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A NOTE ON SOME GENERALIZED CLOSURE AND INTERIOR OPERATORS IN A TOPOLOGICAL SPACE

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Abstract. If X is a topological space and $A \subseteq X$, then the number of distinct sets that can be obtained from A by using all possible compositions for operators i_γ, c_γ (where $\gamma = \sigma, \pi, \alpha, \beta$) introduced by Császár is at the most 25. Explicit expressions for these sets are provided. An example is provided where all the 25 different sets are determined. The result is also discussed for special cases such as when the space is extremally disconnected, resolvable, open-unresolvable, and partition spaces.

1. Introduction

Kuratowski's closure complement theorem (also known as 14 set theorem) [11] has been a guiding source of research not only in topology, but also in various other fields such as Approximation Theory, Relational algebra, Formal Language, Computer programming [4, 5, 9, 15], etc. Peleg [15], while investigating the transitive closure of a binary relation, came across several closure operators which do not satisfy some of the four of Kuratowski's closure axioms, though their properties suffice to maintain "closure complement phenomenon". Similar kind of generalized closure operators are generated by the monotonic mappings introduced by Á. Császár [6]. When several such operators are considered simultaneously and composed, the study of closure complement phenomenon becomes complicated and highly interesting. In the present note, we provide our investigations regarding all possible compositions of four such generalized closure operators and their corresponding interior operators.

Let X be a non empty set and let $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy $\gamma(A) \subseteq \gamma(B)$ for $A \subseteq B$, where $\mathcal{P}(X)$ is the power set of X. According to Á. Császár [6], $A \subseteq X$ is called γ-open if $A \subseteq \gamma(A)$. The γ-open sets of X form a generalized topology [8] (GT in brief) on X in the sense that (i) ϕ is γ-open and (ii) any union of γ-open sets is again γ-open. A set is called γ-closed if its complement is γ-open. For $A \subseteq X$, the largest γ-open set contained in A is called the γ-interior of A and is denoted by $i_\gamma(A)$. Similarly, the smallest γ-closed set containing A is called the γ-closure of A and is denoted by $c_\gamma(A)$.

If X is a topological space and i and c denote the interior and closure operators respectively, then $\gamma = ci$, ici, ic and cic give rise to important families of γ-open sets. The corresponding γ-open sets are known as semi-open, α-open, π-open

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and β-open sets respectively, in the literature. In [7], Á. Császár has shown that in some cases (including the four classical cases mentioned above), the γ-interior $i_\gamma(A)$ and the γ-closure $c_\gamma(A)$ of A are easily obtained by explicit formulas. In the present note, we provide explicit formulas arising out of all possible compositions of different i_γ's and c_γ's, where $\gamma = ci$, ici, ic, or cic in the topological framework. We have found that for $A \subseteq X$, from the possible compositions of i_α, i_σ, i_π, i_β, c_α, c_σ, c_π and c_β, we get at most 25 different sets. We have provided explicit expressions of all these sets. We have also provided an example where this bound is achieved. If the topology satisfies some extra properties such as in an extremally disconnected space, *OU*-space and Partition space etc. the upper bound is less than 25. We have provided a discussion on such particular cases in the paper. Unlike in Kuratowski's 14 set theorem, these 25 operators do not form a monoid, but, rather, a semi-group. Algebraic properties of each of i_γ, c_γ are investigated in our paper [17].

2. The main result

First of all, we provide some definitions and notations:

Definition 2.1. Let (X, τ) be a topological space and $A \subseteq X$. Then, A is called

(i) *semi-open* [12] if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$;
(ii) *α-open* [14] if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
(iii) *pre-open* [13] if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$;
(iv) *β-open* [1] if $A \subseteq \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$.

The complement of a *semi-open* set is called *semi-closed.* The largest semi-open set contained in A is denoted by $i_\sigma(A)$. The smallest semi-closed set containing A is denoted by $c_\sigma(A)$. Thus, the concept of a semi-open set gives rise to the operators c_σ and i_σ. In an analogous way, the concept of an α-open set gives rise to the operators c_α and i_α, the concept of a pre-open set gives rise to the operators c_π and i_π, and the concept of a β-open set gives rise to the operators c_β and i_β.

