theta^{3241} = \theta \circ f_{3241}, \theta^{3412} = \theta \circ f_{3412}, \theta^{3421} = \theta \circ f_{3421},$$
$$\theta^{4123} = \theta \circ f_{4123}, \theta^{4132} = \theta \circ f_{4132}, \theta^{4213} = \theta \circ f_{4213},$$
$$\theta^{4231} = \theta \circ f_{4231}, \theta^{4312} = \theta \circ f_{4312}, \theta^{4321} = \theta \circ f_{4321}.$$

The proof is now complete. □

Remark 3.2. For our convenience, we display the positions 5–30 of $f_{ijkl}(d_1, d_2, d_3, d_4)$ as tables as follows:

	$5/d_1d_2$	$6/d_1d_3$	$7/d_1d_4$	$8/d_2d_3$	$9/d_2d_4$	$10/d_3d_4$
1234	0	0	0	0	0	0
1243	0	0	0	0	0	d_3d_4
1324	0	0	0	d_2d_3	0	0
1342	0	0	0	d_2d_3	d_2d_4	0
1423	0	0	0	0	d_2d_4	d_3d_4
1432	0	0	0	d_2d_3	d_2d_4	d_3d_4
2134	d_1d_2	0	0	0	0	0
2143	d_1d_2	0	0	0	0	d_3d_4
2314	d_1d_2	d_1d_3	0	0	0	0
2341	d_1d_2	d_1d_3	d_1d_4	0	0	0
2413	d_1d_2	0	d_1d_4	0	0	d_3d_4
2431	d_1d_2	d_1d_3	d_1d_4	0	0	d_3d_4
3124	0	d_1d_3	0	d_2d_3	0	0
3142	0	d_1d_3	0	d_2d_3	d_2d_4	0
3214	d_1d_2	d_1d_3	0	d_2d_3	0	0
3241	d_1d_2	d_1d_3	d_1d_4	d_2d_3	0	0
3412	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0
3421	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	0

4123	0	0	d_1d_4	0	d_2d_4	d_3d_4
4132	0	0	d_1d_4	d_2d_3	d_2d_4	d_3d_4
4213	d_1d_2	0	d_1d_4	0	d_2d_4	d_3d_4
4231	d_1d_2	d_1d_3	d_1d_4	0	d_2d_4	d_3d_4
4312	0	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4
4321	d_1d_2	d_1d_3	d_1d_4	d_2d_3	d_2d_4	d_3d_4

	$11-15/d_1d_2d_3$	$16-20/d_1d_2d_4$	$21-25/d_1d_3d_4$	$26-30/d_2d_3d_4$
1234				
1243			21	26
1324	11			27
1342	11	16		28
1423		16	21	29
1432	11	16	21	30
2134	12	17		
2143	12	17	21	26
2314	13	17	22	
2341	13	18	23	
2413	12	18	24	26
2431	13	18	25	26
3124	14		22	27
3142	14	16	22	28
3214	15	17	22	27
3241	15	18	23	27
3412	14	19	23	28
3421	15	20	23	28
4123		19	24	29
4132	11	19	24	30
4213	12	20	24	29
4231	13	20	25	29
4312	14	19	25	30
4321	15	20	25	30

Corollary 3.3. *Let M be a microlinear space with mappings*

$$\gamma_{1234}:Q^{1234}\rightarrow M,\gamma_{1243}:Q^{1243}\rightarrow M,\gamma_{1324}:Q^{1324}\rightarrow M,$$
$$\gamma_{1342}:Q^{1342}\rightarrow M,\gamma_{1423}:Q^{1423}\rightarrow M,\gamma_{1432}:Q^{1432}\rightarrow M,$$
$$\gamma_{2134}:Q^{2134}\rightarrow M,\gamma_{2143}:Q^{2143}\rightarrow M,\gamma_{2314}:Q^{2314}\rightarrow M,$$
$$\gamma_{2341}:Q^{2341}\rightarrow M,\gamma_{2413}:Q^{2413}\rightarrow M,\gamma_{2431}:Q^{2431}\rightarrow M,$$
$$\gamma_{3124}:Q^{3124}\rightarrow M,\gamma_{3142}:Q^{3142}\rightarrow M,\gamma_{3214}:Q^{3214}\rightarrow M,$$
$$\gamma_{3241}:Q^{3241}\rightarrow M,\gamma_{3412}:Q^{3412}\rightarrow M,\gamma_{3421}:Q^{3421}\rightarrow M,$$
$$\gamma_{4123}:Q^{4123}\rightarrow M,\gamma_{4132}:Q^{4132}\rightarrow M,\gamma_{4213}:Q^{4213}\rightarrow M,$$
$$\gamma_{4231}:Q^{4231}\rightarrow M,\gamma_{4312}:Q^{4312}\rightarrow M,\gamma_{4321}:Q^{4321}\rightarrow M$$

abiding by

$$\begin{aligned}
&\gamma_{1234}\circ g_{12}^{1234,1243}=\gamma_{1243}\circ h_{12}^{1234,1243},\gamma_{1342}\circ g_{12}^{1342,1432}=\gamma_{1432}\circ h_{12}^{1342,1432},\\
&\gamma_{2341}\circ g_{12}^{2341,2431}=\gamma_{2431}\circ h_{12}^{2341,2431},\gamma_{3421}\circ g_{12}^{3421,4321}=\gamma_{4321}\circ h_{12}^{3421,4321},\\
&\gamma_{2134}\circ g_{12}^{2134,2143}=\gamma_{2143}\circ h_{12}^{2134,2143},\gamma_{3412}\circ g_{12}^{3412,4312}=\gamma_{4312}\circ h_{12}^{3412,4312},\\
&\gamma_{1324}\circ g_{13}^{1324,1342}=\gamma_{1342}\circ h_{13}^{1324,1342},\gamma_{1243}\circ g_{13}^{1243,1423}=\gamma_{1423}\circ h_{13}^{1243,1423},\\
&\gamma_{3241}\circ g_{13}^{3241,3421}=\gamma_{3421}\circ h_{13}^{3241,3421},\gamma_{2431}\circ g_{13}^{2431,4231}=\gamma_{4231}\circ h_{13}^{2431,4231},\\
&\gamma_{3124}\circ g_{13}^{3124,3142}=\gamma_{3142}\circ h_{13}^{3124,3142},\gamma_{2413}\circ g_{13}^{2413,4213}=\gamma_{4213}\circ h_{13}^{2413,4213},\\
&\gamma_{1423}\circ g_{14}^{1423,1432}=\gamma_{1432}\circ h_{14}^{1423,1432},\gamma_{1234}\circ g_{14}^{1234,1324}=\gamma_{1324}\circ h_{14}^{1234,1324},\\
&\gamma_{4231}\circ g_{14}^{4231,4321}=\gamma_{4321}\circ h_{14}^{4231,4321},\gamma_{2341}\circ g_{14}^{2341,3241}=\gamma_{3241}\circ h_{14}^{2341,3241},\\
&\gamma_{4123}\circ g_{14}^{4123,4132}=\gamma_{4132}\circ h_{14}^{4123,4132},\gamma_{2314}\circ g_{14}^{2314,3214}=\gamma_{3214}\circ h_{14}^{2314,3214},\\
&\gamma_{2314}\circ g_{23}^{2314,2341}=\gamma_{2341}\circ h_{23}^{2314,2341},\gamma_{2143}\circ g_{23}^{2143,2413}=\gamma_{2413}\circ h_{23}^{2143,2413},\\
&\gamma_{3142}\circ g_{23}^{3142,3412}=\gamma_{3412}\circ h_{23}^{3142,3412},\gamma_{1432}\circ g_{23}^{1432,4132}=\gamma_{4132}\circ h_{23}^{1432,4132},\\
&\gamma_{3214}\circ g_{23}^{3214,3241}=\gamma_{3241}\circ h_{23}^{3214,3241},\gamma_{1423}\circ g_{23}^{1423,4123}=\gamma_{4123}\circ h_{23}^{1423,4123},\\
&\gamma_{2413}\circ g_{24}^{2413,2431}=\gamma_{2431}\circ h_{24}^{2413,2431},\gamma_{2134}\circ g_{24}^{2134,2314}=\gamma_{2314}\circ h_{24}^{2134,2314},\\
&\gamma_{4132}\circ g_{24}^{4132,4312}=\gamma_{4312}\circ h_{24}^{4132,4312},\gamma_{1342}\circ g_{24}^{1342,3142}=\gamma_{3142}\circ h_{24}^{1342,3142},\\
&\gamma_{4213}\circ g_{24}^{4213,4231}=\gamma_{4231}\circ h_{24}^{4213,4231},\gamma_{1324}\circ g_{24}^{1324,3124}=\gamma_{3124}\circ h_{24}^{1324,3124},\\
&\gamma_{3412}\circ g_{34}^{3412,3421}=\gamma_{3421}\circ h_{34}^{3412,3421},\gamma_{3124}\circ g_{34}^{3124,3214}=\gamma_{3214}\circ h_{34}^{3124,3214},\\
&\gamma_{4123}\circ g_{34}^{4123,4213}=\gamma_{4213}\circ h_{34}^{4123,4213},\gamma_{1243}\circ g_{34}^{1243,2143}=\gamma_{2143}\circ h_{34}^{1243,2143},\\
&\gamma_{4312}\circ g_{34}^{4312,4321}=\gamma_{4321}\circ h_{34}^{4312,4321},\gamma_{1234}\circ g_{34}^{1234,2134}=\gamma_{2134}\circ h_{34}^{1234,2134}.
\end{aligned}$$

Then, there exists a unique mapping

$$\mathfrak{m}:P\rightarrow M$$

such that

$$\begin{aligned}
&\mathfrak{m}\circ f_{1234}=\gamma_{1234},\ \mathfrak{m}\circ f_{1243}=\gamma_{1243},\ \mathfrak{m}\circ f_{1324}=\gamma_{1324},\ \mathfrak{m}\circ f_{1342}=\gamma_{1342},\\
&\mathfrak{m}\circ f_{1423}=\gamma_{1423},\ \mathfrak{m}\circ f_{1432}=\gamma_{1432},\ \mathfrak{m}\circ f_{2134}=\gamma_{2134},\ \mathfrak{m}\circ f_{2143}=\gamma_{2143},\\
&\mathfrak{m}\circ f_{2314}=\gamma_{2314},\ \mathfrak{m}\circ f_{2341}=\gamma_{2341},\ \mathfrak{m}\circ f_{2413}=\gamma_{2413},\ \mathfrak{m}\circ f_{2431}=\gamma_{2431},\\
&\mathfrak{m}\circ f_{3124}=\gamma_{3124},\ \mathfrak{m}\circ f_{3142}=\gamma_{3142},\ \mathfrak{m}\circ f_{3214}=\gamma_{3214},\ \mathfrak{m}\circ f_{3241}=\gamma_{3241},\\
&\mathfrak{m}\circ f_{3412}=\gamma_{3412},\ \mathfrak{m}\circ f_{3421}=\gamma_{3421},\ \mathfrak{m}\circ f_{4123}=\gamma_{4123},\ \mathfrak{m}\circ f_{4132}=\gamma_{4132},\\
&\mathfrak{m}\circ f_{4213}=\gamma_{4213},\ \mathfrak{m}\circ f_{4231}=\gamma_{4231},\ \mathfrak{m}\circ f_{4312}=\gamma_{4312},\ \mathfrak{m}\circ f_{4321}=\gamma_{4321}.
\end{aligned}$$

Theorem 3.4. *Let M be a microlinear space. Let*

$$\begin{array}{l}\gamma_{1234},\gamma_{1243},\gamma_{1324},\gamma_{1342},\gamma_{1423},\gamma_{1432},\gamma_{2134},\gamma_{2143},\\ \gamma_{2314},\gamma_{2341},\gamma_{2413},\gamma_{2431},\gamma_{3124},\gamma_{3142},\gamma_{3214},\gamma_{3241},\\ \gamma_{3124},\gamma_{3142},\gamma_{4123},\gamma_{4132},\gamma_{4213},\gamma_{4231},\gamma_{4312},\gamma_{4321}\end{array}:D^4\rightarrow M$$

with

$$\gamma_{1234}\mid D^4\{(3,4)\}=\gamma_{1243}\mid D^4\{(3,4)\},\gamma_{1342}\mid D^4\{(3,4)\}=\gamma_{1432}\mid D^4\{(3,4)\},$$

$$\gamma_{2341} \mid D^4\{(3,4)\} = \gamma_{2431} \mid D^4\{(3,4)\}, \gamma_{3421} \mid D^4\{(3,4)\} = \gamma_{4321} \mid D^4\{(3,4)\},$$
$$\gamma_{2143} \mid D^4\{(3,4)\} = \gamma_{2134} \mid D^4\{(3,4)\}, \gamma_{4312} \mid D^4\{(3,4)\} = \gamma_{3412} \mid D^4\{(3,4)\},$$
$$\gamma_{1342} \mid D^4\{(2,4)\} = \gamma_{1324} \mid D^4\{(2,4)\}, \gamma_{1423} \mid D^4\{(2,4)\} = \gamma_{1243} \mid D^4\{(2,4)\},$$
$$\gamma_{3421} \mid D^4\{(2,4)\} = \gamma_{3241} \mid D^4\{(2,4)\}, \gamma_{4231} \mid D^4\{(2,4)\} = \gamma_{2431} \mid D^4\{(2,4)\},$$
$$\gamma_{3124} \mid D^4\{(2,4)\} = \gamma_{3142} \mid D^4\{(2,4)\}, \gamma_{2413} \mid D^4\{(2,4)\} = \gamma_{4213} \mid D^4\{(2,4)\},$$
$$\gamma_{1423} \mid D^4\{(2,3)\} = \gamma_{1432} \mid D^4\{(2,3)\}, \gamma_{1234} \mid D^4\{(2,3)\} = \gamma_{1324} \mid D^4\{(2,3)\},$$
$$\gamma_{4231} \mid D^4\{(2,3)\} = \gamma_{4321} \mid D^4\{(2,3)\}, \gamma_{2341} \mid D^4\{(2,3)\} = \gamma_{3241} \mid D^4\{(2,3)\},$$
$$\gamma_{4132} \mid D^4\{(2,3)\} = \gamma_{4123} \mid D^4\{(2,3)\}, \gamma_{3214} \mid D^4\{(2,3)\} = \gamma_{2314} \mid D^4\{(2,3)\},$$
$$\gamma_{2314} \mid D^4\{(1,4)\} = \gamma_{2341} \mid D^4\{(1,4)\}, \gamma_{2143} \mid D^4\{(1,4)\} = \gamma_{2413} \mid D^4\{(1,4)\},$$
$$\gamma_{3142} \mid D^4\{(1,4)\} = \gamma_{3412} \mid D^4\{(1,4)\}, \gamma_{1432} \mid D^4\{(1,4)\} = \gamma_{4132} \mid D^4\{(1,4)\},$$
$$\gamma_{3241} \mid D^4\{(1,4)\} = \gamma_{3214} \mid D^4\{(1,4)\}, \gamma_{4123} \mid D^4\{(1,4)\} = \gamma_{1423} \mid D^4\{(1,4)\},$$
$$\gamma_{2431} \mid D^4\{(1,3)\} = \gamma_{2413} \mid D^4\{(1,3)\}, \gamma_{2314} \mid D^4\{(1,3)\} = \gamma_{2134} \mid D^4\{(1,3)\},$$
$$\gamma_{4312} \mid D^4\{(1,3)\} = \gamma_{4132} \mid D^4\{(1,3)\}, \gamma_{3142} \mid D^4\{(1,3)\} = \gamma_{1342} \mid D^4\{(1,3)\},$$
$$\gamma_{4213} \mid D^4\{(1,3)\} = \gamma_{4231} \mid D^4\{(1,3)\}, \gamma_{1324} \mid D^4\{(1,3)\} = \gamma_{3124} \mid D^4\{(1,3)\},$$
$$\gamma_{3412} \mid D^4\{(1,2)\} = \gamma_{3421} \mid D^4\{(1,2)\}, \gamma_{3124} \mid D^4\{(1,2)\} = \gamma_{3214} \mid D^4\{(1,2)\},$$
$$\gamma_{4123} \mid D^4\{(1,2)\} = \gamma_{4213} \mid D^4\{(1,2)\}, \gamma_{1243} \mid D^4\{(1,2)\} = \gamma_{2143} \mid D^4\{(1,2)\},$$
$$\gamma_{4321} \mid D^4\{(1,2)\} = \gamma_{4312} \mid D^4\{(1,2)\}, \gamma_{2134} \mid D^4\{(1,2)\} = \gamma_{1234} \mid D^4\{(1,2)\}.$$

