

CLASSIFICATION OF ORTHOGROUPS AS SEMILATTICES OF VARIOUS RECTANGULAR GROUPS FULFILLING $x^{p+1} \approx x$

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Abstract. Completely regular semigroups whose idempotents form a subsemigroup are called orthodox completely regular semigroups, or briefly orthogroups. In this paper, we characterize orthogroups which are semilattices of various rectangular groups fulfilling $x^{p+1} \approx x$ for a fixed prime number p by disjunctions of identities.

1. INTRODUCTION

A semigroup S is called a *completely regular semigroup* if S is a union of groups. Due to the work of Clifford ([2]), Fantham ([5]), Petrich ([10–12]) and others, there are a great deal of facts known about this special class of semigroups. Completely regular semigroups whose idempotents form a subsemigroup are called *orthodox completely regular semigroups*, or briefly *orthogroups*. For this special class of semigroups, there is a characterization given by Yamada ([16]), which uses a more general result of Clifford ([3]). Yamada showed that an orthogroup is a semilattice of *rectangular groups*. In particular, any band is a semilattice of rectangular bands as McLean has proven ([8]).

In 2013, R. A. R. Monzo ([9]) characterized a semilattice of rectangular bands and groups with exponent 2 by the so called *inclusion classes*. This improves the decomposition of orthogroups in a special case (see also [1], [14]).

At this point, it seems to be natural to ask for the characterization of semigroups which are semilattices of certain classes of orthogroups. We will introduce some notation for classes of orthogroups, to be used in this paper:

- L_0 : the trivial semigroup,
- L_1 : class of all right zero semigroups,
- L_2 : class of all left zero semigroups,
- L_3 : class of all rectangular bands,
- L_4 : class of all groups with prime exponent p ,
- L_5 : class of all right groups with prime exponent p ,
- L_6 : class of all left groups with prime exponent p ,
- L_7 : class of all rectangular groups with exponent p ,

where $p \in \mathbb{N}$ is an arbitrary but fixed prime number. Based on the result by Monzo, for a fixed prime number p and a nonempty subset $I \subseteq \{1, 2, 3, 4, 5, 6\}$, we

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characterize the class C of all semigroups S such that every S is a semilattice Y of semigroups S_α with $S_\alpha \in \bigcup_{i \in I} L_i$ for $\alpha \in Y$ by *disjunctions of identities*. More precisely, we give a set $A = \{u_j \approx v_j : j \in J\}$ where the equations $u_j \approx v_j$, $j \in J$ are generated from the variables x and y such that

$$C = MOD(A, \{x^{p+1} \approx x\})$$

holds.

Thus, we start our investigations in section 2 with some basic definitions and results concerning semilattices of semigroups, completely simple semigroups (in particular Rees matrix semigroups) and disjunctions of identities. In section 3, we give some conditions for sets of equations to describe orthodox semigroups fulfilling $x^{p+1} \approx x$. Furthermore, we investigate some properties of disjunctions of identities and concentrate on their connection to classes of rectangular groups. Finally, we prove the main result by giving an explicit characterization.

2. PRELIMINARIES

For the standard terminology and notation of semigroup theory, we refer the reader to [6]. In particular, we denote by E_S the set of all idempotent elements for a semigroup S .

Let A be a non-empty subset of a semigroup S , then A is called *left ideal* if $s \in S$, $a \in A$ imply $sa \in A$; A is called a *right ideal* if $s \in S$, $a \in A$ imply $as \in A$; and A is called an *ideal*, if A is both a left ideal and a right ideal. If S is the only ideal of the semigroup S and there exists an element $e \in E_S$ such that, whenever $ef = fe = f$ for an element $f \in E_S$, we have $e = f$, then S is called a *completely simple semigroup*. For investigations of this latter kind of semigroups, we use the well-known result of Suschkewitsch (1928) based on the Rees Theorem ([13]):

Theorem 2.1. [6] *Let G be a group, let I, Λ be non-empty sets and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G . Let $S = (I \times G \times \Lambda)$, with a multiplication on S defined by*

$$(i, a, \lambda)(j, b, \mu) := (i, ap_{\mu j}b, \mu).$$

Then, S is a completely simple semigroup.