Theorem 2.2. [7] *In a topological space (X, τ) with $A \subseteq X$, we have,*

(i) $c_\sigma(A) = A \cup \mathrm{int}(\mathrm{cl}(A))$;
(ii) $i_\sigma(A) = A \cap \mathrm{cl}(\mathrm{int}(A))$;
(iii) $c_\alpha(A) = A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$;
(iv) $i_\alpha(A) = A \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
(v) $c_\pi(A) = A \cup \mathrm{cl}(\mathrm{int}(A))$;
(vi) $i_\pi(A) = A \cap \mathrm{int}(\mathrm{cl}(A))$;
(vii) $c_\beta(A) = A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$;
(viii) $i_\beta(A) = A \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))$.

The following result will be used to prove the main theorem:

Lemma 2.3. *In a topological space (X, τ) with $A \subseteq X$, the following hold:*

$$\mathrm{cl}[A \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))] = \mathrm{cl}(\mathrm{int}(A));$$
$$\mathrm{int}[A \cup \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))] = \mathrm{int}(\mathrm{cl}(A));$$

$$\begin{aligned}
\mathrm{cl}[A \cap \mathrm{cl}(\mathrm{int}(A))] &= \mathrm{cl}(\mathrm{int}(A));\\
\mathrm{int}[A \cup \mathrm{int}(\mathrm{cl}(A))] &= \mathrm{int}(\mathrm{cl}(A));\\
\mathrm{cl}[A \cap \mathrm{int}(\mathrm{cl}(A))] &= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)));\\
\mathrm{int}[A \cup \mathrm{cl}(\mathrm{int}(A))] &= \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)));\\
\mathrm{cl}[A \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)))] &= \mathrm{cl}(\mathrm{int}(\mathrm{cl}(A)));\\
\mathrm{int}[A \cup \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))] &= \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)));\\
\mathrm{cl}[A \cap \mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(\mathrm{cl}(A))] &= \mathrm{cl}(\mathrm{int}(A));\\
\mathrm{int}[A \cup \mathrm{int}(\mathrm{cl}(A)) \cup \mathrm{cl}(\mathrm{int}(A))] &= \mathrm{int}(\mathrm{cl}(A));\\
\mathrm{cl}[\mathrm{cl}(\mathrm{int}(A)) \cap \mathrm{int}(\mathrm{cl}(A))] &= \mathrm{cl}(\mathrm{int}(A));\\
\mathrm{int}[\mathrm{cl}(\mathrm{int}(A)) \cup \mathrm{int}(\mathrm{cl}(A))] &= \mathrm{int}(\mathrm{cl}(A)).
\end{aligned}$$

Proof. Let (X,τ) be a topological space and $A \subseteq X$. Then, $\mathrm{cl}[A\cap\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))] \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))) \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(\mathrm{int}(A)) = \mathrm{cl}(\mathrm{int}(A))$. Again, consider $\mathrm{int}(A) \subseteq A$ and $\mathrm{int}(A) \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, thus $\mathrm{int}(A) \subseteq A\cap\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$. Hence, $\mathrm{cl}(\mathrm{int}(A)) \subseteq \mathrm{cl}[A\cap\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))]$. Therefore, $\mathrm{cl}[A\cap\mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))] = \mathrm{cl}(\mathrm{int}(A))$. One can prove all the other equalities in an analogous way. □

Now we come to the main result:

Theorem 2.4. *Let (X,τ) be a topological space and $A \subseteq X$. Then, the total number of distinct sets that can be obtained from A by repeatedly using the operators i_γ, c_γ (where $\gamma = \sigma, \pi, \alpha, \beta$) is at most 25 and there exists a topological space (X,τ) and a set $A \subseteq X$ from which all these 25 sets can be realized.*