Then, we have

$$\left[\left[\left[\gamma_{1234} \underset{12}{\dot{-}} \gamma_{1243}\right] \underset{1}{\dot{-}} \left[\gamma_{1342} \underset{12}{\dot{-}} \gamma_{1432}\right]\right] \dot{-} \left[\left[\gamma_{2341} \underset{12}{\dot{-}} \gamma_{2431}\right] \underset{1}{\dot{-}} \left[\gamma_{3421} \underset{12}{\dot{-}} \gamma_{4321}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{1342} \underset{13}{\dot{-}} \gamma_{1324}\right] \underset{1}{\dot{-}} \left[\gamma_{1423} \underset{13}{\dot{-}} \gamma_{1243}\right]\right] \dot{-} \left[\left[\gamma_{3421} \underset{13}{\dot{-}} \gamma_{3241}\right] \underset{1}{\dot{-}} \left[\gamma_{4231} \underset{13}{\dot{-}} \gamma_{2431}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right] \underset{1}{\dot{-}} \left[\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right]\right] \dot{-} \left[\left[\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right] \underset{1}{\dot{-}} \left[\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{2143} \underset{21}{\dot{-}} \gamma_{2134}\right] \underset{1}{\dot{-}} \left[\gamma_{2431} \underset{21}{\dot{-}} \gamma_{2341}\right]\right] \dot{-} \left[\left[\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342}\right] \underset{1}{\dot{-}} \left[\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{2314} \underset{23}{\dot{-}} \gamma_{2341}\right] \underset{1}{\dot{-}} \left[\gamma_{2143} \underset{23}{\dot{-}} \gamma_{2413}\right]\right] \dot{-} \left[\left[\gamma_{3142} \underset{23}{\dot{-}} \gamma_{3412}\right] \underset{1}{\dot{-}} \left[\gamma_{1432} \underset{23}{\dot{-}} \gamma_{4132}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{2431} \underset{24}{\dot{-}} \gamma_{2413}\right] \underset{1}{\dot{-}} \left[\gamma_{2314} \underset{24}{\dot{-}} \gamma_{2134}\right]\right] \dot{-} \left[\left[\gamma_{4312} \underset{24}{\dot{-}} \gamma_{4132}\right] \underset{1}{\dot{-}} \left[\gamma_{3142} \underset{24}{\dot{-}} \gamma_{1342}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{3124} \underset{31}{\dot{-}} \gamma_{3142}\right] \underset{1}{\dot{-}} \left[\gamma_{3241} \underset{31}{\dot{-}} \gamma_{3421}\right]\right] \dot{-} \left[\left[\gamma_{1243} \underset{31}{\dot{-}} \gamma_{1423}\right] \underset{1}{\dot{-}} \left[\gamma_{2413} \underset{31}{\dot{-}} \gamma_{4213}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{3241} \underset{32}{\dot{-}} \gamma_{3214}\right] \underset{1}{\dot{-}} \left[\gamma_{3412} \underset{32}{\dot{-}} \gamma_{3142}\right]\right] \dot{-} \left[\left[\gamma_{2413} \underset{32}{\dot{-}} \gamma_{2143}\right] \underset{1}{\dot{-}} \left[\gamma_{4123} \underset{32}{\dot{-}} \gamma_{1423}\right]\right]\right] +$$
$$\left[\left[\left[\gamma_{3412} \underset{34}{\dot{-}} \gamma_{3421}\right] \underset{1}{\dot{-}} \left[\gamma_{3124} \underset{34}{\dot{-}} \gamma_{3214}\right]\right] \dot{-} \left[\left[\gamma_{4123} \underset{34}{\dot{-}} \gamma_{4213}\right] \underset{1}{\dot{-}} \left[\gamma_{1243} \underset{34}{\dot{-}} \gamma_{2143}\right]\right]\right] +$$

$$\left[\left[\left[\gamma_{4132}\underset{41}{\dot{-}}\gamma_{4123}\right]\underset{1}{\dot{-}}\left[\gamma_{4321}\underset{41}{\dot{-}}\gamma_{4231}\right]\right]\dot{-}\left[\left[\gamma_{1324}\underset{41}{\dot{-}}\gamma_{1234}\right]\underset{1}{\dot{-}}\left[\gamma_{3214}\underset{41}{\dot{-}}\gamma_{2314}\right]\right]\right]+$$
$$\left[\left[\left[\gamma_{4213}\underset{42}{\dot{-}}\gamma_{4231}\right]\underset{1}{\dot{-}}\left[\gamma_{4132}\underset{42}{\dot{-}}\gamma_{4312}\right]\right]\dot{-}\left[\left[\gamma_{2134}\underset{42}{\dot{-}}\gamma_{2314}\right]\underset{1}{\dot{-}}\left[\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right]\right]\right]+$$
$$\left[\left[\left[\gamma_{4321}\underset{43}{\dot{-}}\gamma_{4312}\right]\underset{1}{\dot{-}}\left[\gamma_{4213}\underset{43}{\dot{-}}\gamma_{4123}\right]\right]\dot{-}\left[\left[\gamma_{3214}\underset{43}{\dot{-}}\gamma_{3124}\right]\underset{1}{\dot{-}}\left[\gamma_{2134}\underset{43}{\dot{-}}\gamma_{1234}\right]\right]\right]$$
$$=0. \tag{3.1}$$

Proof. The proof is divided into thirteen steps.

(1) Since

$$\gamma_{1234}(d_1,d_2,d_3,d_4)=\mathfrak{m}\left(d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{53}{0}\right)$$

and

$$\gamma_{1243}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0},\\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{1234}\underset{12}{\dot{-}}\gamma_{1243}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,\underset{3}{0},...,\underset{9}{0},\underset{10}{-d_3},\underset{11}{0},...,\underset{20}{0},-d_1d_3,\underset{22}{0},...,\underset{25}{0},-d_2d_3,\underset{27}{0},...,\underset{30}{0},\\ -d_1d_2d_3,\underset{32}{0},...,\underset{53}{0}\end{array}\right). \tag{3.2}$$

For we have

$$\mathfrak{n}^4_{(\gamma_{1234},\gamma_{1243})}(d_1,d_2,d_3,d_4,d_5)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4-d_5},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4-d_1d_5,\underset{22}{0},...,\underset{25}{0},\\ d_2d_3d_4-d_2d_5,\underset{27}{0},...,\underset{30}{0},d_1d_2d_3d_4-d_1d_2d_5,\underset{32}{0},...,\underset{53}{0}\end{array}\right).$$

Since

$$\gamma_{1342}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0},\\ d_2d_3d_4,0,0,0,0,d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{1432}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c}d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0},\\ d_1d_3d_4,\underset{22}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{34}{0},d_1d_2d_3d_4,\underset{36}{0},...,\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{1342}\underset{12}{\dot{-}}\gamma_{1432}\right)(d_1,d_2,d_3)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, \underset{3}{0}, ..., \underset{9}{0}, -d_3, \underset{11}{0}, ..., \underset{20}{0}, -d_1d_3, \underset{22}{0}, ..., \underset{27}{0}, d_2d_3, \underset{29}{0}, \\ -d_2d_3, \underset{31}{0}, \underset{32}{0}, d_1d_2d_3, \underset{34}{0}, -d_1d_2d_3, \underset{36}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.3}$$

For we have

$$\mathfrak{n}^4_{(\gamma_{1342}, \gamma_{1432})}(d_1, d_2, d_3, d_4, d_5)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{9}{0}, \underset{10}{d_3d_4 - d_5}, \underset{11}{0}, ..., \underset{20}{0}, d_1d_3d_4 - d_1d_5, \underset{22}{0}, ..., \underset{27}{0}, \\ d_2d_5, \underset{29}{0}, d_2d_3d_4 - d_2d_5, \underset{31}{0}, \underset{32}{0}, d_1d_2d_5, \underset{34}{0}, d_1d_2d_3d_4 - d_1d_2d_5, \underset{36}{0}, ..., \underset{53}{0} \end{pmatrix}.$$

For the sake of the completeness of our proof, we have provided the reason why (3.2) and (3.3) are derivable, but we will omit such reasoning from now on. (3.2) and (3.3) imply that

$$\left(\left(\gamma_{1234} \underset{12}{\dot{-}} \gamma_{1243}\right) \underset{1}{\dot{-}} \left(\gamma_{1342} \underset{12}{\dot{-}} \gamma_{1432}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(d_1, \underset{2}{0}, ..., \underset{25}{0}, -d_2, \underset{27}{0}, -d_2, \underset{29}{0}, d_2, -d_1d_2, \underset{32}{0}, -d_1d_2, \underset{34}{0}, d_1d_2, \underset{36}{0}, ..., \underset{53}{0}\right). \tag{3.4}$$

Since

$$\gamma_{2341}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, ..., \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{38}{0}, d_1d_2d_3d_4, \underset{40}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2431}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{24}{0}, d_1d_3d_4, d_2d_3d_4, \underset{27}{0}, ..., \underset{40}{0}, d_1d_2d_3d_4, \underset{42}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{2341} \underset{12}{\dot{-}} \gamma_{2431}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, \underset{3}{0}, ..., \underset{9}{0}, -d_3, \underset{11}{0}, ..., \underset{22}{0}, d_1d_3, 0, -d_1d_3, -d_2d_3, \\ \underset{27}{0}, ..., \underset{38}{0}, d_1d_2d_3, 0, -d_1d_2d_3, \underset{42}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.5}$$

Since

$$\gamma_{3421}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{46}{0}, d_1d_2d_3d_4, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4321}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{52}{0}, d_1d_2d_3d_4 \end{pmatrix},$$

we have

$$\begin{aligned}&\left(\gamma_{3421} \underset{12}{\overset{\cdot}{-}} \gamma_{4321}\right)(d_1, d_2, d_3)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, \underset{3}{0}, ..., \underset{9}{0}, -d_3, \underset{11}{0}, ..., \underset{22}{0}, d_1 d_3, \underset{24}{0}, -d_1 d_3, \underset{26}{0}, \underset{27}{0}, d_2 d_3, \underset{29}{0}, -d_2 d_3, \\ \underset{31}{0}, ..., \underset{46}{0}, d_1 d_2 d_3, \underset{48}{0}, ..., \underset{52}{0}, -d_1 d_2 d_3 \end{array}\right).\end{aligned} \tag{3.6}$$

(3.5) and (3.6) imply that

$$\begin{aligned}&\left(\left(\gamma_{2341} \underset{12}{\overset{\cdot}{-}} \gamma_{2431}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{3421} \underset{12}{\overset{\cdot}{-}} \gamma_{4321}\right)\right)(d_1, d_2)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, ..., \underset{25}{0}, -d_2, \underset{27}{0}, -d_2, \underset{29}{0}, d_2, \underset{31}{0}, ..., \underset{38}{0}, d_1 d_2, \underset{40}{0}, \\ -d_1 d_2, \underset{42}{0}, ..., \underset{46}{0}, -d_1 d_2, \underset{48}{0}, ..., \underset{52}{0}, d_1 d_2 \end{array}\right).\end{aligned} \tag{3.7}$$

(3.4) and (3.7) imply that

$$\begin{aligned}&\left(\begin{array}{c} \left(\left(\gamma_{1234} \underset{12}{\overset{\cdot}{-}} \gamma_{1243}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{1342} \underset{12}{\overset{\cdot}{-}} \gamma_{1432}\right)\right) \overset{\cdot}{-} \\ \left(\left(\gamma_{2341} \underset{12}{\overset{\cdot}{-}} \gamma_{2431}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{3421} \underset{12}{\overset{\cdot}{-}} \gamma_{4321}\right)\right) \end{array}\right)(d)\\&= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{30}{0}, -d, \underset{32}{0}, -d, \underset{34}{0}, d, \underset{36}{0}, ..., \underset{38}{0}, -d, 0, d, \underset{42}{0}, ..., \underset{46}{0}, d, \underset{48}{0}, ..., \underset{52}{0}, -d\right).\end{aligned} \tag{3.8}$$

(2) Since

$$\begin{aligned}&\gamma_{1342}(d_1, d_2, d_3, d_4)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2 d_3, d_2 d_4, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, ..., \underset{27}{0}, \\ d_2 d_3 d_4, 0, 0, 0, 0, d_1 d_2 d_3 d_4, \underset{34}{0}, ..., \underset{53}{0} \end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{1324}(d_1, d_2, d_3, d_4)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2 d_3, \underset{9}{0}, \underset{10}{0}, d_1 d_2 d_3, \underset{12}{0}, ..., \underset{26}{0}, d_2 d_3 d_4, \underset{28}{0}, ..., \underset{31}{0}, \\ d_1 d_2 d_3 d_4, \underset{33}{0}, ..., \underset{53}{0} \end{array}\right),\end{aligned}$$

we have

$$\begin{aligned}&\left(\gamma_{1342} \underset{13}{\overset{\cdot}{-}} \gamma_{1324}\right)(d_1, d_2, d_3)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, 0, d_2, \underset{4}{0}, ..., \underset{8}{0}, d_3, \underset{10}{0}, ..., \underset{15}{0}, d_1 d_3, \underset{17}{0}, ..., \underset{26}{0}, -d_2 d_3, d_2 d_3, \underset{29}{0}, ..., \underset{31}{0}, \\ -d_1 d_2 d_3, d_1 d_2 d_3, \underset{34}{0}, ..., \underset{53}{0} \end{array}\right).\end{aligned} \tag{3.9}$$

