ON AN EQUATION RELATED TO NONADDITIVE ENTROPIES IN INFORMATION THEORY

PREM NATH and DHIRAJ KUMAR SINGH*

Abstract. The paper provides the general solutions of a sum form functional equation containing three unknown mappings with some of the solutions related to the nonadditive entropies in information theory.

1. Introduction

For \( n = 1, 2, \ldots \), let \( \Gamma_n = \left\{ (p_1, \ldots, p_n) : p_i \geq 0, i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1 \right\} \) denote the set of all \( n \)-component finite discrete complete probability distributions with nonnegative elements.

Behara and Nath [1] considered the functional equation

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} f(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} f(p_i) \quad (1.1)
\]

with \( f : I \rightarrow \mathbb{R} \) an unknown mapping, \( I = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \} \), \( \mathbb{R} \) denoting the set of all real numbers; \( (p_1, \ldots, p_n) \in \Gamma_n \), \( (q_1, \ldots, q_m) \in \Gamma_m \); \( n = 1, 2, \ldots \); \( m = 1, 2, \ldots \); \( \alpha \) and \( \beta \) being fixed positive real powers; \( 0^\alpha := 0 \), \( 0^\beta := 0 \) and \( 1^\alpha := 1 \), \( 1^\beta := 1 \). They found the continuous solutions of (1.1).

Losonczi and Maksa ([3], p. 263) considered the functional equation (1.1) with \( f : I \rightarrow \mathbb{R} \) an unknown mapping, \( (p_1, \ldots, p_n) \in \Gamma_n \), \( (q_1, \ldots, q_m) \in \Gamma_m \); \( n \geq 3 \) and \( m \geq 2 \) being fixed integers; \( 1 \neq \alpha \in \mathbb{R} \), \( 1 \neq \beta \in \mathbb{R} \); \( 0^\alpha := 0 \), \( 0^\beta := 0 \); \( 1^\alpha := 1 \), \( 1^\beta := 1 \); and they found the general solutions of (1.1) without imposing any regularity condition on the mapping \( f : I \rightarrow \mathbb{R} \).

The object of this paper is to determine the general solutions of the functional equation

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i) \quad (A)
\]

with \( (p_1, \ldots, p_n) \in \Gamma_n \), \( (q_1, \ldots, q_m) \in \Gamma_m \); \( n \geq 3 \) and \( m \geq 3 \) being fixed integers; \( \beta \) a fixed positive real power satisfying the conventions \( 0^\beta := 0 \), \( 1^\beta := 1 \); \( f, g, h \) being unknown real-valued mappings each with domain \( I \).


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*Corresponding author.
If $g(x) = x$ and $\beta = 1$, then (A) reduces to the equation
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} h(q_j)
\]
which has been discussed in [5].

If $g(x) = x^\alpha$ for all $x \in I$, $\alpha$ being a fixed positive real power; $0^\alpha := 0$, $1^\alpha := 1$, then (A) reduces to the functional equation
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i)
\] (1.2)
which is a Pexider-type generalization of (1.1). Notice that (1.2) reduces to (1.1) when $h(x) = f(x)$ for all $x \in I$. Some results concerning the functional equation (1.2) will be presented elsewhere.

Now, we mention below some definitions and results needed for the development of subsequent sections of this paper.

A mapping $a : \mathbb{R} \to \mathbb{R}$ is said to be additive if $a(x + y) = a(x) + a(y)$ for all $x, y \in \mathbb{R}$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative if $M(xy) = M(x)M(y)$ for all $x, y \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if \( \ell(0) = 0 \) and $\ell(xy) = \ell(x) + \ell(y)$ for all $x \in [0,1]$, $y \in [0,1]$, $[0,1] = \{ x \in \mathbb{R} : 0 < x \leq 1 \}$.