Proof. In a topological space X with $A \subseteq X$, we have

$$\begin{aligned}
i_\sigma \circ i_\sigma(A) = i_\sigma(i_\sigma(A)) &= i_\sigma\,(A \cap \mathrm{cl}(\mathrm{int}(A)))\\
&= (A \cap \mathrm{cl}(\mathrm{int}(A))) \cap \mathrm{cl}[\mathrm{int}(A \cap \mathrm{cl}(\mathrm{int}(A)))]\\
&= (A \cap \mathrm{cl}(\mathrm{int}(A))) \cap \mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))]\\
&= (A \cap \mathrm{cl}(\mathrm{int}(A))) \cap \mathrm{cl}[\mathrm{int}(A)]\\
&= A \cap \mathrm{cl}(\mathrm{int}(A))\\
&= i_\sigma(A).
\end{aligned}$$

We express this by $i_\sigma \circ i_\sigma = i_\sigma$. In the case of i_σ and i_π, we have

$$\begin{aligned}
i_\sigma \circ i_\pi(A) &= i_\sigma\,(A \cap \mathrm{int}(\mathrm{cl}(A)))\\
&= (A \cap \mathrm{int}(\mathrm{cl}(A))) \cap \mathrm{cl}[\mathrm{int}(A \cap \mathrm{int}(\mathrm{cl}(A)))]\\
&= (A \cap \mathrm{int}(\mathrm{cl}(A))) \cap \mathrm{cl}[\mathrm{int}(A) \cap \mathrm{int}(\mathrm{int}(\mathrm{cl}(A)))]\\
&= A \cap \mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{cl}(\mathrm{int}(A)).
\end{aligned}$$

We express this by $i_\sigma \circ i_\pi = A \cap \mathrm{int}(\mathrm{cl}(A)) \cap \mathrm{cl}(\mathrm{int}(A))$.

We put together similar results by using the following composition tables. For convenience, we write c and i for cl and int respectively.

From Tables 1–4, it is clear that so far 21 distinct sets have been realized from a given set A by the compositions listed in the tables. We enumerate them as

Table 1

$\circ$	i_σ	i_α	i_π	i_β
i_σ	i_σ	i_α	$A \cap \mathrm{ic}(A) \cap \mathrm{ci}(A)$	i_σ
i_α	i_α	i_α	i_α	i_α
i_π	i_α	i_α	i_π	i_π
i_β	i_σ	i_α	i_π	i_β

Table 2

$\circ$	c_σ	c_α	c_π	c_β
c_σ	c_σ	c_α	$A \cup \mathrm{ic}(A) \cup \mathrm{ci}(A)$	c_σ
c_α	c_α	c_α	c_α	c_α
c_π	c_α	c_α	c_π	c_π
c_β	c_σ	c_α	c_π	c_β

Table 3

$\circ$	c_σ	c_α	c_π	c_β
i_σ	$[A \cap \mathrm{cic}(A)] \cup \mathrm{ic}(A)$	cic	ci	$[A \cap \mathrm{ci}(A)] \cup \mathrm{ici}(A)$
i_α	ic	ic	ici	ici
i_π	ic	ic	$[A \cup \mathrm{ci}(A)] \cap \mathrm{ic}(A)$	$[A \cap \mathrm{ic}(A)] \cup \mathrm{ici}(A)$
i_β	$i_\sigma c_\sigma$	cic	$[A \cap \mathrm{cic}(A)] \cup \mathrm{ci}(A)$	$[A \cap \mathrm{cic}(A)] \cup \mathrm{ici}(A)$

Table 4

$\circ$	i_σ	i_α	i_π	i_β
c_σ	$i_\sigma c_\beta$	ici	ic	$i_\sigma c_\sigma$
c_α	ci	ci	cic	cic
c_π	ci	ci	$[A \cap \mathrm{ic}(A)] \cup \mathrm{ci}(A)$	$i_\beta c_\pi$
c_β	$i_\sigma c_\beta$	ici	$i_\pi c_\beta$	$i_\beta c_\beta$

follows. For our convenience we avoid writing A, that is, $i_\sigma(A)$ is written as simply i_σ and so on.