Since

$$\begin{aligned}&\gamma_{1423}(d_1, d_2, d_3, d_4)\\&= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, ..., \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, ..., \underset{20}{0}, d_1 d_3 d_4, \\ \underset{22}{0}, ..., \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, ..., \underset{33}{0}, d_1 d_2 d_3 d_4, \underset{35}{0}, ..., \underset{53}{0} \end{array}\right)\end{aligned}$$

and

$$\gamma_{1243}(d_1,d_2,d_3,d_4) = \begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{9}{0},\underset{10}{d_3d_4},\underset{11}{0},...,\underset{20}{0},d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{30}{0}, \\ d_1d_2d_3d_4,\underset{32}{0},...,\underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{1423}\underset{13}{\dot{-}}\gamma_{1243}\right)(d_1,d_2,d_3) = \mathfrak{m}\begin{pmatrix} d_1,0,d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{15}{0},d_1d_3,\underset{17}{0},...,\underset{25}{0},-d_2d_3,\underset{27}{0},\underset{28}{0},d_2d_3,\underset{30}{0}, \\ -d_1d_2d_3,\underset{32}{0},\underset{33}{0},d_1d_2d_3,\underset{35}{0},...,\underset{53}{0} \end{pmatrix}. \tag{3.10}$$

(3.9) and (3.10) imply that

$$\left(\left(\gamma_{1324}\underset{13}{\dot{-}}\gamma_{1342}\right)\underset{1}{\dot{-}}\left(\gamma_{1243}\underset{13}{\dot{-}}\gamma_{1423}\right)\right)(d_1,d_2) = \mathfrak{m}\left(d_1,\underset{2}{0},...,\underset{25}{0},d_2,-d_2,d_2,-d_2,\underset{30}{0},d_1d_2,-d_1d_2,d_1d_2,-d_1d_2,\underset{35}{0},...,\underset{53}{0}\right). \tag{3.11}$$

Since

$$\gamma_{3421}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0}, \\ d_1d_2d_4,0,0,d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{46}{0},d_1d_2d_3d_4,\underset{48}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3241}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4, \\ \underset{19}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{3421}\underset{13}{\dot{-}}\gamma_{3241}\right)(d_1,d_2,d_3) = \mathfrak{m}\begin{pmatrix} d_1,0,d_2,\underset{4}{0},...,\underset{8}{0},d_3,\underset{10}{0},...,\underset{17}{0},-d_1d_3,\underset{19}{0},d_1d_3,\underset{21}{0},...,\underset{26}{0},-d_2d_3, \\ d_2d_3,\underset{29}{0},...,\underset{44}{0},-d_1d_2d_3,\underset{46}{0},d_1d_2d_3,\underset{48}{0},...,\underset{53}{0} \end{pmatrix}. \tag{3.12}$$

Since

$$\gamma_{4231}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,0,\underset{53}{0} \end{pmatrix}$$

and

$\gamma_{2431}(d_1,d_2,d_3,d_4)$

$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, 0, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{24}{0}, d_1d_3d_4, d_2d_3d_4, \underset{27}{0}, ..., \underset{40}{0}, d_1d_2d_3d_4, \underset{42}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{4231} \underset{13}{\dot{-}} \gamma_{2431}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, d_2, \underset{4}{0}, ..., \underset{8}{0}, d_3, \underset{10}{0}, ..., \underset{17}{0}, -d_1d_3, \underset{19}{0}, d_1d_3, \underset{21}{0}, ..., \underset{25}{0}, -d_2d_3, \\ \underset{27}{0}, \underset{28}{0}, d_2d_3, \underset{30}{0}, ..., \underset{40}{0}, -d_1d_2d_3, \underset{42}{0}, ..., \underset{50}{0}, d_1d_2d_3, \underset{52}{0}, \underset{53}{0} \end{array}\right). \quad (3.13)$$

(3.12) and (3.13) imply that

$$\left(\left(\gamma_{3421} \underset{13}{\dot{-}} \gamma_{3241}\right) \underset{1}{\dot{-}} \left(\gamma_{4231} \underset{13}{\dot{-}} \gamma_{2431}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, ..., \underset{25}{0}, d_2, -d_2, d_2, -d_2, \underset{30}{0}, ..., \underset{40}{0}, d_1d_2, \underset{42}{0}, ..., \underset{44}{0}, \\ -d_1d_2, \underset{46}{0}, d_1d_2, \underset{48}{0}, ..., \underset{50}{0}, -d_1d_2, \underset{52}{0}, \underset{53}{0} \end{array}\right). \quad (3.14)$$

(3.11) and (3.14) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{1324} \underset{13}{\dot{-}} \gamma_{1342}\right) \underset{1}{\dot{-}} \left(\gamma_{1243} \underset{13}{\dot{-}} \gamma_{1423}\right)\right) \dot{-} \\ \left(\left(\gamma_{3241} \underset{13}{\dot{-}} \gamma_{3421}\right) \underset{1}{\dot{-}} \left(\gamma_{2431} \underset{13}{\dot{-}} \gamma_{4231}\right)\right) \end{array}\right)(d)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{30}{0}, d, -d, d, -d, \underset{35}{0}, ..., \underset{40}{0}, -d, \underset{42}{0}, ... \underset{44}{0}, d, \underset{46}{0}, -d, \underset{48}{0}, ..., \underset{50}{0}, d, \underset{52}{0}, \underset{53}{0}\right). \quad (3.15)$$

(3) Since

$$\gamma_{1423}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, d_1d_3d_4, \\ \underset{22}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{33}{0}, d_1d_2d_3d_4, \underset{35}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{1432}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, \underset{2}{0}, \underset{3}{0}, d_2, \underset{5}{0}, ..., \underset{7}{0}, -d_3, \underset{9}{0}, \underset{10}{0}, -d_1d_3, \underset{12}{0}, ..., \underset{28}{0}, d_2d_3, -d_2d_3, \\ \underset{31}{0}, ..., \underset{33}{0}, d_1d_2d_3, -d_1d_2d_3, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right). \quad (3.16)$$

Since

$$\gamma_{1234}(d_1, d_2, d_3, d_4) = \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{53}{0}\right)$$

and

$$\gamma_{1324}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,\underset{9}{0},\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{30}{0},\\ 0,d_1d_2d_3d_4,\underset{33}{0},...,\underset{53}{0}\end{pmatrix},$$

we have

$$\left(\gamma_{1234}\underset{14}{\overset{\cdot}{-}}\gamma_{1324}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,\underset{2}{0},\underset{3}{0},d_2,\underset{5}{0},...,\underset{7}{0},-d_3,\underset{9}{0},\underset{10}{0},-d_1d_3,\underset{12}{0},...,\underset{26}{0},-d_2d_3,\\ \underset{28}{0},...,\underset{31}{0},-d_1d_2d_3,\underset{33}{0},...,\underset{53}{0}\end{pmatrix}. \tag{3.17}$$

(3.16) and (3.17) imply that

$$\left(\left(\gamma_{1423}\underset{14}{\overset{\cdot}{-}}\gamma_{1432}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{1234}\underset{14}{\overset{\cdot}{-}}\gamma_{1324}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(d_1,\underset{2}{0},...,\underset{26}{0},d_2,\underset{28}{0},d_2,-d_2,\underset{31}{0},d_1d_2,\underset{33}{0},d_1d_2,-d_1d_2,\underset{36}{0},...,\underset{53}{0}\right). \tag{3.18}$$

Since

$$\gamma_{4231}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0},\\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,\underset{52}{0},\underset{53}{0}\end{pmatrix}$$

and

$$\gamma_{4321}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0},\\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{52}{0},d_1d_2d_3d_4\end{pmatrix},$$

we have

$$\left(\gamma_{4231}\underset{14}{\overset{\cdot}{-}}\gamma_{4321}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,\underset{2}{0},\underset{3}{0},d_2,\underset{5}{0},...,\underset{7}{0},-d_3,\underset{9}{0},...,\underset{12}{0},d_1d_3,\underset{14}{0},-d_1d_3,\underset{16}{0},...,\underset{28}{0},\\ d_2d_3,-d_2d_3,\underset{31}{0},...,\underset{50}{0},d_1d_2d_3,\underset{52}{0},-d_1d_2d_3\end{pmatrix}. \tag{3.19}$$

Since

$$\gamma_{2341}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0},\\ d_1d_3d_4,\underset{24}{0},...,\underset{38}{0},d_1d_2d_3d_4,\underset{40}{0},...,\underset{53}{0}\end{pmatrix}$$

and

$$\gamma_{3241}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0}\end{pmatrix},$$

we have

$$\begin{aligned}&\left(\gamma_{2341}\underset{14}{\overset{\cdot}{-}}\gamma_{3241}\right)(d_1,d_2,d_3)\\&=\mathfrak{m}\left(\begin{array}{c} d_1,\underset{2}{0},\underset{3}{0},d_2,\underset{5}{0},...,\underset{7}{0},-d_3,\underset{9}{0},...,\underset{12}{0},d_1d_3,\underset{14}{0},-d_1d_3,\underset{16}{0},...,\underset{26}{0},-d_2d_3,\\ \underset{28}{0},...,\underset{38}{0},d_1d_2d_3,\underset{40}{0},...,\underset{44}{0},-d_1d_2d_3,\underset{46}{0},...,\underset{53}{0}\end{array}\right).\end{aligned} \tag{3.20}$$

(3.19) and (3.20) imply that

$$\begin{aligned}&\left(\left(\gamma_{4231}\underset{14}{\overset{\cdot}{-}}\gamma_{4321}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2341}\underset{14}{\overset{\cdot}{-}}\gamma_{3241}\right)\right)(d_1,d_2)\\&=\mathfrak{m}\left(\begin{array}{c} d_1,\underset{2}{0},...,\underset{26}{0},d_2,\underset{28}{0},d_2,-d_2,\underset{31}{0},...,\underset{38}{0},-d_1d_2,\\ \underset{40}{0},...,\underset{44}{0},d_1d_2,\underset{46}{0},...,\underset{50}{0},d_1d_2,\underset{52}{0},-d_1d_2\end{array}\right).\end{aligned} \tag{3.21}$$

(3.18) and (3.21) imply that

$$\begin{aligned}&\left(\begin{array}{c}\left(\left(\gamma_{1423}\underset{14}{\overset{\cdot}{-}}\gamma_{1432}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{1234}\underset{14}{\overset{\cdot}{-}}\gamma_{1324}\right)\right)\overset{\cdot}{-}\\ \left(\left(\gamma_{4231}\underset{14}{\overset{\cdot}{-}}\gamma_{4321}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2341}\underset{14}{\overset{\cdot}{-}}\gamma_{3241}\right)\right)\end{array}\right)(d)\\&=\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},d,-d,\underset{36}{0},...,\underset{38}{0},d,\underset{40}{0},...,\underset{44}{0},-d,\underset{46}{0},...,\underset{50}{0},-d,\underset{52}{0},d\right).\end{aligned} \tag{3.22}$$

(4) Since

$$\begin{aligned}&\gamma_{2143}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\underset{18}{0},...,\underset{20}{0},\\ d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\begin{aligned}&\gamma_{2134}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,\\ \underset{18}{0},...,\underset{35}{0},d_1d_2d_3d_4,\underset{37}{0},...,\underset{53}{0}\end{array}\right),\end{aligned}$$

we have

$$\begin{aligned}&\left(\gamma_{2143}\underset{21}{\overset{\cdot}{-}}\gamma_{2134}\right)(d_1,d_2,d_3)\\&=\mathfrak{m}\left(\begin{array}{c} d_2,d_1,\underset{3}{0},\underset{4}{0},d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3,\underset{11}{0},...,\underset{20}{0},d_2d_3,\underset{22}{0},...,\underset{25}{0},d_1d_3,\\ \underset{27}{0},...,\underset{35}{0},-d_1d_2d_3,d_1d_2d_3,\underset{38}{0},...,\underset{53}{0}\end{array}\right).\end{aligned} \tag{3.23}$$

Since

$$\begin{aligned}&\gamma_{2431}(d_1,d_2,d_3,d_4)\\&=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0}\end{array}\right)\end{aligned}$$

and

$$\gamma_{2341}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},...,\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\underset{19}{0},...,\underset{22}{0}, \\ d_1d_3d_4,\underset{24}{0},...,\underset{38}{0},d_1d_2d_3d_4,\underset{40}{0},...,\underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{2431}\underset{21}{\dot{-}}\gamma_{2341}\right)(d_1,d_2,d_3) = \mathfrak{m}\begin{pmatrix} d_2,d_1,\underset{3}{0},\underset{4}{0},d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3,\underset{11}{0},...,\underset{22}{0},-d_2d_3,\underset{24}{0},d_2d_3,d_1d_3, \\ \underset{27}{0},...,\underset{38}{0},-d_1d_2d_3,\underset{40}{0},d_1d_2d_3,\underset{42}{0},...,\underset{53}{0} \end{pmatrix}. \tag{3.24}$$

(3.23) and (3.24) imply that

$$\left(\left(\gamma_{2143}\underset{21}{\dot{-}}\gamma_{2134}\right)\underset{1}{\dot{-}}\left(\gamma_{2431}\underset{21}{\dot{-}}\gamma_{2341}\right)\right)(d_1,d_2) = \mathfrak{m}\begin{pmatrix} \underset{1}{0},d_1,\underset{3}{0},...,\underset{20}{0},d_2,\underset{22}{0},d_2,\underset{24}{0},-d_2,\underset{26}{0},...,\underset{35}{0}, \\ -d_1d_2,d_1d_2,\underset{38}{0},d_1d_2,\underset{40}{0},-d_1d_2,\underset{42}{0},...,\underset{53}{0} \end{pmatrix}. \tag{3.25}$$

Since

$$\gamma_{1432}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{20}{0}, \\ d_1d_3d_4,\underset{22}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{34}{0},d_1d_2d_3d_4,\underset{36}{0},...,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1342}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0}, \\ d_2d_3d_4,0,0,0,0,d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{1432}\underset{21}{\dot{-}}\gamma_{1342}\right)(d_1,d_2,d_3) = \mathfrak{m}\begin{pmatrix} d_2,d_1,\underset{3}{0},\underset{4}{0},d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3,\underset{11}{0},...,\underset{20}{0},d_1d_3,\underset{22}{0},...,\underset{27}{0},-d_2d_3,\underset{29}{0}, \\ d_2d_3,\underset{31}{0},\underset{32}{0},-d_1d_2d_3,\underset{34}{0},d_1d_2d_3,\underset{36}{0},...,\underset{53}{0} \end{pmatrix}. \tag{3.26}$$