**Result 1.1** ([4]). Let a mapping $\phi : I \to \mathbb{R}$ satisfy the functional equation
\[
\sum_{i=1}^{n} \phi(p_i) = c \quad \text{for all} \quad (p_1, \ldots, p_n) \in \Gamma_n, \quad n \geq 3 \quad \text{a fixed integer and} \quad c \quad \text{a given real constant. Then, there exists an additive mapping} \quad A : \mathbb{R} \to \mathbb{R} \quad \text{such that} \quad \phi(p) = A(p) - \frac{1}{n} A(1) + \frac{c}{n} \quad \text{for all} \quad p \in I.
\]

**Result 1.2** ([6]). Let $n \geq 3$, $m \geq 3$ be fixed integers. If mappings $H : I \to \mathbb{R}$, $G : I \to \mathbb{R}$ satisfy the equation
\[
\sum_{i=1}^{n} \sum_{j=1}^{m} H(p_i q_j) = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} H(p_i) \sum_{j=1}^{m} q_j^\beta \\
+ (m - n)H(0) \sum_{j=1}^{m} q_j^\beta + m(n - 1)H(0)
\] (B)
for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$ a fixed positive real power such that $0^\beta := 0$ and $1^\beta := 1$. Then, any general solution of (B) is of the form
\[
\begin{align*}
(i) \quad & H(p) = [H(1) + (m - 1)H(0)]p^\beta + b_1(p) + H(0); \quad b_1(1) = -mH(0) \\
(ii) \quad & G(p) = a_1(p) + G(0); \quad a_1(1) = -nG(0) \\
& \quad \begin{cases} 
\beta_1 
\end{cases}
\end{align*}
\]
or
\[
\begin{align*}
(i) \quad & H(p) = p^\beta \ell(p) - \bar{b}(p) + H(0); \quad \bar{b}(1) = mH(0) \\
(ii) \quad & G(p) = p^\beta + \bar{a}(p) + G(0); \quad \bar{a}(1) = -nG(0) \\
& \quad \begin{cases} 
\beta_2 
\end{cases}
\end{align*}
\]
The main result of this paper is the following:

\[
\begin{align*}
&\text{(i) } H(p) = b_2(p) + H(0); \quad b_2(1) = -mH(0) \quad \text{(\(\beta_3\))} \\
&\text{(ii) } G \text{ an arbitrary real-valued mapping}
\end{align*}
\]

or

\[
\begin{align*}
&\text{(i) } H(p) = d[a(p) - p^\beta] + H(0); \quad a(1) = 1 - \frac{m}{d}H(0) \\
&\text{(ii) } G(p) = a(p) + \bar{B}(p) + G(0); \quad \bar{B}(1) = -nG(0) + \frac{m}{d}H(0) \quad \text{(\(\beta_4\))}
\end{align*}
\]

or

\[
\begin{align*}
&\text{(i) } H(p) = d[M(p) - b(p) - p^\beta] + H(0); \quad b(1) = \frac{m}{d}H(0) \\
&\text{(ii) } G(p) = M(p) - b(p) + \bar{B}(p) + G(0); \quad \bar{B}(1) = -nG(0) + \frac{m}{d}H(0) \quad \text{(\(\beta_5\))}
\end{align*}
\]

where \(M : I \to \mathbb{R}\) is a nonadditive multiplicative mapping with \(M(0) = 0, M(1) = 1; \ell : I \to \mathbb{R}\) is a logarithmic mapping; \(d\) is an arbitrary nonzero real constant; \(a_1 : \mathbb{R} \to \mathbb{R}, b_i : \mathbb{R} \to \mathbb{R}\) \((i = 1, 2), a : \mathbb{R} \to \mathbb{R}, b : \mathbb{R} \to \mathbb{R}, \) and \(\bar{B} : \mathbb{R} \to \mathbb{R}\) are additive mappings.

Note. If \(G(p) = p^\alpha, \alpha > 0, \alpha \neq 1, p \in I,\) then (B) reduces to the functional equation

\[
\sum_{i=1}^{n} \sum_{j=1}^{m} H(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} H(p_i) \sum_{j=1}^{m} q_j^\beta \\
+ (m-n)H(0) \sum_{j=1}^{m} q_j^\beta + mn(1-H(0)),
\]

which may be regarded as an enlargement of (1.1) (with \(H\) in place of \(f\)). Its solutions will be presented elsewhere.