$$1 := \mathrm{ici}, \quad 2 := \mathrm{ci}, \quad 3 := \mathrm{ic},$$
$$4 := \mathrm{cic}, \quad 5 := i_\alpha, \quad 6 := i_\sigma,$$
$$7 := i_\pi, \quad 8 := i_\beta, \quad 9 := c_\alpha,$$
$$10 := c_\pi, \quad 11 := c_\sigma, \quad 12 := c_\beta,$$

$$13 := i_\sigma i_\pi = A \cap \mathrm{ic}(A) \cap \mathrm{ci}(A), \qquad 14 := i_\sigma c_\sigma = [A \cap \mathrm{cic}(A)] \cup \mathrm{ic}(A),$$

$15 := i_\sigma c_\beta = [A \cap \mathrm{ci}(A)] \cup \mathrm{ici}(A)$, $16 := i_\pi c_\pi = [A \cup \mathrm{ci}(A)] \cap \mathrm{ic}(A)$,
$17 := i_\pi c_\beta = [A \cap \mathrm{ic}(A)] \cup \mathrm{ici}(A)$, $18 := i_\beta c_\pi = [A \cap \mathrm{cic}(A)] \cup \mathrm{ci}(A)$,
$19 := i_\beta c_\beta = [A \cap \mathrm{cic}(A)] \cup \mathrm{ici}(A)$, $20 := c_\sigma c_\pi = A \cup \mathrm{ic}(A) \cup \mathrm{ci}(A)$,
$21 := c_\pi i_\pi = [A \cap \mathrm{ic}(A)] \cup \mathrm{ci}(A)$.

Now the question is: How many more new sets can we further obtain? Our investigation shows that at most 4 more new sets can be obtained by composing the above operators further. They are

$$
\begin{aligned}
22 &:= i_\sigma i_\pi c_\pi = \mathrm{ci}(A) \cap \mathrm{ic}(A),\\
23 &:= c_\sigma c_\pi i_\pi = \mathrm{ci}(A) \cup \mathrm{ic}(A),\\
24 &:= i_\sigma c_\sigma c_\pi = [A \cap \mathrm{cic}(A)] \cup \mathrm{ci}(A) \cup \mathrm{ic}(A),\\
25 &:= i_\sigma i_\pi c_\beta = [A \cap \mathrm{ci}(A) \cap \mathrm{ic}(A)] \cup \mathrm{ici}(A).
\end{aligned}
$$

In the final exhaustive table, we show that these 25 sets are all the possible sets that can be obtained by using the operators i_α, i_σ, i_π, i_β, c_α, c_σ, c_π and c_β. For our convenience, we use only the numerals to represent a set in the table. For example, $1, 2$ and 3 represent $\mathrm{ici}(A), \mathrm{ci}(A)$and $\mathrm{ic}(A)$, respectively as they have been listed above. Verification of the calculations involved in preparing the composition table is not much difficult (it is based on Lemma 2.3, the above tables and the Kuratowski's closure-complement theorem) and is left to the reader. The authors have also verified the associativity of the compositions using "Light's associativity test" [3].