Since

$$\gamma_{4312}(d_1,d_2,d_3,d_4) = \mathfrak{m}\begin{pmatrix} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0}, \\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3412}(d_1,d_2,d_3,d_4)$$

$$= \left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, ..., \underset{53}{0} \end{array} \right),$$

we have

$$\left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412} \right) (d_1, d_2, d_3)$$
$$= \mathfrak{m} \left(\begin{array}{c} d_2, d_1, \underset{3}{0}, \underset{4}{0}, d_1d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3, \underset{11}{0}, ..., \underset{20}{0}, \underset{21}{0}, \underset{22}{0}, -d_2d_3, \underset{24}{0}, d_2d_3, \underset{26}{0}, \underset{27}{0}, \\ -d_1d_3, \underset{29}{0}, d_1d_3, \underset{31}{0}, ..., \underset{45}{0}, -d_1d_2d_3, \underset{47}{0}, ..., \underset{51}{0}, d_1d_2d_3, \underset{53}{0} \end{array} \right). \tag{3.27}$$

(3.26) and (3.27) imply that

$$\left(\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342} \right) \underset{1}{\dot{-}} \left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412} \right) \right) (d_1, d_2)$$
$$= \mathfrak{m} \left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{20}{0}, d_2, \underset{22}{0}, d_2, \underset{24}{0}, -d_2, \underset{26}{0}, ..., \underset{32}{0}, -d_1d_2, \underset{34}{0}, d_1d_2, \\ \underset{36}{0}, ..., \underset{45}{0}, d_1d_2, \underset{47}{0}, ..., \underset{51}{0}, -d_1d_2, \underset{53}{0} \end{array} \right). \tag{3.28}$$

(3.25) and (3.28) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{2143} \underset{21}{\dot{-}} \gamma_{2134} \right) \underset{1}{\dot{-}} \left(\gamma_{2431} \underset{21}{\dot{-}} \gamma_{2341} \right) \right) \dot{-} \\ \left(\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342} \right) \underset{1}{\dot{-}} \left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412} \right) \right) \end{array} \right) (d)$$
$$= \mathfrak{m} \left(\underset{1}{0}, ..., \underset{32}{0}, d, \underset{34}{0}, -d, -d, d, \underset{38}{0}, d, \underset{40}{0}, -d, \underset{42}{0}, ..., \underset{45}{0}, -d, \underset{47}{0}, ..., \underset{51}{0}, d, \underset{53}{0} \right). \tag{3.29}$$

(5) Since

$$\gamma_{2314} (d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, ..., \underset{12}{0}, d_1d_2d_3, 0, 0, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{37}{0}, d_1d_2d_3d_4, \underset{39}{0}, ..., \underset{53}{0} \end{array} \right)$$

and

$$\gamma_{2341} (d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m} \left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, \underset{8}{0}, ..., \underset{12}{0}, d_1d_2d_3, \underset{14}{0}, ..., \underset{17}{0}, d_1d_2d_4, \underset{19}{0}, ..., \underset{22}{0}, \\ d_1d_3d_4, \underset{24}{0}, ..., \underset{38}{0}, d_1d_2d_3d_4, \underset{40}{0}, ..., \underset{53}{0} \end{array} \right),$$

we have

$$\left(\gamma_{2314} \underset{23}{\dot{-}} \gamma_{2341} \right) (d_1, d_2, d_3)$$
$$= \mathfrak{m} \left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{16}{0}, d_1d_3, -d_1d_3, \underset{19}{0}, ..., \underset{21}{0}, d_2d_3, -d_2d_3, \\ \underset{24}{0}, ..., \underset{37}{0}, d_1d_2d_3, -d_1d_2d_3, \underset{40}{0}, ..., \underset{53}{0} \end{array} \right). \tag{3.30}$$

Since

$$\gamma_{2143} (d_1, d_2, d_3, d_4)$$

$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3d_4, 0, d_1d_2d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1d_2d_4, 0, 0, 0, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, ..., \underset{36}{0}, d_1d_2d_3d_4, \underset{38}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2413}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, ..., \underset{39}{0}, d_1d_2d_3d_4, \underset{41}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{2143} \underset{23}{\dot{-}} \gamma_{2413}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{16}{0}, d_1d_3, -d_1d_3, \underset{19}{0}, \underset{20}{0}, d_2d_3, \underset{22}{0}, \underset{23}{0}, \\ -d_2d_3, \underset{25}{0}, ..., \underset{36}{0}, d_1d_2d_3, \underset{38}{0}, \underset{39}{0}, -d_1d_2d_3, \underset{41}{0}, ..., \underset{53}{0} \end{array}\right). \tag{3.31}$$

(3.30) and (3.31) imply that

$$\left(\left(\gamma_{2314} \underset{23}{\dot{-}} \gamma_{2341}\right) \underset{1}{\dot{-}} \left(\gamma_{2143} \underset{23}{\dot{-}} \gamma_{2413}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{20}{0}, -d_2, d_2, -d_2, d_2, \underset{25}{0}, ..., \underset{36}{0}, -d_1d_2, d_1d_2, \\ -d_1d_2, d_1d_2, \underset{41}{0}, ..., \underset{53}{0} \end{array}\right). \tag{3.32}$$

Since

$$\gamma_{3142}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{17}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{42}{0}, d_1d_2d_3d_4, \underset{44}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3412}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1d_2d_3d_4, \underset{47}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3142} \underset{23}{\dot{-}} \gamma_{3412}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, d_1, d_2, \underset{4}{0}, ..., \underset{6}{0}, -d_3, \underset{8}{0}, ..., \underset{15}{0}, d_1d_3, \underset{17}{0}, \underset{18}{0}, -d_1d_3, \underset{20}{0}, \underset{21}{0}, d_2d_3, \\ -d_2d_3, \underset{24}{0}, ..., \underset{42}{0}, d_1d_2d_3, \underset{44}{0}, \underset{45}{0}, -d_1d_2d_3, \underset{47}{0}, ..., \underset{53}{0} \end{array}\right). \tag{3.33}$$

Since

$$\gamma_{1432}\left(d_1, d_2, d_3, d_4\right)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{7}{0}, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, \\ d_1d_3d_4, \underset{22}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{34}{0}, d_1d_2d_3d_4, \underset{36}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4132}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,d_2d_3,d_2d_4,d_3d_4,d_1d_2d_3,\underset{12}{0},...,\underset{18}{0},d_1d_2d_4,\\ \underset{20}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{48}{0},d_1d_2d_3d_4,\underset{50}{0},...,\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{1432}\underset{23}{\dot{-}}\gamma_{4132}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,d_2,\underset{4}{0},...,\underset{6}{0},-d_3,\underset{8}{0},...,\underset{15}{0},d_1d_3,\underset{17}{0},\underset{18}{0},-d_1d_3,\underset{20}{0},d_2d_3,\underset{22}{0},\underset{23}{0},\\ -d_2d_3,\underset{25}{0},...,\underset{34}{0},d_1d_2d_3,\underset{36}{0},...,\underset{48}{0},-d_1d_2d_3,\underset{50}{0},...,\underset{53}{0}\end{array}\right). \tag{3.34}$$

(3.33) and (3.34) imply that

$$\left(\left(\gamma_{3142}\underset{23}{\dot{-}}\gamma_{3412}\right)\underset{1}{\dot{-}}\left(\gamma_{1432}\underset{23}{\dot{-}}\gamma_{4132}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,\underset{3}{0},...,\underset{20}{0},-d_2,d_2,-d_2,d_2,\underset{25}{0},...,\underset{34}{0},-d_1d_2,\underset{36}{0},...,\underset{42}{0},\\ d_1d_2,\underset{44}{0},\underset{45}{0},-d_1d_2,\underset{47}{0},\underset{48}{0},d_1d_2,\underset{50}{0},...,\underset{53}{0}\end{array}\right). \tag{3.35}$$

(3.32) and (3.35) imply that

$$\left(\begin{array}{c}\left(\left(\gamma_{2314}\underset{23}{\dot{-}}\gamma_{2341}\right)\underset{1}{\dot{-}}\left(\gamma_{2143}\underset{23}{\dot{-}}\gamma_{2413}\right)\right)\dot{-}\\ \left(\left(\gamma_{3142}\underset{23}{\dot{-}}\gamma_{3412}\right)\underset{1}{\dot{-}}\left(\gamma_{1432}\underset{23}{\dot{-}}\gamma_{4132}\right)\right)\end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{34}{0},d,\underset{36}{0},-d,d,-d,d,\underset{41}{0},\underset{42}{0},-d,\underset{44}{0},\underset{45}{0},d,\underset{47}{0},\underset{48}{0},-d,\underset{50}{0},...,\underset{53}{0}\right). \tag{3.36}$$

(6) Since

$$\gamma_{2431}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{24}{0},d_1d_3d_4,d_2d_3d_4,\underset{27}{0},...,\underset{40}{0},d_1d_2d_3d_4,\underset{42}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{2413}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{17}{0},d_1d_2d_4,\\ \underset{19}{0},...,\underset{23}{0},d_1d_3d_4,0,d_2d_3d_4,\underset{27}{0},...,\underset{39}{0},d_1d_2d_3d_4,\underset{41}{0},...,\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{2431}\underset{24}{\dot{-}}\gamma_{2413}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,\underset{3}{0},d_2,\underset{5}{0},d_3,\underset{7}{0},...,\underset{11}{0},-d_1d_3,d_1d_3,\underset{14}{0},...,\underset{23}{0},\\ -d_2d_3,d_2d_3,\underset{26}{0},...,\underset{39}{0},-d_1d_2d_3,d_1d_2d_3,\underset{42}{0},...,\underset{53}{0}\end{array}\right). \tag{3.37}$$

Since

$$\gamma_{2314}(d_1,d_2,d_3,d_4)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, ..., \underset{12}{0}, d_1d_2d_3, 0, 0, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{37}{0}, d_1d_2d_3d_4, \underset{39}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2134}\left(d_1, d_2, d_3, d_4\right) = \begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, ..., \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1d_2d_4, \\ \underset{18}{0}, ..., \underset{35}{0}, d_1d_2d_3d_4, \underset{37}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{2314} \overset{\cdot}{\underset{24}{-}} \gamma_{2134}\right)\left(d_1, d_2, d_3\right) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_1, \underset{3}{0}, d_2, \underset{5}{0}, d_3, \underset{7}{0}, ..., \underset{11}{0}, -d_1d_3, d_1d_3, \underset{14}{0}, ..., \underset{21}{0}, d_2d_3, \\ \underset{23}{0}, ..., \underset{35}{0}, -d_1d_2d_3, \underset{37}{0}, d_1d_2d_3, \underset{39}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.38}$$

(3.37) and (3.38) imply that

$$\left(\left(\gamma_{2431} \overset{\cdot}{\underset{24}{-}} \gamma_{2413}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{2314} \overset{\cdot}{\underset{24}{-}} \gamma_{2134}\right)\right)\left(d_1, d_2\right) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_1, \underset{3}{0}, ..., \underset{21}{0}, -d_2, \underset{23}{0}, -d_2, d_2, \underset{26}{0}, ..., \underset{35}{0}, d_1d_2, \underset{37}{0}, -d_1d_2, \\ \underset{39}{0}, -d_1d_2, d_1d_2, \underset{42}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.39}$$

Since

$$\gamma_{4312}\left(d_1, d_2, d_3, d_4\right) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, \\ d_1d_2d_4, \underset{20}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{51}{0}, d_1d_2d_3d_4, \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4132}\left(d_1, d_2, d_3, d_4\right) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1d_4, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{48}{0}, d_1d_2d_3d_4, \underset{50}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{4312} \overset{\cdot}{\underset{24}{-}} \gamma_{4132}\right)\left(d_1, d_2, d_3\right) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_1, \underset{3}{0}, d_2, \underset{5}{0}, d_3, \underset{7}{0}, ..., \underset{10}{0}, -d_1d_3, \underset{12}{0}, \underset{13}{0}, d_1d_3, \underset{15}{0}, ..., \underset{23}{0}, \\ -d_2d_3, d_2d_3, \underset{26}{0}, ..., \underset{48}{0}, -d_1d_2d_3, \underset{50}{0}, \underset{51}{0}, d_1d_2d_3, \underset{53}{0} \end{pmatrix}. \tag{3.40}$$

Since

$$\gamma_{3142}\left(d_1, d_2, d_3, d_4\right) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{17}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{42}{0}, d_1d_2d_3d_4, \underset{44}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1342}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},d_1d_2d_3,\underset{12}{0},...,\underset{15}{0},d_1d_2d_4,\underset{17}{0},...,\underset{27}{0}, \\ d_2d_3d_4,0,0,0,0,d_1d_2d_3d_4,\underset{34}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3142}\underset{24}{\overset{\cdot}{-}}\gamma_{1342}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,\underset{3}{0},d_2,\underset{5}{0},d_3,\underset{7}{0},...,\underset{10}{0},-d_1d_3,\underset{12}{0},\underset{13}{0},d_1d_3,\underset{15}{0},...,\underset{21}{0},d_2d_3, \\ \underset{23}{0},...,\underset{32}{0},-d_1d_2d_3,\underset{34}{0},...,\underset{42}{0},d_1d_2d_3,\underset{44}{0},...,\underset{53}{0} \end{array}\right). \tag{3.41}$$

(3.40) and (3.41) imply that

$$\left(\left(\gamma_{4312}\underset{24}{\overset{\cdot}{-}}\gamma_{4132}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{3142}\underset{24}{\overset{\cdot}{-}}\gamma_{1342}\right)\right)\left(d_1,d_2\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_1,\underset{3}{0},...,\underset{21}{0},-d_2,\underset{23}{0},-d_2,d_2,\underset{26}{0},...,\underset{32}{0},d_1d_2,\underset{34}{0},...,\underset{42}{0}, \\ -d_1d_2,\underset{44}{0},...,\underset{48}{0},-d_1d_2,\underset{50}{0},\underset{51}{0},d_1d_2,\underset{53}{0} \end{array}\right). \tag{3.42}$$

(3.39) and (3.42) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{2431}\underset{24}{\overset{\cdot}{-}}\gamma_{2413}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2314}\underset{24}{\overset{\cdot}{-}}\gamma_{2134}\right)\right)\overset{\cdot}{-} \\ \left(\left(\gamma_{4312}\underset{24}{\overset{\cdot}{-}}\gamma_{4132}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{3142}\underset{24}{\overset{\cdot}{-}}\gamma_{1342}\right)\right) \end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},-d,\underset{34}{0},\underset{35}{0},d,\underset{37}{0},-d,\underset{39}{0},-d,d,\underset{42}{0},d,\underset{44}{0},...,\underset{48}{0},d,\underset{50}{0},\underset{51}{0},-d,\underset{53}{0}\right). \tag{3.43}$$