2. The main result

The main result of this paper is the following:

Theorem 2.1. Let \(n \geq 3, m \geq 3\) be fixed integers and \(f : I \to \mathbb{R}, g : I \to \mathbb{R}, h : I \to \mathbb{R}\) be mappings which satisfy the functional equation (A) for all \((p_1, \ldots, p_n) \in \Gamma_n, (q_1, \ldots, q_m) \in \Gamma_m; \beta \neq 1\) being a fixed positive real power such that \(0^\beta := 0, 1^\beta := 1.\) Then, any general solution \((f, g, h)\) of (A) is of the form

\[
\begin{align*}
&\text{(i) } f(p) = [h(1) + (n-1)h(0)]p^\beta + A_1(p) + f(0) \\
&\text{(ii) } g(p) = A_2(p) + g(0) \\
&\text{(iii) } h(p) = [h(1) + (n-1)h(0)]p^\beta + \bar{A}(p) + h(0) \quad \text{(\(\alpha_1\))}
\end{align*}
\]

or

\[
\begin{align*}
&\text{(i) } f(p) = [g(1) + (n-1)g(0)][[h(1) + (m-1)h(0)]p^\beta + A_3(p)] \\
&\quad + [h(1) + (n-1)h(0)]p^\beta + A_1(p) + f(0) \\
&\text{(ii) } g \text{ an arbitrary real-valued mapping} \\
&\text{(iii) } h(p) = [h(1) + (m-1)h(0)]p^\beta + A_3(p) + h(0) \quad \text{(\(\alpha_2\))}
\end{align*}
\]
or

\[
\begin{align*}
(i) \quad f(p) &= \{[g(1) + (n-1)g(0)]\ell(p) + [f(1) + (nm-1)f(0)]p^\beta \\
& \quad - [g(1) + (n-1)g(0)]\bar{b}(p) + A_1(p) + f(0) \\
(ii) \quad g(p) &= [g(1) + (n-1)g(0)][p^\beta + \bar{a}(p)] + g(0) \\
(iii) \quad h(p) &= \{\ell(p) + [h(1) + (m-1)h(0)]\}p^\beta - \bar{b}(p) + h(0)
\end{align*}
\]

(\alpha_3)

with \([g(1) + (n-1)g(0)] \neq 0\) or

\[
\begin{align*}
(i) \quad f(p) &= \{[f(1) + (nm-1)f(0)] - d[g(1) + (n-1)g(0)]p^\beta \\
& \quad + d[g(1) + (n-1)g(0)]a(p) + A_1(p) + f(0) \\
(ii) \quad g(p) &= [g(1) + (n-1)g(0)][a(p) + \bar{B}(p)] + g(0) \\
(iii) \quad h(p) &= \{[h(1) + (m-1)h(0)] - d\}p^\beta + da(p) + h(0)
\end{align*}
\]

(\alpha_4)

with \([g(1) + (n-1)g(0)] \neq 0\) or

\[
\begin{align*}
(i) \quad f(p) &= d[g(1) + (n-1)g(0)][M(p) - b(p)] + \{[f(1) + (nm-1)f(0)] \\
& \quad - d[g(1) + (n-1)g(0)]p^\beta + A_1(p) + f(0) \\
(ii) \quad g(p) &= [g(1) + (n-1)g(0)][M(p) - b(p) + \bar{B}(p)] + g(0) \\
(iii) \quad h(p) &= d[M(p) - b(p)] + \{[h(1) + (m-1)h(0)] - d\}p^\beta + h(0)
\end{align*}
\]

(\alpha_5)

with \([g(1) + (n-1)g(0)] \neq 0\). Moreover,

\[
\begin{align*}
(i) \quad [g(1) + (n-1)g(0)] - [h(p) - h(0)] \\
& = [h(1) + (m-1)h(0)][g(p) - g(0)] - [h(1) + (n-1)h(0)]p^\beta + B(p) \\
(ii) \quad f(1) + (nm-1)f(0) &= [g(1) + (n-1)g(0)][h(1) + (m-1)h(0)] \\
& \quad + [h(1) + (n-1)h(0)];
\end{align*}
\]

(\gamma)

