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A REMARK ON EXTREMALLY μ-DISCONNECTED GENERALIZED TOPOLOGICAL SPACES

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Abstract. A more general definition of extremally μ-disconnected generalized topological space [3] is introduced and its properties are studied. We have further improved the definitions of generalized open sets [1] and upper(lower) semi-continuous functions defined for a generalized topological space in [5]. In this generalized framework we obtain the analogues of results in [1, 3, 5]. Examples of extremally μ-disconnected generalized topological spaces are given.

1. Introduction

Extremally disconnected topological spaces defined by Stone [6] turned out to be non-trivial generalization of the class of discrete spaces. A topological space is said to be extremally disconnected if the closure of every open set is open. The same definition is adapted by Császár [3] in generalized topological spaces as follows: Let X be a set and $\mathcal{P}(X)$ be the power set of X. A subset μ of $\mathcal{P}(X)$ is called generalized topology (GT) on X if μ is closed under arbitrary unions and, in that case, (X, μ) is called a generalized topological space (GTS). The elements of μ are called μ-open sets and their complements are called μ-closed sets. The closure of a set A, denoted by $c_\mu A$, is the intersection of μ-closed sets containing A. A GTS (X, μ) is called extremally μ-disconnected if $c_\mu U \in \mu$ for each μ-open set U. Our main argument on which this entire paper is based is that $c_\mu U$, for any $U \in \mu$, is never in μ unless $X \in \mu$. Therefore, if (X, μ) is not strong, that is, $X \notin \mu$, then (X, μ) is not extremally μ-disconnected since $c_\mu \emptyset = X - M_\mu$ where $M_\mu = \cup\{U : U \in \mu\}$ is not μ-open. Hence, the notion of extremally μ-disconnectedness does not act as a classification device in the class of non strong generalized topological spaces. This does not seem to be a very satisfactory situation. To rectify the situation, we have modified the above definition as follows: A GTS (X, μ) is said to be extremally μ-disconnected if $c_\mu U \cap M_\mu \in \mu$ for each μ-open set U. This definition of course reduces to the standard one if μ is a topology on X. The present paper discusses analogues of various properties of extremally μ-disconnected generalized topological spaces.

There is another direction in which this paper achieves further generalizations. In a GTS (X, μ), a subset A of X is called μ-semi-open if $A \subseteq c_\mu i_\mu A$, μ-preopen if

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$A \subseteq i_\mu c_\mu A$, μ-α-open if $A \subseteq i_\mu c_\mu i_\mu A$, and μ-λ-open if $A \subseteq c_\mu i_\mu c_\mu A$ [1] : here, $i_\mu A$, the interior of A, is the union of all μ-open sets contained in A. In a topological space, the union of open sets, semi-open sets, preopen sets, α-open sets, and λ-open sets is the same set, which is equal to X. If a GTS (X, μ) is not strong, the union of μ-open sets is $M_\mu \neq X$ whereas, since X is μ-semi-open, the union of μ-semi-open sets is X. The situation for the class of μ-λ-open sets will be similar. Consequently, we modify the definition of a μ-semi-open set A as follows: $A \subseteq c_\mu i_\mu A \cap M_\mu$ [8] and that of a μ-λ-open set as follows : $A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$. Note that, if a GT space is strong, the newly defined notions coincide with the corresponding notion defined above. In this more general framework, we have obtained every result in Sharma [5] and related results in Császár [1, 3].

There is a third direction in which further generalization is achieved. Sharma [5] has defined generalized upper semi-continuous(lower semi-continuous) function from a GTS (X, μ) to the real line $\mathbb{R}$ and gave an example showing that these two notions together cannot be equivalent to the notion of generalized continuity [2]. We have replaced the real line $\mathbb{R}$ with a generalized topological space $(R, \tau[a, b])$, where R denotes the set of real numbers, $a, b \in R$ and $\tau[a, b]$ is the GT generated by the generalized basis $\mathcal{B}$ that consist of left open rays $[a, c)$ and right open rays $(c, b]$, $a < c < b$. This GT is appropriate to obtain the above equivalence. This also ensures that, in an extremally μ-disconnected GTS (X, μ), there is a rich supply of μ-upper(lower) semi-continuous functions.

The paper is organized as follows. Section 2 contains basic notions and notation used in the paper. In Section 3, we obtain equivalences shown in [5]. Section 4 deals with the μ-upper(μ-lower) semi-continuous functions. Section 5 provides examples of extremally μ-disconnected generalized topological spaces.

2. Preliminaries

Let X be a set. A subset $\mathcal{B}$ of $\mathcal{P}(X)$ is called a *generalized basis* for X [4]. The collection μ of all unions of elements of $\mathcal{B}$ is a GT on X called the *generalized topology* generated by $\mathcal{B}$. (X, μ) shall be used generically to denote a generalized topological space.