(7) Since

$$\gamma_{3124}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3142}\left(d_1,d_2,d_3,d_4\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{17}{0},...,\underset{21}{0}, \\ d_1d_3d_4,\underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3124}\underset{31}{\overset{\cdot}{-}}\gamma_{3142}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},...,\underset{26}{0},d_1d_3,-d_1d_3, \\ \underset{29}{0},...,\underset{41}{0},d_1d_2d_3,-d_1d_2d_3,\underset{44}{0},...,\underset{53}{0} \end{array}\right). \tag{3.44}$$

Since

$$\gamma_{3241}\left(d_1,d_2,d_3,d_4\right)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{22}{0}, d_1d_3d_4, \underset{24}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{44}{0}, d_1d_2d_3d_4, \underset{46}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3421}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, \underset{10}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, 0, d_1d_3d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2d_3d_4, \underset{29}{0}, ..., \underset{46}{0}, d_1d_2d_3d_4, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{3241} \underset{31}{\dot{-}} \gamma_{3421}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, d_1, \underset{4}{0}, ..., \underset{8}{0}, -d_3, \underset{10}{0}, ..., \underset{17}{0}, d_2d_3, \underset{19}{0}, -d_2d_3, \underset{21}{0}, ..., \underset{26}{0}, \\ d_1d_3, -d_1d_3, \underset{29}{0}, ..., \underset{44}{0}, d_1d_2d_3, \underset{46}{0}, -d_1d_2d_3, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.45}$$

(3.44) and (3.45) imply that

$$\left(\left(\gamma_{3124} \underset{31}{\dot{-}} \gamma_{3142}\right) \underset{1}{\dot{-}} \left(\gamma_{3241} \underset{31}{\dot{-}} \gamma_{3421}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_1, \underset{4}{0}, ..., \underset{15}{0}, -d_2, \underset{17}{0}, -d_2, \underset{19}{0}, d_2, \underset{21}{0}, ..., \underset{41}{0}, \\ d_1d_2, -d_1d_2, \underset{44}{0}, -d_1d_2, \underset{46}{0}, d_1d_2, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.46}$$

Since

$$\gamma_{1243}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{9}{0}, \underset{10}{d_3d_4}, \underset{11}{0}, ..., \underset{20}{0}, d_1d_3d_4, \underset{22}{0}, ..., \underset{25}{0}, d_2d_3d_4, \underset{27}{0}, ..., \underset{30}{0}, \\ d_1d_2d_3d_4, \underset{32}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1423}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{15}{0}, d_1d_2d_4, \underset{17}{0}, ..., \underset{20}{0}, d_1d_3d_4, \\ \underset{22}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{33}{0}, d_1d_2d_3d_4, \underset{35}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{1243} \underset{31}{\dot{-}} \gamma_{1423}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, d_1, \underset{4}{0}, ..., \underset{8}{0}, -d_3, \underset{10}{0}, ..., \underset{15}{0}, -d_2d_3, \underset{17}{0}, ..., \underset{25}{0}, d_1d_3, \\ \underset{27}{0}, \underset{28}{0}, -d_1d_3, \underset{30}{0}, d_1d_2d_3, \underset{32}{0}, \underset{33}{0}, -d_1d_2d_3, \underset{35}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.47}$$

Since

$$\gamma_{2413}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, \underset{9}{0}, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{17}{0}, d_1d_2d_4, \\ \underset{19}{0}, ..., \underset{23}{0}, d_1d_3d_4, 0, d_2d_3d_4, \underset{27}{0}, ..., \underset{39}{0}, d_1d_2d_3d_4, \underset{41}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4213}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} d_2,\underset{2}{0},d_1,\underset{4}{0},...,\underset{8}{0},-d_3,\underset{10}{0},...,\underset{17}{0},d_2d_3,\underset{19}{0},-d_2d_3,\underset{21}{0},...,\underset{25}{0},d_1d_3, \\ \underset{27}{0},\underset{28}{0},-d_1d_3,\underset{30}{0},...,\underset{39}{0},d_1d_2d_3,\underset{41}{0},...,\underset{49}{0},-d_1d_2d_3,\underset{51}{0},...,\underset{53}{0} \end{array}\right). \quad (3.48)$$

(3.47) and (3.48) imply that

$$\left(\left(\gamma_{1243}\underset{31}{\dot{-}}\gamma_{1423}\right)\underset{1}{\dot{-}}\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)\right)(d_1,d_2) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},-d_2,\underset{17}{0},-d_2,\underset{19}{0},d_2,\underset{21}{0},...,\underset{30}{0},d_1d_2,\underset{32}{0},\underset{33}{0},-d_1d_2, \\ \underset{35}{0},...,\underset{39}{0},-d_1d_2,\underset{41}{0},...,\underset{49}{0},d_1d_2,\underset{51}{0},...,\underset{53}{0} \end{array}\right). \quad (3.49)$$

(3.46) and (3.49) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{3124}\underset{31}{\dot{-}}\gamma_{3142}\right)\underset{1}{\dot{-}}\left(\gamma_{3241}\underset{31}{\dot{-}}\gamma_{3421}\right)\right)\dot{-} \\ \left(\left(\gamma_{1243}\underset{31}{\dot{-}}\gamma_{1423}\right)\underset{1}{\dot{-}}\left(\gamma_{2413}\underset{31}{\dot{-}}\gamma_{4213}\right)\right) \end{array}\right)(d) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},\underset{33}{0},d,\underset{35}{0},...,\underset{39}{0},d,\underset{41}{0},d,-d,\underset{44}{0},-d,\underset{46}{0},d, \\ \underset{48}{0},\underset{49}{0},-d,\underset{51}{0},...,\underset{53}{0} \end{array}\right). \quad (3.50)$$

(8) Since

$$\gamma_{3241}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},\underset{17}{0},d_1d_2d_4, \\ \underset{19}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{44}{0},d_1d_2d_3d_4,\underset{46}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3214}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,0,d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0}, \\ d_1d_3d_4,\underset{23}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{43}{0},d_1d_2d_3d_4,\underset{45}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3241}\underset{32}{\dot{-}}\gamma_{3214}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{16}{0},-d_2d_3,d_2d_3,\underset{19}{0},...,\underset{21}{0},-d_1d_3,d_1d_3, \\ \underset{24}{0},...,\underset{43}{0},-d_1d_2d_3,d_1d_2d_3,\underset{46}{0},...,\underset{53}{0} \end{array}\right). \quad (3.51)$$

Since

$$\gamma_{3412}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0},d_1d_2d_4, \\ \underset{20}{0},...,\underset{22}{0},d_1d_3d_4,\underset{24}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{45}{0},d_1d_2d_3d_4,\underset{47}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3142}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,d_2d_4,\underset{10}{0},...,\underset{13}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{17}{0},...,\underset{21}{0}, \\ d_1d_3d_4,\underset{23}{0},...,\underset{27}{0},d_2d_3d_4,\underset{29}{0},...,\underset{42}{0},d_1d_2d_3d_4,\underset{44}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3412}\underset{32}{\dot{-}}\gamma_{3142}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{15}{0},-d_2d_3,\underset{17}{0},\underset{18}{0},d_2d_3,\underset{20}{0},\underset{21}{0}, \\ -d_1d_3,d_1d_3,\underset{24}{0},...,\underset{42}{0},-d_1d_2d_3,\underset{44}{0},\underset{45}{0},d_1d_2d_3,\underset{47}{0},...,\underset{53}{0} \end{array}\right). \tag{3.52}$$

(3.51) and (3.52) imply that

$$\left(\left(\gamma_{3241}\underset{32}{\dot{-}}\gamma_{3214}\right)\underset{1}{\dot{-}}\left(\gamma_{3412}\underset{32}{\dot{-}}\gamma_{3142}\right)\right)(d_1,d_2) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{15}{0},d_2,-d_2,d_2,-d_2,\underset{20}{0},...,\underset{42}{0},d_1d_2,-d_1d_2, \\ d_1d_2,-d_1d_2,\underset{47}{0},...,\underset{53}{0} \end{array}\right). \tag{3.53}$$

Since

$$\gamma_{2413}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},\underset{9}{0},d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{17}{0},d_1d_2d_4, \\ \underset{19}{0},...,\underset{23}{0},d_1d_3d_4,0,d_2d_3d_4,\underset{27}{0},...,\underset{39}{0},d_1d_2d_3d_4,\underset{41}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{2143}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},...,\underset{9}{0},d_3d_4,0,d_1d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_2d_4,0,0,0, \\ d_1d_3d_4,\underset{22}{0},...,\underset{25}{0},d_2d_3d_4,\underset{27}{0},...,\underset{36}{0},d_1d_2d_3d_4,\underset{38}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{2413}\underset{32}{\dot{-}}\gamma_{2143}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,d_1,\underset{4}{0},...,\underset{6}{0},d_3,\underset{8}{0},...,\underset{16}{0},-d_2d_3,d_2d_3,\underset{19}{0},\underset{20}{0},-d_1d_3, \\ \underset{22}{0},\underset{23}{0},d_1d_3,\underset{25}{0},...,\underset{36}{0},-d_1d_2d_3,\underset{38}{0},\underset{39}{0},d_1d_2d_3,\underset{41}{0},...,\underset{53}{0} \end{array}\right). \tag{3.54}$$

Since

$$\gamma_{4123}(d_1,d_2,d_3,d_4)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1 d_4, \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, ..., \underset{18}{0}, d_1 d_2 d_4, \underset{20}{0}, ..., \underset{23}{0}, d_1 d_3 d_4, \\ \underset{25}{0}, ..., \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, ..., \underset{47}{0}, d_1 d_2 d_3 d_4, \underset{49}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1423}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, ..., \underset{15}{0}, d_1 d_2 d_4, \underset{17}{0}, ..., \underset{20}{0}, d_1 d_3 d_4, \\ \underset{22}{0}, ..., \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, ..., \underset{33}{0}, d_1 d_2 d_3 d_4, \underset{35}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{4123} \underset{32}{\overset{\cdot}{-}} \gamma_{1423}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_2, d_1, \underset{4}{0}, ..., \underset{6}{0}, d_3, \underset{8}{0}, ..., \underset{15}{0}, -d_2 d_3, \underset{17}{0}, \underset{18}{0}, d_2 d_3, \underset{20}{0}, -d_1 d_3, \\ \underset{22}{0}, \underset{23}{0}, d_1 d_3, \underset{25}{0}, ..., \underset{33}{0}, -d_1 d_2 d_3, \underset{35}{0}, ..., \underset{47}{0}, d_1 d_2 d_3, \underset{49}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.55}$$

(3.54) and (3.55) imply that

$$\left(\left(\gamma_{2413} \underset{32}{\overset{\cdot}{-}} \gamma_{2143}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{4123} \underset{32}{\overset{\cdot}{-}} \gamma_{1423}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_1, \underset{4}{0}, ..., \underset{15}{0}, d_2, -d_2, d_2, -d_2, \underset{20}{0}, ..., \underset{33}{0}, d_1 d_2, \underset{35}{0}, \underset{36}{0}, -d_1 d_2, \\ \underset{38}{0}, \underset{39}{0}, d_1 d_2, \underset{41}{0}, ..., \underset{47}{0}, -d_1 d_2, \underset{49}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.56}$$

(3.53) and (3.56) imply that

$$\begin{pmatrix} \left(\left(\gamma_{3241} \underset{32}{\overset{\cdot}{-}} \gamma_{3214}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{3412} \underset{32}{\overset{\cdot}{-}} \gamma_{3142}\right)\right) \overset{\cdot}{-} \\ \left(\left(\gamma_{2413} \underset{32}{\overset{\cdot}{-}} \gamma_{2143}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{4123} \underset{32}{\overset{\cdot}{-}} \gamma_{1423}\right)\right) \end{pmatrix}(d)$$
$$= \mathfrak{m}\left(\underset{1}{0}, ..., \underset{33}{0}, -d, \underset{35}{0}, \underset{36}{0}, d, \underset{38}{0}, \underset{39}{0}, -d, \underset{41}{0}, \underset{42}{0}, d, -d, d, -d, \underset{47}{0}, d, \underset{49}{0}, ..., \underset{53}{0}\right). \tag{3.57}$$

(9) Since

$$\gamma_{3412}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, \underset{10}{0}, ..., \underset{13}{0}, d_1 d_2 d_3, \underset{15}{0}, ..., \underset{18}{0}, d_1 d_2 d_4, \\ \underset{20}{0}, ..., \underset{22}{0}, d_1 d_3 d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2 d_3 d_4, \underset{29}{0}, ..., \underset{45}{0}, d_1 d_2 d_3 d_4, \underset{47}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3421}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, d_1 d_3, d_1 d_4, d_2 d_3, d_2 d_4, \underset{10}{0}, ..., \underset{14}{0}, d_1 d_2 d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1 d_2 d_4, \underset{21}{0}, \underset{22}{0}, d_1 d_3 d_4, \underset{24}{0}, ..., \underset{27}{0}, d_2 d_3 d_4, \underset{29}{0}, ..., \underset{46}{0}, d_1 d_2 d_3 d_4, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{3412} \underset{34}{\overset{\cdot}{-}} \gamma_{3421}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_1, d_2, -d_3, \underset{6}{0}, ..., \underset{13}{0}, d_1 d_3, -d_1 d_3, \underset{16}{0}, ..., \underset{18}{0}, d_2 d_3, -d_2 d_3, \\ \underset{21}{0}, ..., \underset{45}{0}, d_1 d_2 d_3, -d_1 d_2 d_3, \underset{48}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.58}$$

Since

$$\gamma_{3124}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3214}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{14}{0},d_1d_2d_3,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0}, \\ d_1d_3d_4,\underset{23}{0},...,\underset{26}{0},d_2d_3d_4,\underset{28}{0},...,\underset{43}{0},d_1d_2d_3d_4,\underset{45}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3124}\underset{34}{\dot{-}}\gamma_{3214}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{13}{0},d_1d_3,-d_1d_3,\underset{16}{0},-d_2d_3,\underset{18}{0},...,\underset{41}{0}, \\ d_1d_2d_3,\underset{43}{0},-d_1d_2d_3,\underset{45}{0},...,\underset{53}{0} \end{array}\right). \tag{3.59}$$