\(M : I \to \mathbb{R}\) is a nonadditive multiplicative mapping with \(M(0) = 0\) and \(M(1) = 1\); \(\ell : I \to \mathbb{R}\) is a logarithmic mapping; \(d \neq 0\) is an arbitrary real constant; \(A_i : \mathbb{R} \to \mathbb{R}\) \((i = 1, 2, 3)\), \(\bar{b} : \mathbb{R} \to \mathbb{R}\), \(\bar{a} : \mathbb{R} \to \mathbb{R}\), \(a : \mathbb{R} \to \mathbb{R}\), \(b : \mathbb{R} \to \mathbb{R}\), \(A : \mathbb{R} \to \mathbb{R}\), \(B : \mathbb{R} \to \mathbb{R}\).
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By Result 1.1, there exists an additive mapping $B : \mathbb{R} \to \mathbb{R}$ such that $(\gamma)(i)$ holds with $B(1)$ given by $(\delta)(ix)$.

From equations (2.2) and (2.3), we obtain

$$[g(1) + (n - 1)g(0)] \sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j) \sum_{j=1}^{m} \{h(q_j) - [h(1) + (m - 1)h(0)]q_j^\beta\}$$

$$+ [g(1) + (n - 1)g(0)] \left\{ \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} q_j^\beta + (m - n)h(0) \sum_{j=1}^{m} q_j^\beta + m(n - 1)h(0) \right\}.$$  

(2.4)

Case 1. $g(1) + (n - 1)g(0) = 0$.

In this case, equation (2.4) reduces to the equation

$$\sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} \{h(q_j) - [h(1) + (m - 1)h(0)]q_j^\beta\} = 0 \quad (2.5)$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n, (q_1, \ldots, q_m) \in \Gamma_m; n \geq 3, m \geq 3$ being fixed integers.

Consider the situation of $\sum_{i=1}^{n} g(p_i) = 0$ for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 1.1, there exists an additive mapping $A_2 : \mathbb{R} \to \mathbb{R}$ such that $(\alpha_1)(ii)$ follows with $A_2(1)$ given by $(\delta)(ii)$. Also, from $(\alpha_1)(ii)$ and (2.1); $(\alpha_1)(i)$ follows with $A_1(1) = -nmf(0)$ which can be obtained from $(\delta)(i)$ when $g(1) + (n - 1)g(0) = 0$. Now, let us put $q_1 = 1, q_2 = \ldots = q_m = 0$ in (A) and use $(\alpha_1)(i), (\alpha_1)(ii)$. We obtain

$$\sum_{i=1}^{n} \{h(p_i) - [h(1) + (n - 1)h(0)]p_i^\beta\} = 0.$$  

By Result 1.1, there exists an additive mapping $\tilde{A} : \mathbb{R} \to \mathbb{R}$ such that $(\alpha_1)(iii)$ holds with $\tilde{A}(1)$ given by $(\delta)(x)$. So, the solution $(\alpha_1)$ has been obtained with $A_1(1) = -nmf(0)$. Now, consider the situation of $\sum_{j=1}^{m} \{h(q_j) - [h(1) + (m - 1)h(0)]q_j^\beta\} = 0$ for all $(q_1, \ldots, q_m) \in \Gamma_m$.

Making use of Result 1.1, it follows that there exists an additive mapping $A_3 : \mathbb{R} \to \mathbb{R}$ with $A_3(1)$ given by $(\delta)(iii)$, such that $(\alpha_2)(iii)$ holds. Now, from $(\alpha_2)(iii)$ and (2.5), one can easily conclude that $g$ is an arbitrary real-valued mapping with $g(1) + (n - 1)g(0) = 0$. Also, from (2.1), it follows that $f(p) = [h(1) + (n - 1)h(0)]p^\beta + A_1(p) + f(0)$ with $A_1(1) = -nmf(0)$. This solution is included in $(\alpha_2)$.

Case 2. $g(1) + (n - 1)g(0) \neq 0$.