Conversely, every completely simple semigroup is isomorphic to a semigroup constructed in this way.

These so defined semigroups are called *Rees matrix semigroups (over the group G)*. Because of our focus on orthodox semigroups, there is a simpler form of the previous theorem:

Theorem 2.2. *Let G be a group, let I, Λ be non-empty sets. Let $S = I \times G \times \Lambda$, and define a multiplication on S by*

$$(i, a, \lambda)(j, b, \mu) = (i, ab, \mu).$$

Then, S is an orthodox completely simple semigroup.

Conversely, every orthodox completely simple semigroup is isomorphic to a semigroup constructed in this way.

An orthodox completely simple semigroup is also called a *rectangular group*. So, every $S \in L_i$ for $i \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ is isomorphic to an orthodox Rees matrix semigroup. We can easily see that the previous set of semigroup classes is partially ordered, and thus can be illustrated in the following way:

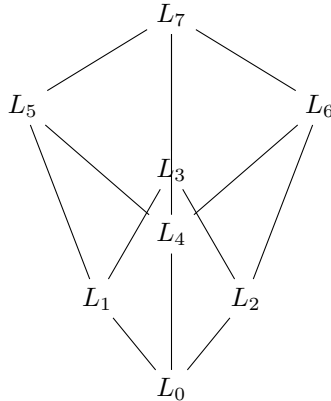


Figure 1. Lattice of elements in L_7 .

Recall that a semigroup S is called a *semilattice* if S is commutative and every element is idempotent. A semigroup S is a semilattice Y of semigroups S_α if there exists a homomorphism ϕ of S onto the semilattice Y such that $S_\alpha = \phi^{-1}(\alpha)$ for all $\alpha \in Y$. The link between the completely simple semigroups and the completely regular semigroups is a fundamental theorem by Clifford, which is basic for the below results:

Theorem 2.3. [6] *Every completely regular semigroup is a semilattice of completely simple semigroups.*

For more detailed information on completely regular semigroups and orthogroups, we refer the reader to [6].

Our classification of particular orthogroups as semilattices of special rectangular groups is based on the concept of *disjunctions of identities*. The following definition of this concept is taken from Thron and Koppitz ([15]) and related to the inclusion classes of Monzo as well as to the definitions by Ljapin ([7]) and Evseev ([4]).

Let $X = \{x, y, z, \dots\}$ be a non-empty set and let X^+ be the set of all words generated by the elements of X . For every semigroup S and every map $h : X \rightarrow S$, there is a unique extension of h to a homomorphism $\bar{h} : X^+ \rightarrow S$.

A set $D \subseteq X^+ \times X^+$ is said to be a *disjunction of identities* in S (in symbols: $S \vDash D$) if for all maps $h : X \rightarrow S$ there is $u \approx v \in D$ with $\bar{h}(u) = \bar{h}(v)$ where we use $u \approx v$ instead of $(u, v) \in X^+ \times X^+$. A set $\mathcal{D} \subseteq \mathcal{P}(X^+ \times X^+)$ holds in S (in symbols: $S \in MOD(\mathcal{D})$) if $S \vDash D$ for all $D \in \mathcal{D}$.