(3.58) and (3.59) imply that

$$\left(\left(\gamma_{3412}\underset{34}{\dot{-}}\gamma_{3421}\right)\underset{1}{\dot{-}}\left(\gamma_{3124}\underset{34}{\dot{-}}\gamma_{3214}\right)\right)(d_1,d_2) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,\underset{4}{0},...,\underset{16}{0},d_2,\underset{18}{0},d_2,-d_2,\underset{21}{0},...\underset{41}{0},-d_1d_2,\underset{43}{0},d_1d_2, \\ \underset{45}{0},d_1d_2,-d_1d_2,\underset{48}{0},...,\underset{53}{0} \end{array}\right). \tag{3.60}$$

Since

$$\gamma_{4123}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},...,\underset{18}{0},d_1d_2d_4,\underset{20}{0},...,\underset{23}{0},d_1d_3d_4, \\ \underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{47}{0},d_1d_2d_3d_4,\underset{49}{0},...,\underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4213}(d_1,d_2,d_3,d_4) = \mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,0,d_1d_4,0,d_2d_4,d_3d_4,0,d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4, \\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{4123}\underset{34}{\dot{-}}\gamma_{4213}\right)(d_1,d_2,d_3) = \mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_1,d_2,-d_3,\underset{6}{0},...,\underset{11}{0},-d_1d_3,\underset{13}{0},...,\underset{18}{0},d_2d_3,-d_2d_3, \\ \underset{21}{0},...,\underset{47}{0},d_1d_2d_3,\underset{49}{0},-d_1d_2d_3,\underset{51}{0},...,\underset{53}{0} \end{array}\right). \tag{3.61}$$

Since

$$\gamma_{1243}(d_1,d_2,d_3,d_4)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{9}{0}, \underset{10}{d_3 d_4}, \underset{11}{0}, ..., \underset{20}{0}, d_1 d_3 d_4, \underset{22}{0}, ..., \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{30}{0}, \\ d_1 d_2 d_3 d_4, \underset{32}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2143}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1 d_2, \underset{6}{0}, ..., \underset{9}{0}, d_3 d_4, 0, d_1 d_2 d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1 d_2 d_4, \underset{18}{0}, ..., \underset{20}{0}, \\ d_1 d_3 d_4, \underset{22}{0}, ..., \underset{25}{0}, d_2 d_3 d_4, \underset{27}{0}, ..., \underset{36}{0}, d_1 d_2 d_3 d_4, \underset{38}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{1243} \underset{34}{\overset{\cdot}{-}} \gamma_{2143}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_1, d_2, -d_3, \underset{6}{0}, ..., \underset{11}{0}, -d_1 d_3, \underset{13}{0}, ..., \underset{16}{0}, -d_2 d_3, \\ \underset{18}{0}, ..., \underset{30}{0}, d_1 d_2 d_3, \underset{32}{0}, ..., \underset{36}{0}, -d_1 d_2 d_3, \underset{38}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.62}$$

(3.61) and (3.62) imply that

$$\left(\left(\gamma_{4123} \underset{34}{\overset{\cdot}{-}} \gamma_{4213}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{1243} \underset{34}{\overset{\cdot}{-}} \gamma_{2143}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, \underset{2}{0}, d_1, \underset{4}{0}, ..., \underset{16}{0}, d_2, \underset{18}{0}, d_2, -d_2, \underset{21}{0}, ..., \underset{30}{0}, -d_1 d_2, \underset{32}{0}, ..., \underset{36}{0}, \\ d_1 d_2, \underset{38}{0}, ..., \underset{47}{0}, d_1 d_2, \underset{49}{0}, -d_1 d_2, \underset{51}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.63}$$

(3.60) and (3.63) imply that

$$\begin{pmatrix} \left(\left(\gamma_{3412} \underset{34}{\overset{\cdot}{-}} \gamma_{3421}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{3124} \underset{34}{\overset{\cdot}{-}} \gamma_{3214}\right)\right) \overset{\cdot}{-} \\ \left(\left(\gamma_{4123} \underset{34}{\overset{\cdot}{-}} \gamma_{4213}\right) \underset{1}{\overset{\cdot}{-}} \left(\gamma_{1243} \underset{34}{\overset{\cdot}{-}} \gamma_{2143}\right)\right) \end{pmatrix}(d)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, ..., \underset{30}{0}, d, \underset{32}{0}, ..., \underset{36}{0}, -d, \underset{38}{0}, ..., \underset{41}{0}, -d, \underset{43}{0}, d, \underset{45}{0}, \\ d, -d, -d, \underset{49}{0}, d, \underset{51}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.64}$$

(10) Since

$$\gamma_{4132}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1 d_4, d_2 d_3, d_2 d_4, d_3 d_4, d_1 d_2 d_3, \underset{12}{0}, ..., \underset{18}{0}, d_1 d_2 d_4, \\ \underset{20}{0}, ..., \underset{23}{0}, d_1 d_3 d_4, \underset{25}{0}, ..., \underset{29}{0}, d_2 d_3 d_4, \underset{31}{0}, ..., \underset{48}{0}, d_1 d_2 d_3 d_4, \underset{50}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4123}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1 d_4, \underset{8}{0}, d_2 d_4, d_3 d_4, \underset{11}{0}, ..., \underset{18}{0}, d_1 d_2 d_4, \underset{20}{0}, ..., \underset{23}{0}, d_1 d_3 d_4, \\ \underset{25}{0}, ..., \underset{28}{0}, d_2 d_3 d_4, \underset{30}{0}, ..., \underset{47}{0}, d_1 d_2 d_3 d_4, \underset{49}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{4132} \underset{41}{\overset{\cdot}{-}} \gamma_{4123}\right)(d_1, d_2, d_3)$$

$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, ..., \underset{10}{0}, d_2d_3, \underset{12}{0}, ..., \underset{28}{0}, -d_1d_3, d_1d_3, \\ \underset{31}{0}, ..., \underset{47}{0}, -d_1d_2d_3, d_1d_2d_3, \underset{50}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.65}$$

Since

$$\gamma_{4321}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{14}{0}, d_1d_2d_3, \underset{16}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{52}{0}, d_1d_2d_3d_4 \end{pmatrix}$$

and

$$\gamma_{4231}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, d_1d_4, 0, d_2d_4, d_3d_4, 0, 0, d_1d_2d_3, \underset{14}{0}, ..., \underset{19}{0}, \\ d_1d_2d_4, \underset{21}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{50}{0}, d_1d_2d_3d_4, 0, \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, ..., \underset{12}{0}, -d_2d_3, 0, d_2d_3, \underset{16}{0}, ..., \underset{28}{0}, \\ -d_1d_3, d_1d_3, \underset{31}{0}, ..., \underset{50}{0}, -d_1d_2d_3, \underset{52}{0}, d_1d_2d_3 \end{pmatrix}. \tag{3.66}$$

(3.65) and (3.66) imply that

$$\left(\left(\gamma_{4132} \overset{\cdot}{\underset{41}{-}} \gamma_{4123}\right) \overset{\cdot}{\underset{1}{-}} \left(\gamma_{4321} \overset{\cdot}{\underset{41}{-}} \gamma_{4231}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\begin{pmatrix} \underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{10}{0}, d_2, \underset{12}{0}, d_2, \underset{14}{0}, -d_2, \underset{16}{0}, ..., \underset{47}{0}, \\ -d_1d_2, d_1d_2, \underset{50}{0}, d_1d_2, \underset{52}{0}, -d_1d_2 \end{pmatrix}. \tag{3.67}$$

Since

$$\gamma_{1324}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, \underset{7}{0}, d_2d_3, \underset{9}{0}, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{31}{0}, \\ d_1d_2d_3d_4, \underset{33}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{1234}(d_1, d_2, d_3, d_4) = \mathfrak{m}\left(d_1, d_2, d_3, d_4, \underset{5}{0}, ..., \underset{53}{0}\right),$$

we have

$$\left(\gamma_{1324} \overset{\cdot}{\underset{41}{-}} \gamma_{1234}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\begin{pmatrix} d_2, \underset{2}{0}, \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{7}{0}, d_3, \underset{9}{0}, \underset{10}{0}, d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, \\ d_1d_3, \underset{28}{0}, ..., \underset{31}{0}, d_1d_2d_3, \underset{33}{0}, ..., \underset{53}{0} \end{pmatrix}. \tag{3.68}$$

Since

$$\gamma_{3214}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{43}{0}, d_1d_2d_3d_4, \underset{45}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2314}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,\underset{7}{0},...,\underset{12}{0},d_1d_2d_3,0,0,0,d_1d_2d_4,\underset{18}{0},...,\underset{21}{0},d_1d_3d_4,\\ \underset{23}{0},...,\underset{37}{0},d_1d_2d_3d_4,\underset{39}{0},...,\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{3214}\underset{41}{\dot{-}}\gamma_{2314}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_2,\underset{2}{0},\underset{3}{0},d_1,\underset{5}{0},...,\underset{7}{0},d_3,\underset{9}{0},...,\underset{12}{0},-d_2d_3,\underset{14}{0},d_2d_3,\underset{16}{0},...,\underset{26}{0},d_1d_3,\\ \underset{28}{0},...,\underset{37}{0},-d_1d_2d_3,\underset{39}{0},...,\underset{43}{0},d_1d_2d_3,\underset{45}{0},...,\underset{53}{0}\end{array}\right). \quad (3.69)$$

(3.68) and (3.69) imply that

$$\left(\left(\gamma_{1324}\underset{41}{\dot{-}}\gamma_{1234}\right)\underset{1}{\dot{-}}\left(\gamma_{3214}\underset{41}{\dot{-}}\gamma_{2314}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{10}{0},d_2,\underset{12}{0},d_2,\underset{14}{0},-d_2,\underset{16}{0},...,\underset{31}{0},d_1d_2,\\ \underset{33}{0},...,\underset{37}{0},d_1d_2,\underset{39}{0},...,\underset{43}{0},-d_1d_2,\underset{45}{0},...,\underset{53}{0}\end{array}\right). \quad (3.70)$$

(3.67) and (3.70) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{4132}\underset{41}{\dot{-}}\gamma_{4123}\right)\underset{1}{\dot{-}}\left(\gamma_{4321}\underset{41}{\dot{-}}\gamma_{4231}\right)\right)\dot{-}\\ \left(\left(\gamma_{1324}\underset{41}{\dot{-}}\gamma_{1234}\right)\underset{1}{\dot{-}}\left(\gamma_{3214}\underset{41}{\dot{-}}\gamma_{2314}\right)\right)\end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},-d,\underset{33}{0},...,\underset{37}{0},-d,\underset{39}{0},...,\underset{43}{0},d,\underset{45}{0},...,\underset{47}{0},-d,d,\underset{50}{0},d,\underset{52}{0},\underset{53}{-d}\right). \quad (3.71)$$

(11) Since

$$\gamma_{4213}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,\underset{6}{0},d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},d_1d_2d_3,\underset{13}{0},...,\underset{19}{0},d_1d_2d_4,\\ \underset{21}{0},...,\underset{23}{0},d_1d_3d_4,\underset{25}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{49}{0},d_1d_2d_3d_4,\underset{51}{0},...,\underset{53}{0}\end{array}\right)$$

and

$$\gamma_{4231}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,\underset{8}{0},d_2d_4,d_3d_4,\underset{11}{0},\underset{12}{0},d_1d_2d_3,\underset{14}{0},...,\underset{19}{0},\\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{28}{0},d_2d_3d_4,\underset{30}{0},...,\underset{50}{0},d_1d_2d_3d_4,\underset{52}{0},\underset{53}{0}\end{array}\right),$$

we have

$$\left(\gamma_{4213}\underset{42}{\dot{-}}\gamma_{4231}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{11}{0},d_2d_3,-d_2d_3,\underset{14}{0},...,\underset{23}{0},\\ d_1d_3,-d_1d_3,\underset{26}{0},...,\underset{49}{0},d_1d_2d_3,\underset{51}{-d_1d_2d_3},0,0\end{array}\right). \quad (3.72)$$

Since

$$\gamma_{4132}(d_1,d_2,d_3,d_4)$$

$$= \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, d_1d_4, d_2d_3, d_2d_4, d_3d_4, d_1d_2d_3, \underset{12}{0}, ..., \underset{18}{0}, d_1d_2d_4, \\ \underset{20}{0}, ..., \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{48}{0}, d_1d_2d_3d_4, \underset{50}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{4312}(d_1, d_2, d_3, d_4) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, 0, d_1d_3, d_1d_4, d_2d_3, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{18}{0}, \\ d_1d_2d_4, \underset{20}{0}, ..., \underset{24}{0}, d_1d_3d_4, \underset{26}{0}, ..., \underset{29}{0}, d_2d_3d_4, \underset{31}{0}, ..., \underset{51}{0}, d_1d_2d_3d_4, \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{4132} \underset{42}{\dot{-}} \gamma_{4312}\right)(d_1, d_2, d_3) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_2, \underset{3}{0}, d_1, \underset{5}{0}, -d_3, \underset{7}{0}, ..., \underset{10}{0}, d_2d_3, \underset{12}{0}, \underset{13}{0}, -d_2d_3, \underset{15}{0}, ..., \underset{23}{0}, d_1d_3, \\ -d_1d_3, \underset{26}{0}, ..., \underset{48}{0}, d_1d_2d_3, \underset{50}{0}, \underset{51}{0}, -d_1d_2d_3, \underset{53}{0} \end{pmatrix}. \quad (3.73)$$

(3.72) and (3.73) imply that

$$\left(\left(\gamma_{4213} \underset{42}{\dot{-}} \gamma_{4231}\right) \underset{1}{\dot{-}} \left(\gamma_{4132} \underset{42}{\dot{-}} \gamma_{4312}\right)\right)(d_1, d_2) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{10}{0}, -d_2, d_2, -d_2, d_2, \underset{15}{0}, ..., \underset{48}{0}, -d_1d_2, \\ d_1d_2, -d_1d_2, d_1d_2, \underset{53}{0} \end{pmatrix}. \quad (3.74)$$

Since

$$\gamma_{2134}(d_1, d_2, d_3, d_4) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, ..., \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1d_2d_4, \\ \underset{18}{0}, ..., \underset{35}{0}, d_1d_2d_3d_4, \underset{37}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{2314}(d_1, d_2, d_3, d_4) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, \underset{7}{0}, ..., \underset{12}{0}, d_1d_2d_3, 0, 0, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, d_1d_3d_4, \\ \underset{23}{0}, ..., \underset{37}{0}, d_1d_2d_3d_4, \underset{39}{0}, ..., \underset{53}{0} \end{pmatrix},$$

we have

$$\left(\gamma_{2134} \underset{42}{\dot{-}} \gamma_{2314}\right)(d_1, d_2, d_3) = \mathfrak{m}\begin{pmatrix} \underset{1}{0}, d_2, \underset{3}{0}, d_1, \underset{5}{0}, -d_3, \underset{7}{0}, ..., \underset{11}{0}, d_2d_3, -d_2d_3, \underset{14}{0}, ..., \underset{21}{0}, -d_1d_3, \\ \underset{23}{0}, ..., \underset{35}{0}, d_1d_2d_3, 0, -d_1d_2d_3, \underset{39}{0}, ..., \underset{53}{0} \end{pmatrix}. \quad (3.75)$$