Table 5

∘	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1	3	3	1	1	1	1	3	1	3	1	1	3	1	1	1	1	1	3	1	1	3	3	1
2	2	2	4	4	2	2	2	2	4	2	4	2	2	4	2	2	2	2	2	4	2	2	4	4	2
3	1	1	3	3	1	1	3	3	3	3	3	3	1	3	1	3	3	3	3	3	3	1	3	3	1
4	2	2	4	4	2	2	4	4	4	4	4	4	2	4	2	4	4	4	4	4	4	2	4	4	2
5	1	1	3	3	5	5	5	5	3	1	3	1	5	3	1	1	1	1	1	3	1	1	3	3	1
6	1	2	3	4	5	6	13	6	4	2	14	15	13	14	15	22	25	2	15	24	2	22	23	24	25
7	1	1	3	3	5	5	7	7	3	16	3	17	5	3	1	16	17	16	17	3	16	1	3	3	1
8	1	2	3	4	5	6	7	8	4	18	14	19	13	14	15	16	17	18	19	24	21	22	23	24	25
9	2	2	4	4	2	2	4	4	9	9	9	9	2	4	2	4	4	4	4	9	4	2	4	4	2
10	2	2	4	4	2	2	21	18	9	10	9	10	2	4	2	21	21	18	18	9	21	2	4	4	2
11	1	2	3	4	1	15	3	14	9	20	11	11	25	14	15	3	3	24	14	20	23	22	23	24	25
12	1	2	3	4	1	15	17	19	9	10	11	12	25	14	15	16	17	18	19	20	21	22	23	24	25
13	1	1	3	3	5	5	13	13	3	22	3	25	5	3	1	22	25	22	25	3	22	1	3	3	1
14	1	2	3	4	1	15	3	14	4	24	14	14	25	14	15	3	3	24	14	24	23	22	23	24	25
15	1	2	3	4	1	15	25	15	4	2	14	15	25	14	15	22	25	2	15	24	2	22	23	24	25
16	1	1	3	3	1	1	16	16	3	16	3	16	1	3	1	16	16	16	16	3	16	1	3	3	1
17	1	1	3	3	1	1	17	17	3	16	3	17	1	3	1	16	17	16	17	3	16	1	3	3	1
18	2	2	4	4	2	2	21	18	4	18	4	18	2	4	2	21	21	18	18	4	21	2	4	4	2
19	1	2	3	4	1	15	17	19	4	18	14	19	25	14	15	16	17	18	19	24	21	22	23	24	25
20	2	2	4	4	2	2	23	24	9	20	9	20	2	4	2	23	23	24	24	9	23	2	4	4	2
21	2	2	4	4	2	2	21	21	4	21	4	21	2	4	2	21	21	21	21	4	21	2	4	4	2
22	1	1	3	3	1	1	22	22	3	22	3	22	1	3	1	22	22	22	22	3	22	1	3	3	1
23	2	2	4	4	2	2	23	23	4	23	4	23	2	4	2	23	23	23	23	4	23	2	4	4	2
24	2	2	4	4	2	2	23	24	4	24	4	24	2	4	2	23	23	24	24	4	23	2	4	4	2
25	1	1	3	3	1	1	25	25	3	22	3	25	1	3	1	22	25	22	25	3	22	1	3	3	1

□

Below we provide an example of a topological space, where the bound of 25 sets has been demonstrated.

Example 2.5. Let $X = \mathbb{R}$ be the set of real numbers with the usual topology. Let $A \subseteq X$ be defined by

$$A = \{-1/n,\ n \in \mathbb{N}\} \cup \Big[[1,3] \setminus \{2+1/n, n \in \mathbb{N}\}\Big] \cup \Big[(5,7] \cap (\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big)\Big] \cup (-3,-2].$$

Then, we have

(i)

$$\mathrm{ic}(A) = (1,3) \cup (5,7) \cup (-3,-2);$$

(ii)

$$\mathrm{cic}(A) = [1,3] \cup [5,7] \cup [-3,-2];$$

(iii)

$$\mathrm{ci}(A) = [1,3] \cup \{6\} \cup \bigcup_{n=1}^{\infty}\Big[6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big] \cup [-3,-2];$$

(iv)

$$\mathrm{ici}(A) = (1,3) \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big) \cup (-3,-2);$$

(v)

$$A \cap \mathrm{ic}(A) = (1,3) \setminus \{2+1/n,\ n \in \mathbb{N}\} \cup \Big[(5,7) \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big)\Big)\Big] \cup (-3,-2);$$

(vi)

$$A \cup \mathrm{ic}(A) = \{-1/n, n \in \mathbb{N}\} \cup [1,3) \cup (5,7] \cup (-3,-2];$$

(vii)

$$A \cap \mathrm{cic}(A) = [1,3] \setminus \{2+1/n,\ n \in \mathbb{N}\} \cup \Big[(5,7] \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big)\Big] \cup (-3,-2];$$

(viii)

$$A \cup \mathrm{cic}(A) = \{-1/n,\ n \in \mathbb{N}\} \cup [1,3] \cup [5,7] \cup [-3,-2];$$