In this case, let us write (2.4) in the form

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \{h(p_i q_j) - [h(1) + (m - 1)h(0)]p_i q_j^\beta\}$$

$$= \sum_{i=1}^{n} \{[g(1) + (n - 1)g(0)]^{-1}g(p_i)\} \sum_{j=1}^{m} \{h(q_j) - [h(1) + (m - 1)h(0)]q_j^\beta\}.$$
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\[ + \sum_{i=1}^{n} \{h(p_i) - [h(1) + (m - 1)h(0)]p_i^\beta \} \sum_{j=1}^{m} q_j^\beta \]

\[ + (m - n)h(0) \sum_{j=1}^{m} q_j^\beta + m(n - 1)h(0). \]  \hfill (2.6)

Define the mappings \( G : I \to \mathbb{R} \) and \( H : I \to \mathbb{R} \) as

\[ G(p) = [g(1) + (n - 1)g(0)]^{-1}g(p) \]  \hfill (2.7)

and

\[ H(p) = h(p) - [h(1) + (m - 1)h(0)]p^\beta \]  \hfill (2.8)

for all \( p \in I \). From (2.7) and (2.8), it is easy to conclude that

\[ G(0) = \frac{g(0)}{g(1) + (n - 1)g(0)}, \quad g(1) + (n - 1)g(0) \neq 0 \]  \hfill (2.9)

\[ G(1) + (n - 1)G(0) = 1 \]  \hfill (2.10)

\[ H(0) = h(0) \]  \hfill (2.11)

and

\[ H(1) + (m - 1)H(0) = 0. \]  \hfill (2.12)

Also, from (2.6), (2.7) and (2.8), equation (B) follows. Therefore, in Result 1.2, we have to consider only those solutions which satisfy both (2.10) and (2.12). We reject \((\beta_1)\) because, in this case, (2.10) does not hold. Both (2.10) and (2.12) hold in \((\beta_2), (\beta_4)\) and \((\beta_5)\). In \((\beta_3), (2.12)\) holds but \((\beta_3)\) can be accepted provided we consider only those arbitrary mappings \( G \) which also satisfy (2.10). This requirement is met by any arbitrary mapping \( g \) which satisfies the condition \([g(1) + (n - 1)g(0)] \neq 0\). This fact is obvious from (2.7). Keeping in view these observations, \((\alpha_2)\) with \([g(1) + (n - 1)g(0)] \neq 0\) and \((\delta)(\text{i}), (\text{iii})\) follows from \((\beta_3), (2.7), (2.8), (2.9), (2.11)\) and (2.1). Moreover,

\((a_1)\) The solution \((\alpha_3)\) with \([g(1) + (n - 1)g(0)] \neq 0\) and \((\gamma)(\text{ii}), (\delta)(\text{i}), (\text{vi}), (\text{vii})\) follows from \((\beta_2), (2.7), (2.8), (2.9), (2.11)\) and (2.1);

\((a_2)\) The solution \((\alpha_4)\) with \([g(1) + (n - 1)g(0)] \neq 0\) and \((\gamma)(\text{ii}), (\delta)(\text{i}), (\text{iv}), (\text{viii})\) follows from \((\beta_4), (2.7), (2.8), (2.9), (2.11)\) and (2.1);

\((a_3)\) The solution \((\alpha_5)\) with \([g(1) + (n - 1)g(0)] \neq 0\) and \((\gamma)(\text{ii}), (\delta)(\text{i}), (\text{v}), (\text{viii})\) follows from \((\beta_5), (2.7), (2.8), (2.9), (2.11)\) and (2.1).

Making use of \((\gamma)(\text{i}), (\text{ii})\), it can be verified that \((\alpha_1)\) to \((\alpha_5)\) are, indeed, the solutions of (A).