Let $u \in X^+$ be a term with $u = u_1 \dots u_m$ for $u_1, \dots, u_m \in X$ and $m \in \mathbb{N}$. Then, the *leftmost variable* of u is defined by $leftmost(u) := u_1$. Dually, we have

the definition of the *rightmost variable* of u by $\text{rightmost}(u) := u_m$. Furthermore, the *length* of u is given by $l(u) := m$. The *change of a word* u is defined by

$$r(u) := |\{i \in \{1, \dots, m-1\} : u_i \neq u_{i+1}\}|.$$

We use this latter notion to describe the following properties on $X^+ \times X^+$:

$$\begin{aligned} P_1 &= \{u \approx v \in X^+ \times X^+ : \text{leftmost}(u) = \text{leftmost}(v)\}, \\ P_2 &= \{u \approx v \in X^+ \times X^+ : \text{rightmost}(u) = \text{rightmost}(v)\}, \\ P_3 &= \{u \approx v \in X^+ \times X^+ : l(u) \equiv l(v) \pmod{p}\}, \\ P_4 &= \{u \approx v \in X^+ \times X^+ : r(u) \equiv r(v) \pmod{p}\}. \end{aligned}$$

In the following investigations, we only use a two-element set X with the elements x and y , i.e. $X = \{x, y\}$.

3. CLASSIFICATION OF ORTHOGROUPS AS SEMILATTICES OF VARIOUS RECTANGULAR GROUPS FULFILLING $x^{p+1} \approx x$

Lemma 3.1. *Let u be a word built up by the variables x and y .*

If $\text{leftmost}(u) \neq \text{rightmost}(u)$, then $r(u)$ is odd.

If $\text{leftmost}(u) = \text{rightmost}(u)$, then $r(u)$ is even.

Proof. Without loss of generality, assume in the first case $\text{leftmost}(u) = x$. This gives us

$$u = x^{k_1} y^{k_2} \dots x^{k_{r(u)}} y^{k_{r(u)+1}}$$

for $k_1, \dots, k_{r(u)+1} \in \mathbb{N}$, and so $r(u) + 1$ is an even number. Hence $r(u)$ is odd. In an analogous way, we prove the second statement. \square

The following Lemma is based upon some remarks by Howie ([6]):

Lemma 3.2. *Let S be a semigroup with $S \models \{x^{p+1} \approx x\}$, then S is completely regular.*

Proof. Let $S \models \{x^{p+1} \approx x\}$. Then, for an arbitrary mapping $h : \{x, y\} \rightarrow S$ with $x \mapsto a$ for some $a \in S$, there is $\bar{h}(x) = \bar{h}(x^{p+1})$. Hence, we observe $a = a^{p+1}$ and so $a = a^{qp+1}$ for $q \geq 0$. For every $s \in \mathbb{N}$, we can write $s = 1 + qp + u$ where $q \geq 0$ and $0 \leq u \leq p-1$, so it follows $a^s = a^{1+qp+u} = a^{1+qp} a^u = a^{1+u}$.

For the element a , we investigate the set of all powers $K_a = \{a, a^2, a^3, \dots\}$. Using the previous facts, K_a can be limited to $K_a = \{a, a^2, \dots, a^p\}$. Let $a^{u+1}, a^{v+1} \in K_a$ be arbitrary elements with $0 \leq u, v \leq p-1$, then we can find an element a^{w+1} in K_a such that $a^{u+1} a^{w+1} = a^{v+1}$ choosing $w \equiv v - u - 1 \pmod{p}$ with $0 \leq w \leq p-1$. Obviously, K_a is a subgroup of S and so S is an union of groups. Hence, S is a completely regular semigroup. \square

Lemma 3.3. *Let $A := \{u_j \approx v_j : j \in J\}$ be a disjunction of identities where the equations $u_j \approx v_j \in A, j \in J$, built up by the variables x and y . Furthermore, let $u_j \approx v_j \notin P_1 \cap P_2 \cap P_3$ for $j \in J$. If there is at most one $j \in J$ with $u_j \approx v_j \in P_1 \cap P_2$ and $S \in (A, \{x^{p+1} \approx x\})$, then S is orthodox.*

Proof. Let $e, f \in S$ be idempotents. Suppose $u_j \approx v_j \notin P_1 \cap P_2 \cap P_4$ for all $j \in J$. Then, we consider the mapping $h : \{x, y\} \mapsto S$ defined by $h(x) = e$ and $h(y) = f$. So, there is an $j \in J$ such that $\bar{h}(u_j) = \bar{h}(v_j)$.