(ix)

$$A \cap \mathrm{ci}(A) = [1,3] \setminus \{2+1/n,\ n \in \mathbb{N}\} \cup \{6\} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big] \cup (-3,-2];$$

(x)

$$A \cup \mathrm{ci}(A) = \{-1/n, n \in \mathbb{N}\} \cup [1,3] \cup \Big[(5,7] \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big[6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big]\Big)\Big] \cup [-3,-2];$$

(xi)

$$A \cap \mathrm{ici}(A) = (1,3) \setminus \{2+1/n, n \in \mathbb{N}\} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big) \cup (-3,-2);$$

(xii)

$$A \cup \mathrm{ici}(A) = \{-1/n, n \in \mathbb{N}\} \cup [1,3) \cup \Big[(5,7] \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big]\Big)\Big] \cup (-3,-2];$$

(xiii)

$$\mathrm{ic}(A) \cap \mathrm{ci}(A) = (1,3) \cup \{6\} \cup \bigcup_{n=1}^{\infty}\Big[6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big] \cup (-3,-2);$$

(xiv)

$$\mathrm{ic}(A) \cup \mathrm{ci}(A) = [1,3] \cup (5,7) \cup [-3,-2];$$

(xv)

$$A \cap \mathrm{ic}(A) \cap \mathrm{ci}(A) = (1,3) \setminus \{2+1/n, n \in \mathbb{N}\} \cup \{6\} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big] \cup (-3,-2);$$

(xvi)

$$A \cup \mathrm{ic}(A) \cup \mathrm{ci}(A) = \{-1/n, n \in \mathbb{N}\} \cup [1,3] \cup (5,7] \cup [-3,-2];$$

(xvii)

$$[A \cap \mathrm{cic}(A)] \cup \mathrm{ic}(A) = [1,3) \cup (5,7] \cup (-3,-2];$$

(xviii)

$$[A \cap \mathrm{ci}(A)] \cup \mathrm{ici}(A) = [1,3) \cup \{6\} \cup \bigcup_{n=1}^{\infty}\Big(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big] \cup (-3,-2];$$

(xix)

$$[A \cup \mathrm{ci}(A)] \cap \mathrm{ic}(A) = (1,3) \cup \Big[(5,7) \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty}\Big[6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}\Big]\Big)\Big] \cup (-3,-2);$$

(xx)

$$[A \cap \mathrm{ic}(A)] \cup \mathrm{ici}(A) = (1,3) \cup \Big[(5,7) \cap (\mathbb{Q} \cup \bigcup_{n=1}^{\infty}(6+\frac{1}{2n\pi}, 6+\frac{1}{(2n-1)\pi}))\Big] \cup (-3,-2);$$

(xxi)

$$[A \cap \mathrm{cic}(A)] \cup \mathrm{ci}(A) = [1,3] \cup \Big[(5,7] \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty} \Big[6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}\Big]\Big)\Big] \cup [-3,-2];$$

(xxii)

$$[A \cap \mathrm{cic}(A)] \cup \mathrm{ici}(A) = [1,3) \cup \Big[(5,7] \cap (\mathbb{Q} \cup \bigcup_{n=1}^{\infty} (6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}]))\Big] \cup (-3,-2];$$

(xxiii)

$$[A \cap \mathrm{ic}(A)] \cup \mathrm{ci}(A) = [1,3] \cup \Big[(5,7) \cap \Big(\mathbb{Q} \cup \bigcup_{n=1}^{\infty} \Big[6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}\Big]\Big)\Big] \cup [-3,-2];$$

(xxiv)

$$[A \cup \mathrm{ci}(A) \cup \mathrm{ic}(A)] \cap \mathrm{cic}(A) = [1,3] \cup (5,7] \cup [-3,-2];$$

(xxv)

$$[A \cap \mathrm{ci}(A) \cap \mathrm{ic}(A)] \cup \mathrm{ici}(A) = (1,3) \cup \{6\} \cup \bigcup_{n=1}^{\infty} \Big(6 + \frac{1}{2n\pi}, 6 + \frac{1}{(2n-1)\pi}\Big] \cup (-3,-2).$$

3. Some special cases

If the topology on X satisfies some extra conditions, then the upper bound obtained above may be less than 25. We have the following results.