\(\square\)

3. Comments

The object of this section is to comment upon various solutions, mentioned in Theorem 2.1, from the point of view of information theory.
Behara and Nath [1] have defined the entropy $H_n^{(\alpha, \beta)}(p_1, \ldots, p_n)$ of type $(\alpha, \beta)$ ($H_n^{(\alpha, \beta)} : \Gamma_n \to \mathbb{R}, n = 1, 2, \ldots$) as

$$H_n^{(\alpha, \beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1} \left( \sum_{i=1}^{n} p_i^\alpha - \sum_{i=1}^{n} p_i^\beta \right) & \text{if } \alpha \neq \beta \\ -2^{\beta-1} \sum_{i=1}^{n} p_i^\beta \log_2 p_i & \text{if } \alpha = \beta \end{cases}$$

(3.1)

where $0^\beta \log_2 0 := 0$ and $\alpha, \beta$ are fixed positive real powers which satisfy $0^\alpha := 0, 0^\beta := 0, 1^\alpha := 1, 1^\beta := 1$. Havrda and Charvát [2] have defined the entropies of degree $\beta$, $0 < \beta \in \mathbb{R}, \beta \neq 1$ as

$$H_n^\beta(p_1, \ldots, p_n) = (1 - 2^{1-\beta})^{-1} \left[ 1 - \sum_{i=1}^{n} p_i^\beta \right]$$

(3.2)

with $H_n^\beta : \Gamma_n \to \mathbb{R}, n = 1, 2, \ldots$ and $0^\beta := 0, 1^\beta := 1$. Both the entropies mentioned in (3.1) and (3.2) are nonadditive.

I. In the solution $(\alpha_1)$, $\sum_{i=1}^{n} g(p_i) = 0$ follows. Then, equation (A) reduces to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i).$$

(3.3)

The functional equation (3.3) has been discussed by Nath and Singh [6].

II. In the solution $(\alpha_2)$, the mapping $g$ is arbitrary. So, there are three possibilities: $g$ is additive or $g$ is multiplicative or $g$ is logarithmic.

Let us consider the case of $g$ being additive. If $g(p) \equiv 0$, then $g$ is certainly additive. In this case, (A) reduces to (3.3). In general, when $g$ is additive, (A) reduces to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = g(1) \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i).$$

(3.4)

If $g(1) = 0$, we get the functional equation (3.3) again. If $g(1) = 1$, then (3.4) reduces to the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i).$$

(3.5)

Consider the case of $g$ being multiplicative. Here, we discuss some particular cases.

(i) $g(x) \equiv 0$. In this case, we again get (3.3).

(ii) $g(x) \equiv 1$. In this case, (A) reduces to

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = n \sum_{j=1}^{m} h(q_j) + \sum_{j=1}^{m} q_j^\beta \sum_{i=1}^{n} h(p_i)$$

where $n = 1, 2$ are not admissible because we assume $n \geq 3$ throughout the paper.

(iii) $g(x) = x$ for all $x \in I$. In this case, equation (A) reduces to (3.5).
In each case, the functional equation (3.3) arises. Such a discussion concerning (3.6), we have \( \lambda \) where \( \ell \). Let us choose the logarithmic mapping (3).

Finally, we consider the case of (v) Consider the situation \( g(p) = \left\{ \begin{array}{ll} 0 & \text{if } 0 \leq p < 1 \\ 1 & \text{if } p = 1 \end{array} \right. \) In this case, \( \sum_{i=1}^{n} g(p_i) = 1 \) if \( p_i = 1 \) for exactly one \( i, 1 \leq i \leq n \) and 0 if \( 0 \leq p_i < 1, i = 1, 2, \ldots, n \).

Finally, we consider the case of \( g(p) = \ell(p) \) where \( \ell \) is a logarithmic mapping. If \( \ell(p) \equiv 0 \), then (A) reduces to (3.3).

Notice that, if \( g(p) \equiv 0 \), then \( g \) is additive, multiplicative and logarithmic. In each case, the functional equation (3.3) arises. Such a discussion concerning \( g(p) \equiv 0 \) is needed for the sake of completeness in relation to solution (a2).