Admit $u_j \approx v_j \in P_1 \cap P_2$, then we get $u_j \approx v_j \notin P_4$. We put $u := xu_jy$ and $v := xv_jy$. Hence, without loss of generality, we can assume $leftmost(u) = x$ and $rightmost(u) = y$ such that $\bar{h}(u) = (ef)^{\frac{r(u)+1}{2}}$ and $\bar{h}(v) = (ef)^{\frac{r(v)+1}{2}}$. The last fact works due to Lemma 3.1. Since $u_j \approx v_j \notin P_4$, we conclude $u \approx v \notin P_4$. In addition, we have $ef = (ef)^{p+1}$. Thus, $(ef)^{r(u)+1} = (ef)^{r(v)+1}$ by $\bar{h}(u)^2 = \bar{h}(v)^2$ provides $ef = (ef)^2$.

Admit $u_j \approx v_j \notin P_1$. Without loss of generality, let $x = leftmost(u_j) \neq leftmost(v_j) = y$. From $\bar{h}(u_j) = \bar{h}(v_j)$, we obtain $f\bar{h}(u_j) = \bar{h}(v_j)$, i.e. $f\bar{h}(u_j) = \bar{h}(u_j)$ and so $ef\bar{h}(u_j) = \bar{h}(u_j)$. It is easy to see that there is a number $r \in \mathbb{N}$ such that $\bar{h}(u_j)f = (ef)^r \bar{h}(u_j)f = (ef)^{p+1} = ef$. Hence, $ef = \bar{h}(u_j)f = ef\bar{h}(u_j)f = (ef)^2$.

Admit $u_j \approx v_j \notin P_2$. Then, we get $ef = (ef)^2$ by dual arguments.

Suppose $u_j \approx v_j \notin (P_1 \cap P_2) \setminus P_4$ for all $j \in J$. Then, we consider the mapping $h : \{x, y\} \mapsto S$ defined by $h(x) = efe$ and $h(y) = fef$. So, there is an element $j \in J$ such that $\bar{h}(u_j) = \bar{h}(v_j)$.

Admit $u_j \approx v_j \in P_1 \cap P_2$, then we get $u_j \approx v_j \in P_4$. As already mentioned above, without loss of generality, we can assume that $leftmost(u_j) = x$ and $rightmost(u_j) = y$ such that $\bar{h}(u_j) = (ef)^{\frac{r(u_j)+1}{2} + l(u_j)}$ and $\bar{h}(v_j) = (ef)^{\frac{r(v_j)+1}{2} + l(v_j)}$ with $u_j \approx v_j \in P_4$. Furthermore, we have $ef = (ef)^{p+1}$, which provides us $(ef)^{l(u_j)} = (ef)^{l(v_j)}$. So, we get $ef = (ef)^2$ because $u_j \approx v_j \notin P_3$ and $ef = (ef)^{p+1}$.

Admit $u_j \approx v_j \notin P_1 \cap P_2$, then we get $ef = (ef)^2$ by the same arguments as for the case $h(x) = e$ and $h(y) = f$.

Notice that, since there is $u_j \approx v_j \in P_1 \cap P_2$ for at most one $j \in J$, all possible cases are investigated. To sum up, it has been shown that $ef \in E_S$ for $e, f \in E_S$, and so the semigroup S is orthodox. \square

Lemma 3.4. *Let S be an orthodox Rees matrix semigroup over the group G with $G \in L_4$. Further, let $h : \{x, y\} \mapsto S$ and let $u \approx v$ be an equation built up by the variables x and y such that $h(x) = (i_1, g, \lambda_1)$, $h(y) = (i_2, g, \lambda_2)$ and $\bar{h}(u) = \bar{h}(v)$. Then, the following statements are true:*