Since

$$\gamma_{1324}(d_1, d_2, d_3, d_4) = \mathfrak{m}\begin{pmatrix} d_1, d_2, d_3, d_4, \underset{5}{0}, \underset{6}{0}, \underset{7}{0}, d_2d_3, \underset{9}{0}, \underset{10}{0}, d_1d_2d_3, \underset{12}{0}, ..., \underset{26}{0}, d_2d_3d_4, 0, 0, \underset{30}{0}, \\ 0, d_1d_2d_3d_4, \underset{33}{0}, ..., \underset{53}{0} \end{pmatrix}$$

and

$$\gamma_{3124}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,\underset{7}{0},d_2d_3,\underset{9}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{21}{0},d_1d_3d_4,\underset{23}{0},...,\underset{26}{0}, \\ d_2d_3d_4,\underset{28}{0},...,\underset{41}{0},d_1d_2d_3d_4,\underset{43}{0},...,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},d_2,\underset{3}{0},d_1,\underset{5}{0},-d_3,\underset{7}{0},...,\underset{10}{0},d_2d_3,0,0,-d_2d_3,\underset{15}{0},...,\underset{21}{0}, \\ -d_1d_3,\underset{23}{0},...,\underset{31}{0},d_1d_2d_3,\underset{33}{0},...,\underset{41}{0},-d_1d_2d_3,\underset{43}{0},...,\underset{53}{0} \end{array}\right). \quad (3.76)$$

(3.75) and (3.76) imply that

$$\left(\left(\gamma_{2134}\underset{42}{\dot{-}}\gamma_{2314}\right)\underset{1}{\dot{-}}\left(\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right)\right)(d_1,d_2)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{10}{0},-d_2,d_2,-d_2,d_2,\underset{15}{0},...,\underset{31}{0},-d_1d_2,\underset{33}{0},...,\underset{35}{0}, \\ d_1d_2,0,-d_1d_2,\underset{39}{0},...,\underset{41}{0},d_1d_2,\underset{43}{0},...,\underset{53}{0} \end{array}\right). \quad (3.77)$$

(3.74) and (3.77) imply that

$$\left(\begin{array}{c} \left(\left(\gamma_{4213}\underset{42}{\dot{-}}\gamma_{4231}\right)\underset{1}{\dot{-}}\left(\gamma_{4132}\underset{42}{\dot{-}}\gamma_{4312}\right)\right)\dot{-} \\ \left(\left(\gamma_{2134}\underset{42}{\dot{-}}\gamma_{2314}\right)\underset{1}{\dot{-}}\left(\gamma_{1324}\underset{42}{\dot{-}}\gamma_{3124}\right)\right) \end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},...,\underset{35}{0},-d,0,d,\underset{39}{0},...,\underset{41}{0},-d,\underset{43}{0},...,\underset{48}{0},-d,d,-d,d,\underset{53}{0}\right). \quad (3.78)$$

(12) Since

$$\gamma_{4321}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,d_1d_2,d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{14}{0},d_1d_2d_3,\underset{16}{0},...,\underset{19}{0}, \\ d_1d_2d_4,\underset{21}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{52}{0},d_1d_2d_3d_4 \end{array}\right)$$

and

$$\gamma_{4312}(d_1,d_2,d_3,d_4)$$
$$=\mathfrak{m}\left(\begin{array}{c} d_1,d_2,d_3,d_4,\underset{5}{0},d_1d_3,d_1d_4,d_2d_3,d_2d_4,d_3d_4,\underset{11}{0},...,\underset{13}{0},d_1d_2d_3,\underset{15}{0},...,\underset{18}{0}, \\ d_1d_2d_4,\underset{20}{0},...,\underset{24}{0},d_1d_3d_4,\underset{26}{0},...,\underset{29}{0},d_2d_3d_4,\underset{31}{0},...,\underset{51}{0},d_1d_2d_3d_4,\underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{4321}\underset{43}{\dot{-}}\gamma_{4312}\right)(d_1,d_2,d_3)$$
$$=\mathfrak{m}\left(\begin{array}{c} \underset{1}{0},\underset{2}{0},d_2,d_1,d_3,\underset{6}{0},...,\underset{13}{0},-d_2d_3,d_2d_3,\underset{16}{0},...,\underset{18}{0},-d_1d_3,d_1d_3, \\ \underset{21}{0},...,\underset{51}{0},-d_1d_2d_3,d_1d_2d_3 \end{array}\right). \quad (3.79)$$

Since

$$\gamma_{4213}(d_1,d_2,d_3,d_4)$$

$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, d_1d_4, \underset{8}{0}, d_2d_4, d_3d_4, \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{19}{0}, d_1d_2d_4, \\ \underset{21}{0}, ..., \underset{23}{0}, d_1d_3d_4, \underset{25}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{49}{0}, d_1d_2d_3d_4, \underset{51}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{4123}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, 0, 0, d_1d_4, 0, 0, d_2d_4, d_3d_4, \underset{11}{0}, ..., \underset{18}{0}, d_1d_2d_4, \underset{20}{0}, ..., \underset{23}{0}, d_1d_3d_4, \\ \underset{25}{0}, ..., \underset{28}{0}, d_2d_3d_4, \underset{30}{0}, ..., \underset{47}{0}, d_1d_2d_3d_4, \underset{49}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{4213} \underset{43}{\dot{-}} \gamma_{4123}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_2, d_1, d_3, \underset{6}{0}, ..., \underset{11}{0}, d_2d_3, \underset{13}{0}, ..., \underset{18}{0}, -d_1d_3, d_1d_3, \underset{21}{0}, ..., \underset{47}{0}, \\ -d_1d_2d_3, \underset{49}{0}, d_1d_2d_3, \underset{51}{0}, ..., \underset{53}{0} \end{array}\right). \tag{3.80}$$

(3.79) and (3.80) imply that

$$\left(\left(\gamma_{4321} \underset{43}{\dot{-}} \gamma_{4312}\right) \underset{1}{\dot{-}} \left(\gamma_{4213} \underset{43}{\dot{-}} \gamma_{4123}\right)\right)(d_1, d_2)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, ..., \underset{3}{0}, d_1, \underset{5}{0}, ..., \underset{11}{0}, -d_2, 0, -d_2, d_2, \underset{16}{0}, ..., \underset{47}{0}, d_1d_2, \underset{49}{0}, \\ -d_1d_2, \underset{51}{0}, -d_1d_2, d_1d_2 \end{array}\right). \tag{3.81}$$

Since

$$\gamma_{3214}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, d_1d_3, 0, d_2d_3, \underset{9}{0}, ..., \underset{14}{0}, d_1d_2d_3, 0, d_1d_2d_4, \underset{18}{0}, ..., \underset{21}{0}, \\ d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, d_2d_3d_4, \underset{28}{0}, ..., \underset{43}{0}, d_1d_2d_3d_4, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{3124}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, \underset{5}{0}, d_1d_3, \underset{7}{0}, d_2d_3, \underset{9}{0}, ..., \underset{13}{0}, d_1d_2d_3, \underset{15}{0}, ..., \underset{21}{0}, d_1d_3d_4, \underset{23}{0}, ..., \underset{26}{0}, \\ d_2d_3d_4, \underset{28}{0}, ..., \underset{41}{0}, d_1d_2d_3d_4, \underset{43}{0}, ..., \underset{53}{0} \end{array}\right),$$

we have

$$\left(\gamma_{3214} \underset{43}{\dot{-}} \gamma_{3124}\right)(d_1, d_2, d_3)$$
$$= \mathfrak{m}\left(\begin{array}{c} \underset{1}{0}, \underset{2}{0}, d_2, d_1, d_3, \underset{6}{0}, ..., \underset{13}{0}, -d_2d_3, d_2d_3, 0, d_1d_3, \\ \underset{18}{0}, ..., \underset{41}{0}, -d_1d_2d_3, 0, d_1d_2d_3, \underset{45}{0}, ..., \underset{53}{0} \end{array}\right). \tag{3.82}$$

Since

$$\gamma_{2134}(d_1, d_2, d_3, d_4)$$
$$= \mathfrak{m}\left(\begin{array}{c} d_1, d_2, d_3, d_4, d_1d_2, \underset{6}{0}, ..., \underset{11}{0}, d_1d_2d_3, \underset{13}{0}, ..., \underset{16}{0}, d_1d_2d_4, \\ \underset{18}{0}, ..., \underset{35}{0}, d_1d_2d_3d_4, \underset{37}{0}, ..., \underset{53}{0} \end{array}\right)$$

and

$$\gamma_{1234}\left(d_1,d_2,d_3,d_4\right)=\mathfrak{m}\left(d_1,d_2,d_3,d_4,\underset{5}{0},...,\underset{53}{0}\right),$$

we have

$$\left(\gamma_{2134}\underset{43}{\overset{\cdot}{-}}\gamma_{1234}\right)\left(d_1,d_2,d_3\right)$$
$$=\mathfrak{m}\left(\underset{1}{0},\underset{2}{0},d_2,d_1,d_3,\underset{6}{0},...,\underset{11}{0},d_2d_3,\underset{13}{0},...,\underset{16}{0},d_1d_3,\underset{18}{0},...,\underset{35}{0},d_1d_2d_3,\underset{37}{0},...,\underset{53}{0}\right). \tag{3.83}$$

(3.82) and (3.83) imply that

$$\left(\left(\gamma_{3214}\underset{43}{\overset{\cdot}{-}}\gamma_{3124}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2134}\underset{43}{\overset{\cdot}{-}}\gamma_{1234}\right)\right)\left(d_1,d_2\right)$$
$$=\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},...,\underset{3}{0},d_1,\underset{5}{0},...,\underset{11}{0},-d_2,0,-d_2,d_2,\underset{16}{0},...,\underset{35}{0},-d_1d_2,\\ \underset{37}{0},...,\underset{41}{0},-d_1d_2,\underset{43}{0},d_1d_2,\underset{45}{0},...,\underset{53}{0}\end{array}\right). \tag{3.84}$$

(3.81) and (3.84) imply that

$$\left(\begin{array}{c}\left(\left(\gamma_{4321}\underset{43}{\overset{\cdot}{-}}\gamma_{4312}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{4213}\underset{43}{\overset{\cdot}{-}}\gamma_{4123}\right)\right)\overset{\cdot}{-}\\ \left(\left(\gamma_{3214}\underset{43}{\overset{\cdot}{-}}\gamma_{3124}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2134}\underset{43}{\overset{\cdot}{-}}\gamma_{1234}\right)\right)\end{array}\right)(d)$$
$$=\mathfrak{m}\left(\underset{1}{0},...,\underset{35}{0},d,\underset{37}{0},...,\underset{41}{0},d,0,-d,\underset{45}{0},...,\underset{47}{0},d,\underset{49}{0},-d,\underset{51}{0},-d,d\right) \tag{3.85}$$

(13) By (3.8), (3.15), (3.22), (3.29), (3.36), (3.43), (3.50), (3.57), (3.64), (3.71), (3.78) and (3.85), we have

the left-hand side of the equation (3.1)

$$=\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},-d,\underset{34}{0},d,\underset{36}{0},...,\underset{38}{0},-d,0,d,\underset{42}{0},...,\underset{46}{0},d,\underset{48}{0},...,\underset{52}{0},-d\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{30}{0},d,-d,d,-d,\underset{35}{0},...,\underset{40}{0},-d,\underset{42}{0},...\underset{44}{0},d,\underset{46}{0},-d,\underset{48}{0},...,\underset{50}{0},d,\underset{52}{0},\underset{53}{0}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},d,-d,\underset{36}{0},...,\underset{38}{0},d,\underset{40}{0},...,\underset{44}{0},-d,\underset{46}{0},...,\underset{50}{0},-d,\underset{52}{0},d\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},d,\underset{34}{0},-d,-d,d,\underset{38}{0},d,\underset{40}{0},-d,\underset{42}{0},...,\underset{45}{0},-d,\underset{47}{0},...,\underset{51}{0},d,\underset{53}{0}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{34}{0},d,\underset{36}{0},-d,d,-d,d,\underset{41}{0},\underset{42}{0},-d,\underset{44}{0},\underset{45}{0},d,\underset{47}{0},\underset{48}{0},-d,\underset{50}{0},...,\underset{53}{0}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{32}{0},-d,\underset{34}{0},\underset{35}{0},d,\underset{37}{0},-d,\underset{39}{0},-d,d,\underset{42}{0},d,\underset{44}{0},...,\underset{48}{0},d,\underset{50}{0},\underset{51}{0},-d,\underset{53}{0}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},...,\underset{30}{0},-d,\underset{32}{0},\underset{33}{0},d,\underset{35}{0},...,\underset{39}{0},d,\underset{41}{0},d,-d,\underset{44}{0},-d,\underset{46}{0},d,\\ \underset{48}{0},\underset{49}{0},-d,\underset{51}{0},...,\underset{53}{0}\end{array}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{33}{0},-d,\underset{35}{0},\underset{36}{0},d,\underset{38}{0},\underset{39}{0},-d,\underset{41}{0},\underset{42}{0},d,-d,d,-d,\underset{47}{0},d,\underset{49}{0},...,\underset{53}{0}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\begin{array}{c}\underset{1}{0},...,\underset{30}{0},d,\underset{32}{0},...,\underset{36}{0},-d,\underset{38}{0},...,\underset{41}{0},-d,\underset{43}{0},d,\underset{45}{0},d,-d,-d,\\ \underset{49}{0},d,\underset{51}{0},...,\underset{53}{0}\end{array}\right)\right)+$$

$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},-d,\underset{33}{0},...,\underset{37}{0},-d,\underset{39}{0},...,\underset{43}{0},d,\underset{45}{0},...,\underset{47}{0},-d,d,\underset{50}{0},d,0,\underset{53}{-d}\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{31}{0},d,\underset{33}{0},...,\underset{35}{0},-d,0,d,\underset{39}{0},...,\underset{41}{0},-d,\underset{43}{0},...,\underset{48}{0},-d,d,-d,d,0\right)\right)+$$
$$\left(\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{35}{0},d,\underset{37}{0},...,\underset{41}{0},d,0,-d,\underset{45}{0},...,\underset{47}{0},d,0,-d,0,-d,d\right)\right)$$
$$=\lambda d.\mathfrak{m}\begin{pmatrix} \underset{1}{0},...,\underset{30}{0},-d+\underset{31}{d}-d+d,-d+\underset{32}{d}-d+d, \\ -d+\underset{33}{d}+d-d,-d+\underset{34}{d}+d-d,d-\underset{35}{d}-d+d, \\ -d+\underset{36}{d}-d+d,d-\underset{37}{d}+d-d,d-\underset{38}{d}-d+d, \\ -d+\underset{39}{d}+d-d,d-\underset{40}{d}+d-d,d-\underset{41}{d}-d+d, \\ d-\underset{42}{d}-d+d,-d+\underset{43}{d}-d+d,-d+\underset{44}{d}+d-d, \\ d-\underset{45}{d}-d+d,-d+\underset{46}{d}-d+d,d-\underset{47}{d}+d-d, \\ d-\underset{48}{d}-d+d,-d+\underset{49}{d}+d-d,-d+\underset{50}{d}+d-d, \\ d-\underset{51}{d}+d-d,d-\underset{52}{d}+d-d,-d+\underset{53}{d}-d+d \end{pmatrix}$$
$$=\lambda d.\mathfrak{m}\left(\underset{1}{0},...,\underset{53}{0}\right)$$