III. From the solution (a3), it can be easily derived that for all \( (p_1, \ldots, p_n) \in \Gamma_n \),

\[
\sum_{i=1}^{n} f(p_i) = [g(1) + (n-1)g(0)] \sum_{i=1}^{n} p_i^\beta \ell(p_i) + [f(1) + (nm-1)f(0)] \sum_{i=1}^{n} p_i^\beta
\]

\[- n(m-1)f(0);\]

\[
\sum_{i=1}^{n} g(p_i) = [g(1) + (n-1)g(0)] \sum_{i=1}^{n} p_i^\beta;\]

\[
\sum_{i=1}^{n} h(p_i) = \sum_{i=1}^{n} p_i^\beta \ell(p_i) + [h(1) + (m-1)h(0)] \sum_{i=1}^{n} p_i^\beta + (n-m)h(0).\]

Let us choose the logarithmic mapping \( \ell : I \rightarrow \mathbb{R} \) as

\[
\ell(p) = \begin{cases} 
\lambda \log_2 p & \text{if } 0 < p \leq 1 \\
0 & \text{if } p = 0
\end{cases}
\]  \hspace{1cm} (3.6)

where \( \lambda \neq 0 \) is an arbitrary real constant. Then, making use of (3.1), (3.2) and (3.6), we have

\[
\sum_{i=1}^{n} f(p_i) = -[g(1) + (n-1)g(0)]\lambda 2^{1-\beta} H_n^{(\beta,\beta)}(p_1, \ldots, p_n)
\]
Thus, we observe that the mappings \( f_\alpha > 0 \), \( f_\beta > 0 \) of the solution \((\alpha, \beta)\) when \( \alpha \neq 1 \), \( \alpha \neq \beta \), then, from solution \((\alpha_5)\), we obtain

\[
\sum_{j=1}^{m} f(q_j) = d[g(1) + (n - 1)g(0)](2^{1-\alpha} - 2^{1-\beta})H_{m}^{(\alpha, \beta)}(q_1, \ldots, q_m)
- [f(1) + (nm - 1)f(0)](1 - 2^{1-\beta})H_{m}^{(\alpha, \beta)}(q_1, \ldots, q_m)
+ [f(1) + (m - 1)f(0)];
\]

\[
\sum_{j=1}^{m} g(q_j) = [g(1) + (n - 1)g(0)][-(1 - 2^{1-\alpha})H_{m}^{(\alpha, \beta)}(q_1, \ldots, q_m) + 1] + (m - n)g(0);
\]

\[
\sum_{j=1}^{m} h(q_j) = d(2^{1-\alpha} - 2^{1-\beta})H_{m}^{(\alpha, \beta)}(q_1, \ldots, q_m)
- [h(1) + (m - 1)h(0)][(1 - 2^{1-\beta})H_{m}^{(\alpha, \beta)}(q_1, \ldots, q_m) - 1].
\]

Thus, we observe that the mappings \( f \) and \( h \) are related to both the entropies \( H_{n}^{(\beta, \beta)} \) and \( H_{n}^{\beta} \), whereas the mapping \( g \) is related only to the entropies \( H_{n}^{\beta} \).

**IV.** From the solution \((\alpha_4)\), for all \((p_1, \ldots, p_n) \in \Gamma_n\), it can be derived that

\[
\sum_{i=1}^{n} f(p_i) = \{(f(1) + (nm - 1)f(0)) - d[g(1) + (n - 1)g(0)]\}
\times [-(1 - 2^{1-\beta})H_{m}^{(\beta, \beta)}(p_1, \ldots, p_n) + 1] + d[g(1) + (n - 1)g(0)]
- n(m - 1)f(0);
\]

\[
\sum_{i=1}^{n} g(p_i) = [g(1) + (n - 1)g(0)];
\]

\[
\sum_{i=1}^{n} h(p_i) = \{[h(1) + (m - 1)h(0)] - d\}[-(1 - 2^{1-\beta})H_{m}^{\beta}(p_1, \ldots, p_n) + 1]
+ d + (n - m)h(0).
\]

Here, observe that the mappings in the solution \((\alpha_5)\) are related to the entropies of type \((\alpha, \beta)\) when \( \alpha \neq \beta \); the entropies of degree \( \alpha \) and the entropies of degree \( \beta \).
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References


Prem Nath, Department of Mathematics, University of Delhi, Delhi–110007, India
e-mail: pmathmaths@gmail.com

Dhiraj Kumar Singh, Department of Mathematics, Zakir Husain Delhi College (University of Delhi), Jawaharlal Nehru Marg, Delhi–110002, India
e-mail: dhiraj426@rediffmail.com, dksingh@zh.du.ac.in