- (1) *If $u \approx v \notin P_1$, then $i_1 = i_2$,*
- (2) *If $u \approx v \notin P_2$, then $\lambda_1 = \lambda_2$,*
- (3) *If $u \approx v \notin P_3$, then $g = e$ for the identity e of G .*

Proof. (1) Without loss of generality, we assume that $leftmost(u) = x$ and thus $leftmost(v) = y$. Hence, we get $(i_1, g^{l(u)}, \mu_1) = \bar{h}(u) = \bar{h}(v) = (i_2, g^{l(v)}, \mu_2)$ for $\mu_1, \mu_2 \in \{\lambda_1, \lambda_2\}$. This implies $i_1 = i_2$.

(2) This is done dually.

(3) We get by $\bar{h}(u) = \bar{h}(v)$ the equality $(j_1, g^{l(u)}, \mu_1) = (j_2, g^{l(v)}, \mu_2)$ with $j_1, j_2 \in \{i_1, i_2\}$ and $\mu_1, \mu_2 \in \{\lambda_1, \lambda_2\}$. Then, $g^{l(u)} = g^{l(v)}$. It is well

known in group theory that $g = g^{p+1}$ and $g^{l(u)} = g^{l(v)}$ implies that g is the identity in G whenever p is a prime number and $u \approx v \notin P_3$. \square

Definition 3.5. Let

$$\Phi : \{x, y\}^+ \times \{x, y\}^+ \longrightarrow \{0, 1, 2, 3, 4, 5, 6, 7\}$$

be a mapping defined by

$$\begin{aligned} \Phi(u \approx v) &:= 0 \text{ if } u \approx v \notin P_1 \cup P_2 \cup P_3, \\ \Phi(u \approx v) &:= 1 \text{ if } u \approx v \in P_2 \setminus (P_1 \cup P_3), \\ \Phi(u \approx v) &:= 2 \text{ if } u \approx v \in P_1 \setminus (P_2 \cup P_3), \\ \Phi(u \approx v) &:= 3 \text{ if } u \approx v \in (P_1 \cap P_2) \setminus P_3, \\ \Phi(u \approx v) &:= 4 \text{ if } u \approx v \in P_3 \setminus (P_1 \cup P_2), \\ \Phi(u \approx v) &:= 5 \text{ if } u \approx v \in (P_2 \cap P_3) \setminus P_1, \\ \Phi(u \approx v) &:= 6 \text{ if } u \approx v \in (P_1 \cap P_3) \setminus P_2, \\ \Phi(u \approx v) &:= 7 \text{ if } u \approx v \in P_1 \cap P_2 \cap P_3, \end{aligned}$$

for all $u, v \in \{x, y\}^+$.

It is easy to see that the mapping Φ is well-defined.

Lemma 3.6. Let $A := \{u_j \approx v_j : j \in J\}$ be a disjunction of identities where the equations $u_j \approx v_j$, $j \in J$, built up by the variables x and y . If S is a non-trivial rectangular group with

$$S \in \text{MOD}(A, \{x^{p+1} \approx x\}),$$

then

$$S \in \bigcup_{j \in J} L_{\Phi(u_j \approx v_j)}.$$

Proof. Since $S \in \text{MOD}(\{x^{p+1} \approx x\})$ is a rectangular group, S is isomorphic to an orthodox Rees matrix semigroup over the group G with $G \in L_4$, i.e. $S \in L_7$. Assume that $S \notin L_{\Phi(u_j \approx v_j)}$ for all $j \in J$. Since $S \cong I \times G \times \Lambda$ is non-trivial, we have $|I| > 1$ or $|G| > 1$ or $|\Lambda| > 1$, which comprises seven cases:

Case 1: $|I| > 1$, $|G| > 1$ and $|\Lambda| > 1$: There are $i_1, i_2 \in I$ with $i_1 \neq i_2$, $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$ and $g \in G \setminus \{e\}$. Let $h : \{x, y\} \mapsto I \times G \times \Lambda$ with $h : x \mapsto (i_1, g, \lambda_1)$ and $h : y \mapsto (i_2, g, \lambda_2)$. Since $S \cong I \times G \times \Lambda$ with $S \in \text{MOD}(A)$, there is an element $k \in J$ with $\bar{h}(u_k) = \bar{h}(v_k)$.