□

Remark 3.5. For our convenience, we display the positions 31–53 in (3.8), (3.15), (3.22), (3.29), (3.36), (3.43), (3.50), (3.57), (3.64), (3.71), (3.78) and (3.85) as a table:

	$\frac{1}{12}$	$\frac{2}{13}$	$\frac{3}{14}$	$\frac{4}{21}$	$\frac{5}{23}$	$\frac{6}{24}$	$\frac{7}{31}$	$\frac{8}{32}$	$\frac{9}{34}$	$\frac{10}{41}$	$\frac{11}{42}$	$\frac{12}{43}$
31/1243	$-d$	d					$-d$		d			
32/1324		$-d$	d							$-d$	d	
33/1342	$-d$	d		d		$-d$						
34/1423		$-d$	d				d	$-d$				
35/1432	d		$-d$	$-d$	d							
36/2134				$-d$		d					$-d$	d
37/2143				d	$-d$			d	$-d$			
38/2314					d	$-d$				$-d$	d	
39/2341	$-d$		d	d	$-d$							
40/2413					d	$-d$	d	$-d$				
41/2431	d	$-d$		$-d$		d						
42/3124							d		$-d$		$-d$	d
43/3142					$-d$	d	$-d$	d				
44/3214								$-d$	d	d		$-d$
45/3241		d	$-d$				$-d$	d				
46/3412				$-d$	d			$-d$	d			
47/3421	d	$-d$					d		$-d$			
48/4123								d	$-d$	$-d$		d
49/4132					$-d$	d				d	$-d$	
50/4213							$-d$		d		d	$-d$

51/4231		d	$-d$							d	$-d$	
52/4312				d		$-d$					d	$-d$
53/4321	$-d$		d							$-d$		d

Corollary 3.6. *Let M be a microlinear space with*

$$X_1, X_2, X_3, X_4 \in \mathfrak{X}(M).$$

Then, we have

$$\begin{aligned}
&[X_1,[X_2,[X_3,X_4]]]+[X_1,[X_3,[X_4,X_2]]]+[X_1,[X_4,[X_2,X_3]]]+\\
&[X_2,[X_1,[X_4,X_3]]]+[X_2,[X_3,[X_1,X_4]]]+[X_2,[X_4,[X_3,X_1]]]+\\
&[X_3,[X_1,[X_2,X_4]]]+[X_3,[X_2,[X_4,X_1]]]+[X_3,[X_4,[X_1,X_2]]]+\\
&[X_4,[X_1,[X_3,X_2]]]+[X_4,[X_2,[X_1,X_3]]]+[X_4,[X_3,[X_2,X_1]]]\\
&=0.
\end{aligned}$$

Proof. Let

$$\begin{aligned}
&\gamma_{1234}=X_4*X_3*X_2*X_1, \gamma_{1243}=(X_3*X_4*X_2*X_1)^{\sigma_{1243}},\\
&\gamma_{1324}=(X_4*X_2*X_3*X_1)^{\sigma_{1324}}, \gamma_{1342}=(X_2*X_4*X_3*X_1)^{\sigma_{1342}},\\
&\gamma_{1423}=(X_3*X_2*X_4*X_1)^{\sigma_{1423}}, \gamma_{1432}=(X_2*X_3*X_4*X_1)^{\sigma_{1432}},\\
&\gamma_{2134}=(X_4*X_3*X_1*X_2)^{\sigma_{2134}}, \gamma_{2143}=(X_3*X_4*X_1*X_2)^{\sigma_{2143}},\\
&\gamma_{2314}=(X_4*X_1*X_3*X_2)^{\sigma_{2314}}, \gamma_{2341}=(X_1*X_4*X_3*X_2)^{\sigma_{2341}},\\
&\gamma_{2413}=(X_3*X_1*X_4*X_2)^{\sigma_{2413}}, \gamma_{2431}=(X_1*X_3*X_4*X_2)^{\sigma_{2431}},\\
&\gamma_{3124}=(X_4*X_2*X_1*X_3)^{\sigma_{3124}}, \gamma_{3142}=(X_2*X_4*X_1*X_3)^{\sigma_{3142}},\\
&\gamma_{3214}=(X_4*X_1*X_2*X_3)^{\sigma_{3214}}, \gamma_{3241}=(X_1*X_4*X_2*X_3)^{\sigma_{3241}},\\
&\gamma_{3412}=(X_2*X_1*X_4*X_3)^{\sigma_{3412}}, \gamma_{3421}=(X_1*X_2*X_4*X_3)^{\sigma_{3421}},\\
&\gamma_{4123}=(X_3*X_2*X_1*X_4)^{\sigma_{4123}}, \gamma_{4132}=(X_2*X_3*X_1*X_4)^{\sigma_{4132}},\\
&\gamma_{4213}=(X_3*X_1*X_2*X_4)^{\sigma_{4213}}, \gamma_{4231}=(X_1*X_3*X_2*X_4)^{\sigma_{4231}},\\
&\gamma_{4312}=(X_2*X_1*X_3*X_4)^{\sigma_{4312}}, \gamma_{4321}=(X_1*X_2*X_3*X_4)^{\sigma_{4321}}
\end{aligned}$$

with

$$\begin{aligned}
&\sigma_{1243}=\begin{pmatrix}1234\\1243\end{pmatrix}, \sigma_{1324}=\begin{pmatrix}1234\\1324\end{pmatrix}, \sigma_{1342}=\begin{pmatrix}1234\\1423\end{pmatrix}, \sigma_{1423}=\begin{pmatrix}1234\\1342\end{pmatrix},\\
&\sigma_{1432}=\begin{pmatrix}1234\\1432\end{pmatrix}, \sigma_{2134}=\begin{pmatrix}1234\\2134\end{pmatrix}, \sigma_{2143}=\begin{pmatrix}1234\\2143\end{pmatrix}, \sigma_{2314}=\begin{pmatrix}1234\\3124\end{pmatrix},\\
&\sigma_{2341}=\begin{pmatrix}1234\\4123\end{pmatrix}, \sigma_{2413}=\begin{pmatrix}1234\\3142\end{pmatrix}, \sigma_{2431}=\begin{pmatrix}1234\\4132\end{pmatrix}, \sigma_{3124}=\begin{pmatrix}1234\\2314\end{pmatrix},\\
&\sigma_{3142}=\begin{pmatrix}1234\\2413\end{pmatrix}, \sigma_{3214}=\begin{pmatrix}1234\\3214\end{pmatrix}, \sigma_{3241}=\begin{pmatrix}1234\\4213\end{pmatrix}, \sigma_{3412}=\begin{pmatrix}1234\\3412\end{pmatrix},\\
&\sigma_{3421}=\begin{pmatrix}1234\\4312\end{pmatrix}, \sigma_{4123}=\begin{pmatrix}1234\\2341\end{pmatrix}, \sigma_{4132}=\begin{pmatrix}1234\\2431\end{pmatrix}, \sigma_{4213}=\begin{pmatrix}1234\\3241\end{pmatrix},
\end{aligned}$$

$$\sigma_{4231} = \begin{pmatrix} 1234 \\ 4231 \end{pmatrix}, \sigma_{4312} = \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}, \sigma_{4321} = \begin{pmatrix} 1234 \\ 4321 \end{pmatrix}.$$

Then, it is easy to see that

$[X_1, [X_2, [X_3, X_4]]] =$

$$\left(\left(\gamma_{1234} \underset{12}{\dot{-}} \gamma_{1243}\right) \underset{1}{\dot{-}} \left(\gamma_{1342} \underset{12}{\dot{-}} \gamma_{1432}\right)\right) \dot{-} \left(\left(\gamma_{2341} \underset{12}{\dot{-}} \gamma_{2431}\right) \underset{1}{\dot{-}} \left(\gamma_{3421} \underset{12}{\dot{-}} \gamma_{4321}\right)\right),$$

$[X_1, [X_3, [X_4, X_2]]] =$

$$\left(\left(\gamma_{1342} \underset{13}{\dot{-}} \gamma_{1324}\right) \underset{1}{\dot{-}} \left(\gamma_{1423} \underset{13}{\dot{-}} \gamma_{1243}\right)\right) \dot{-} \left(\left(\gamma_{3421} \underset{13}{\dot{-}} \gamma_{3241}\right) \underset{1}{\dot{-}} \left(\gamma_{4231} \underset{13}{\dot{-}} \gamma_{2431}\right)\right),$$

$[X_1, [X_4, [X_2, X_3]]] =$

$$\left(\left(\gamma_{1423} \underset{14}{\dot{-}} \gamma_{1432}\right) \underset{1}{\dot{-}} \left(\gamma_{1234} \underset{14}{\dot{-}} \gamma_{1324}\right)\right) \dot{-} \left(\left(\gamma_{4231} \underset{14}{\dot{-}} \gamma_{4321}\right) \underset{1}{\dot{-}} \left(\gamma_{2341} \underset{14}{\dot{-}} \gamma_{3241}\right)\right),$$

$[X_2, [X_1, [X_4, X_3]]] =$

$$\left(\left(\gamma_{2143} \underset{21}{\dot{-}} \gamma_{2134}\right) \underset{1}{\dot{-}} \left(\gamma_{2431} \underset{21}{\dot{-}} \gamma_{2341}\right)\right) \dot{-} \left(\left(\gamma_{1432} \underset{21}{\dot{-}} \gamma_{1342}\right) \underset{1}{\dot{-}} \left(\gamma_{4312} \underset{21}{\dot{-}} \gamma_{3412}\right)\right),$$

$[X_2, [X_3, [X_1, X_4]]] =$

$$\left(\left(\gamma_{2314} \underset{23}{\dot{-}} \gamma_{2341}\right) \underset{1}{\dot{-}} \left(\gamma_{2143} \underset{23}{\dot{-}} \gamma_{2413}\right)\right) \dot{-} \left(\left(\gamma_{3142} \underset{23}{\dot{-}} \gamma_{3412}\right) \underset{1}{\dot{-}} \left(\gamma_{1432} \underset{23}{\dot{-}} \gamma_{4132}\right)\right),$$

$[X_2, [X_4, [X_3, X_1]]] =$

$$\left(\left(\gamma_{2431} \underset{24}{\dot{-}} \gamma_{2413}\right) \underset{1}{\dot{-}} \left(\gamma_{2314} \underset{24}{\dot{-}} \gamma_{2134}\right)\right) \dot{-} \left(\left(\gamma_{4312} \underset{24}{\dot{-}} \gamma_{4132}\right) \underset{1}{\dot{-}} \left(\gamma_{3142} \underset{24}{\dot{-}} \gamma_{1342}\right)\right),$$

$[X_3, [X_1, [X_2, X_4]]] =$

$$\left(\left(\gamma_{3124} \underset{31}{\dot{-}} \gamma_{3142}\right) \underset{1}{\dot{-}} \left(\gamma_{3241} \underset{31}{\dot{-}} \gamma_{3421}\right)\right) \dot{-} \left(\left(\gamma_{1243} \underset{31}{\dot{-}} \gamma_{1423}\right) \underset{1}{\dot{-}} \left(\gamma_{2413} \underset{31}{\dot{-}} \gamma_{4213}\right)\right),$$

$[X_3, [X_2, [X_4, X_1]]] =$

$$\left(\left(\gamma_{3241} \underset{32}{\dot{-}} \gamma_{3214}\right) \underset{1}{\dot{-}} \left(\gamma_{3412} \underset{32}{\dot{-}} \gamma_{3142}\right)\right) \dot{-} \left(\left(\gamma_{2413} \underset{32}{\dot{-}} \gamma_{2143}\right) \underset{1}{\dot{-}} \left(\gamma_{4123} \underset{32}{\dot{-}} \gamma_{1423}\right)\right),$$

$[X_3, [X_4, [X_1, X_2]]] =$

$$\left(\left(\gamma_{3412} \underset{34}{\dot{-}} \gamma_{3421}\right) \underset{1}{\dot{-}} \left(\gamma_{3124} \underset{34}{\dot{-}} \gamma_{3214}\right)\right) \dot{-} \left(\left(\gamma_{4123} \underset{34}{\dot{-}} \gamma_{4213}\right) \underset{1}{\dot{-}} \left(\gamma_{1243} \underset{34}{\dot{-}} \gamma_{2143}\right)\right),$$

$[X_4, [X_1, [X_3, X_2]]] =$

$$\left(\left(\gamma_{4132} \underset{41}{\dot{-}} \gamma_{4123}\right) \underset{1}{\dot{-}} \left(\gamma_{4321} \underset{41}{\dot{-}} \gamma_{4231}\right)\right) \dot{-} \left(\left(\gamma_{1324} \underset{41}{\dot{-}} \gamma_{1234}\right) \underset{1}{\dot{-}} \left(\gamma_{3214} \underset{41}{\dot{-}} \gamma_{2314}\right)\right),$$

$[X_4, [X_2, [X_1, X_3]]] =$

$$\left(\left(\gamma_{4213}\underset{42}{\overset{\cdot}{-}}\gamma_{4231}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{4132}\underset{42}{\overset{\cdot}{-}}\gamma_{4312}\right)\right)\overset{\cdot}{-}\left(\left(\gamma_{2134}\underset{42}{\overset{\cdot}{-}}\gamma_{2314}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{1324}\underset{42}{\overset{\cdot}{-}}\gamma_{3124}\right)\right),$$

$$[X_4,[X_3,[X_2,X_1]]] =$$

$$\left(\left(\gamma_{4321}\underset{43}{\overset{\cdot}{-}}\gamma_{4312}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{4213}\underset{43}{\overset{\cdot}{-}}\gamma_{4123}\right)\right)\overset{\cdot}{-}\left(\left(\gamma_{3214}\underset{43}{\overset{\cdot}{-}}\gamma_{3124}\right)\underset{1}{\overset{\cdot}{-}}\left(\gamma_{2134}\underset{43}{\overset{\cdot}{-}}\gamma_{1234}\right)\right).$$

□

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