Definition 3.1. A topological space (X,τ) is said to be

(i) *extremally disconnected* [16] if the closure of any open set is open,
(ii) *resolvable* [10] if it contains a dense set with empty interior,
(iii) *open unresolvable* [2] if no open subspace is resolvable,
(iv) *partition space* [18] if its open sets form a Boolean algebra.

For open unresolvable space, we use the notation OU-space.

Corollary 3.2. *In the case of* (X,τ) *being extremally disconnected (OU-space, partition space, extremally disconnected OU-space or discrete space), the upper bound obtained above reduces to* 7 *(resp.* 7, 3, 3 *or* 1*).*

Case 1. Let (X,τ) be extremally disconnected and $A \subseteq X$. Then $\mathrm{ci}(A)$ is an open set. Hence, $\mathrm{ici}(A) = \mathrm{ci}(A)$ and $\mathrm{cic}(A) = \mathrm{ic}(A)$. Thus the distinct sets obtained from the operators i_γ, c_γ(where $\gamma = \alpha, \pi, \sigma, \beta$) are $\mathrm{ici} = \mathrm{ci}, \mathrm{ic} = \mathrm{cic}, i_\alpha = i_\sigma, i_\pi = i_\beta, c_\sigma = c_\alpha, c_\pi = c_\beta$ and $i_\pi c_\pi = [A \cup \mathrm{ci}(A)] \cap \mathrm{ic}(A)$, which are seven in number altogether.

Case 2. Let (X, τ) be an OU-space and $A \subseteq X$. In [2], Aull shows that OU-spaces are exactly those spaces in which every dense set has dense interior and we know that spaces whose dense sets have dense interiors satisfy ici $=$ ic. For the converse part, let ici $=$ ic for every $A \subseteq X$ and let A be a dense set in X. Therefore, $c(A) = X$ and $\mathrm{ic}(A) = X$. Since $\mathrm{ici}(A) = \mathrm{ic}(A) = X$, we have $X \subseteq \mathrm{ci}(A)$. Hence, $i(A)$ is dense in X. Therefore, in an OU-space, we have ici $=$ ic and cic $=$ ci. Thus the distinct sets obtained from the operators i_γ, c_γ(where $\gamma = \alpha, \pi, \sigma, \beta$) are ici $=$ ic, ci $=$ cic, $i_\alpha = i_\pi, i_\sigma = i_\beta, c_\sigma = c_\beta, c_\pi = c_\alpha$ and $i_\sigma c_\beta = [A \cap \mathrm{ci}(A)] \cup \mathrm{ic}(A)$, which are seven in number altogether.

Case 3. Let (X, τ) be a partition-space. Then its open sets are clopen, that is, closed and open simultaneously. Therefore, $\mathrm{ci}(A) = \mathrm{i}(A)$ for $A \subseteq X$. Therefore, in partition-space, we have i $=$ ici $=$ ci and cic $=$ ic $=$ c. Thus the distinct sets obtained from the operators i_γ, c_γ(where $\gamma = \alpha, \pi, \sigma, \beta$) are i, c and A only.

Case 4. Let (X, τ) be an extremally disconnected OU-space. Then we have ici $=$ ci $=$ ic $=$ cic for $A \subseteq X$. Therefore, in extremally disconnected OU-space, the distinct sets obtained from the operators i_γ, c_γ (where $\gamma = \alpha, \pi, \sigma, \beta$) are ici, i_α and c_α only, which are three in number altogether.

Case 5. Let (X, τ) be a discrete space. Then every singleton is closed and open. Therefore, $\mathrm{c}(A) = \mathrm{i}(A) = A$ for $A \subseteq X$. Therefore, in discrete space, we have only 1 set A itself, from the operators i_γ, c_γ(where $\gamma = \alpha, \pi, \sigma, \beta$).

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