Because $S \in L_7$, we conclude $\Phi(u_k \approx v_k) \neq 7$, i.e. the following condition is fulfilled: $u_k \approx v_k \notin P_1 \cap P_2 \cap P_3$. Then, by Lemma 3.4 we get $i_1 = i_2$ or $\lambda_1 = \lambda_2$ or $g = e$. So, there is a contradiction.

Case 2: $|I| = 1$, $|G| > 1$ and $|\Lambda| > 1$: There are elements $i_1 \in I$, $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$ and $g \in G \setminus \{e\}$. Let $h : \{x, y\} \mapsto I \times G \times \Lambda$ with $h : x \mapsto (i_1, g, \lambda_1)$ and $h : y \mapsto (i_1, g, \lambda_2)$. Since $S \cong I \times G \times \Lambda$ with $S \in \text{MOD}(A)$, there is an element $k \in J$ with $\bar{h}(u_k) = \bar{h}(v_k)$. From $|I| = 1$, $|G| > 1$ and $|\Lambda| > 1$, it follows $S \in L_5$ and thus $\Phi(u_k \approx v_k) \neq 5$. Moreover,

$S \in L_5 \subseteq L_7$ implies $\Phi(u_k \approx v_k) \neq 7$. Thus, we have $u_k \approx v_k \notin P_2 \cap P_3$. Lemma 3.4 yields $\lambda_1 = \lambda_2$ or $g = e$, once more a contradiction.

- Case 3: $|I| > 1$, $|G| > 1$ and $|\Lambda| = 1$: Dually, we can verify that this case leads to a contradiction.
- Case 4: $|I| > 1$, $|G| = 1$ and $|\Lambda| > 1$: There are elements $i_1, i_2 \in I$ with $i_1 \neq i_2$, $e \in G$ and $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. Let $h : \{x, y\} \mapsto I \times G \times \Lambda$ with $h : x \mapsto (i_1, e, \lambda_1)$ and $h : y \mapsto (i_2, e, \lambda_2)$. Since $S \cong I \times G \times \Lambda$ with $S \in MOD(A)$, there is an element $k \in J$ with $\bar{h}(u_k) = \bar{h}(v_k)$. From $|I| > 1$, $|G| = 1$ and $|\Lambda| > 1$, it follows $S \in L_3$ and thus $\Phi(u_k \approx v_k) \neq 3$. Moreover, $S \in L_3 \subseteq L_7$ implies $\Phi(u_k \approx v_k) \neq 7$. Thus, we have $u_k \approx v_k \notin P_1 \cap P_2$. And so we have $i_1 = i_2$ or $\lambda_1 = \lambda_2$ by Lemma 3.4, a contradiction.
- Case 5: $|I| = |\Lambda| = 1$ and $|G| > 1$: There are elements $i_1 \in I$, $\lambda_1 \in \Lambda$ and $g \in G \setminus \{e\}$. Let $h : \{x, y\} \mapsto I \times G \times \Lambda$ with $h : x \mapsto (i_1, g, \lambda_1)$ and $h : y \mapsto (i_1, g, \lambda_1)$. Since $S \cong I \times G \times \Lambda$ with $S \in MOD(A)$, there is an element $k \in J$ with $\bar{h}(u_k) = \bar{h}(v_k)$. From $|I| = |\Lambda| = 1$ and $|G| > 1$, it follows $S \in L_4$. Notice that $S \in L_5$, $S \in L_6$ and $S \in L_7$, which leads to $\Phi(u_k \approx v_k) \notin \{4, 5, 6, 7\}$. We conclude $u_k \approx v_k \notin P_3$. By Lemma 3.4, we obtain $g = e$, a contradiction.
- Case 6: $|I| = |G| = 1$ and $|\Lambda| > 1$: There are elements $i_1 \in I$ and $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \neq \lambda_2$. We define a mapping $h : \{x, y\} \mapsto I \times G \times \Lambda$ with $h : x \mapsto (i_1, e, \lambda_1)$ and $h : y \mapsto (i_1, e, \lambda_2)$. Since $S \cong I \times G \times \Lambda$ with $S \in MOD(A)$, there is an element $k \in J$ with $\bar{h}(u_k) = \bar{h}(v_k)$. From $|I| = 1 = |G|$ and $|\Lambda| > 1$ it follows $S \in L_1$, and thus $S \in L_3$, $S \in L_5$ and $S \in L_7$. This requires $\Phi(u_k \approx v_k) \notin \{1, 3, 5, 7\}$ and we conclude $u_k \approx v_k \notin P_2$. This means $\lambda_1 \neq \lambda_2$ by Lemma 3.4, which is a contradiction.
- Case 7: $|\Lambda| = |G| = 1$ and $|I| > 1$: Dually, we can verify that this case leads to a contradiction, too.

Altogether, we have shown that $S \notin L_{\Phi(u_j \approx v_j)}$ for all $j \in J$ provides a contradiction. Thus, there is an element $j \in J$ with $S \in L_{\Phi(u_j \approx v_j)}$, which proves the assertion. \square

Lemma 3.7. *Let S be a semilattice Y of completely simple semigroups S_α ($\alpha \in Y$) such that $S \models \{x^{p+1} \approx x\}$. Further, let $a \in S_\alpha$ and $b \in S_\beta$ for $\alpha, \beta \in Y$.*

- (1) *If $S_{\alpha\beta} \in L_1$, then $ab = bab$,*
- (2) *If $S_{\alpha\beta} \in L_2$, then $ab = aba$,*
- (3) *If $S_{\alpha\beta} \in L_3$, then $ab = aba^{2p-2}b$,*
- (4) *If $S_{\alpha\beta} \in L_5$, then $ab = b^pab$,*
- (5) *If $S_{\alpha\beta} \in L_6$, then $ab = aba^p$,*
- (6) *If $S_{\alpha\beta} \in L_4$, then $ab = b^paba^p$,*
- (7) *If $S_{\alpha\beta} \in L_7$, then $ab = ab^pa^pb$.*

Proof. (1) Notice that $ab, bab \in S_{\alpha\beta}$ and so there is $ab = (bab)ab = b(ab)^2 = bab$.

(2) This is done in much the same way.

(3) Notice that $ab, a^{2p-2}b \in S_{\alpha\beta}$. Thus, we calculate $ab = (ab)^2 = ab(a^{2p-2}b)ab = aba^{2p-3}(ab)^2 = aba^{2p-2}b$.

- (4) Notice that $(ba)^p$ is an idempotent element in $S_{\alpha\beta} \in L_5$ such that $ab = (ba)^p ab = b^p (ba)^p ab = b^p ab$.
- (5) This is done in much the same way.
- (6) We have $ab = aba^p$ because $S_{\alpha\beta} \in L_4 \subseteq L_6$ and $ab = b^p ab$ because $S_{\alpha\beta} \in L_4 \subseteq L_5$, respectively. Thus, $ab = aba^p = b^p aba^p$.
- (7) Notice that $ab, (ba)^p \in S_{\alpha\beta}$ with $(ba)^p$ idempotent, so $ab = (ab)^{p+1} = ab(ab)^{p-1} ab = ab^{p+1} (ab)^{p-1} a^{p+1} b = ab^p (ba)^p a^p b = ab^p a^p b$.

□

In connection with the previous results, we define the following mapping for further purposes:

Definition 3.8. Let

$$\rho : \{1, 2, 3, 4, 5, 6\} \longrightarrow \{x, y\}^+$$

be a mapping defined by

$$\begin{aligned} \rho(1) &:= yxy, & \rho(2) &:= xyx, & \rho(3) &:= xyx^{2p-2}y, \\ \rho(4) &:= y^p xyx^p, & \rho(5) &:= y^p xy, & \rho(6) &:= xyx^p. \end{aligned}$$

The mapping ρ is well defined.

Lemma 3.9. Let $i \in \{1, 2, 3, 4, 5, 6\}$, then

$$L_{\Phi(xy \approx \rho(i))} = L_i.$$

Proof. We consider $\rho(i)$, and observe that $xy \approx \rho(i) \in \Phi^{-1}(i)$ by simple calculations. This implies $\Phi(xy \approx \rho(i)) \in \Phi\Phi^{-1}(i) = \{i\}$ and thus $L_{\Phi(xy \approx \rho(i))} = L_i$. □

Theorem 3.10. Let $I \subseteq \{1, 2, 3, 4, 5, 6\}$ be a non-empty set and let C be a class of all semigroups S such that every S is a semilattice Y of semigroups S_α with $S_\alpha \in \bigcup_{i \in I} L_i$ for $\alpha \in Y$. Then, there is a disjunction of identities

$$A := \{u_j \approx v_j : j \in J\}$$

such that

$$C = \text{MOD}(A, \{x^{p+1} \approx x\}).$$

Proof. Let $I \subseteq \{1, 2, 3, 4, 5, 6\}$ be a non-empty set, C be the class of all semigroups S such that every S is a semilattice Y of semigroups S_α with $S_\alpha \in \bigcup_{i \in I} L_i$ for $\alpha \in Y$ and let A be defined by $A := \{xy \approx \rho(i) : i \in I\}$.

From this definition of A , it follows that $C = \text{MOD}(A, \{x^{p+1} \approx x\})$.

$C \subseteq \text{MOD}(A, \{x^{p+1} \approx x\})$: Let $S \in C$. For some $a, b \in S$ we choose a mapping $h : \{x, y\} \rightarrow S$ with $h(x) = a$ and $h(y) = b$. Then, there are $\alpha, \beta \in Y$ such that $a \in S_\alpha$ and $b \in S_\beta$, and $ab \in S_{\alpha\beta}$. Thus, there is an element $i \in I$ with $S_{\alpha\beta} \in L_i$. Then, we conclude that $ab = \bar{h}(\rho(i))$ by Lemma 3.7, which shows that $S \models A$. Furthermore, all elements $a \in S$ fulfill $a^{p+1} = a$, hence $S \in \text{MOD}(A, \{x^{p+1} \approx x\})$.

$\text{MOD}(A, \{x^{p+1} \approx x\}) \subseteq C$: Let $S \in \text{MOD}(A, \{x^{p+1} \approx x\})$. Then, $S \models \{x^{p+1} \approx x\}$ and S is a completely regular semigroup by Lemma 3.2. Applying Theorem 2.3 shows that S is a semilattice Y of semigroups S_α for $\alpha \in Y$ where

every S_α is isomorphic to a Rees Matrix semigroup (Theorem 2.1). Furthermore, the set $A = \{xy \approx \rho(i) : i \in I\}$ fulfills all requirements of Lemma 3.3 and so S is an orthodox semigroup. Particularly, S is a semilattice Y of orthodox semigroups S_α ($\alpha \in Y$) where every S_α is a rectangular group as described in Theorem 2.2. This fact makes it possible to use Lemma 3.6, which yields $S_\alpha \in \bigcup_{i \in I} L_{\Phi(xy \approx \rho(i))}$ for any $\alpha \in Y$. Hence, by Lemma 3.9 we obtain $S_\alpha \in \bigcup_{i \in I} L_i$ for all $\alpha \in Y$. \square

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