Lemma 2.1. *Let (X, μ) be a GT-space and $A, B \subseteq X$. Then, the following statements hold.*

(i) *$x \in c_\mu A$ if and only if $x \in U \in \mu$ implies $U \cap A \neq \emptyset$.*
(ii) *If $U, V \in \mu$ and $U \cap V = \emptyset$, then $c_\mu U \cap V = \emptyset$ and $U \cap c_\mu V = \emptyset$.*
(iii) *$c_\mu A = X - i_\mu (X - A)$ for any $A \subseteq X$.*
(iv) *$c_\mu A = c_\mu (A \cap M_\mu)$.*
(v) *For any set $A \subseteq X$, $i_\mu c_\mu i_\mu c_\mu A = i_\mu c_\mu A$ and $c_\mu i_\mu c_\mu i_\mu A = c_\mu i_\mu A$.*

Recall that a set A is said to be *μ-semi-open* if $A \subseteq c_\mu i_\mu A \cap M_\mu$, *$\mu$-preopen* if $A \subseteq i_\mu c_\mu A$, *$\mu$-$\alpha$-open* if $A \subseteq i_\mu c_\mu i_\mu A$ and *μ-β-open* if $A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$. The collection of all μ-semi-open (μ-preopen, μ-α-open, μ-β-open) sets are denoted by s_μ,(π_μ,α_μ,β_μ). These sets are GT's on X and the following inclusions hold.

Theorem 2.2. (i) *$\mu \subseteq \alpha_\mu \subseteq s_\mu \subseteq \beta_\mu$.*
(ii) *$\alpha_\mu \subseteq \pi_\mu \subseteq \beta_\mu$.*

The sets U and V in a GT space (X, μ) are said to be *μ-separated* if $c_\mu U \cap V = \emptyset$ and $U \cap c_\mu V = \emptyset$.

A subset S in a GT-space (X, μ) is said to be *μ-connected* [7] if $S \cap M_\mu = U \cup V$ where U and V are μ-separated sets implies $U = \emptyset$ or $V = \emptyset$. (X, μ) is said to be μ-connected if it is a μ-connected subset of itself.

The following lemmas are immediate.

Lemma 2.3. *If μ and μ' are GTs on a set X, then $\mu \subseteq \mu'$ implies $c_{\mu'} A \subseteq c_\mu A$ for all $A \subseteq X$.*

Lemma 2.4. *Let μ and μ' be GTs on a set X and $\mu \subseteq \mu'$. If U and V are μ-separated, then U and V are μ'-separated.*

Theorem 2.5. *Let μ and μ' be GTs on a set X with $\mu \subseteq \mu'$. Then, a μ'-connected set is μ-connected.*

Theorem 2.6 ([7])**.** *The following statements are equivalent.*

(i) *(X, μ) is μ-connected.*
(ii) *If $M_\mu = G \cup G'$, $G, G' \in \mu$, $G \cap G' = \emptyset$, then $G = \emptyset$ or $G' = \emptyset$.*

A GT-space (X, μ) is called *μ-irreducible* [5] if, for each non-empty pair of μ-open sets U and V, $U \cap V \neq \emptyset$.

In view of 2.5, the following implications are immediate.

$$\begin{array}{ccc} \beta_\mu\text{-connectedness} & \Rightarrow & \pi_\mu\text{-connectedness} \\ \Downarrow & & \Downarrow \\ s_\mu\text{-connectedness} & \Rightarrow & \mu\text{-connectedness} \\ & & \Updownarrow \\ & & \alpha_\mu\text{-connectedness} \end{array}$$

Theorem 2.7. $\pi_{\beta_\mu} = \beta_{\beta_\mu} = \beta_\mu$.

Proof. Since $A \subseteq c_\mu A$, $c_{\beta_\mu} A \subseteq c_{\beta_\mu} c_\mu A \subseteq c_\mu c_\mu A = c_\mu A$ by Lemma 2.3. If $B \in \beta_\mu$ and $B \subseteq c_{\beta_\mu} A$, then $B \subseteq c_\mu i_\mu c_\mu B \cap M_\mu$ and $B \subseteq c_\mu A$. So that $B \subseteq c_\mu i_\mu c_\mu c_\mu A \cap M_\mu = c_\mu i_\mu c_\mu A \cap M_\mu$. Hence, $i_{\beta_\mu} c_{\beta_\mu} A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$. On the other hand $X - c_\mu i_\mu c_\mu A \subseteq X - i_\mu c_\mu i_\mu c_\mu A = c_\mu(X - c_\mu i_\mu c_\mu A) = c_\mu i_\mu(X - i_\mu c_\mu A) = c_\mu i_\mu(X - i_\mu c_\mu i_\mu c_\mu A) = c_\mu i_\mu c_\mu(X - c_\mu i_\mu c_\mu A)$. Since $X - c_\mu i_\mu c_\mu A \subseteq M_\mu$, $X - c_\mu i_\mu c_\mu A \in \beta_\mu$. This together with $i_{\beta_\mu} c_{\beta_\mu} A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$ gives the inclusion $c_{\beta_\mu} i_{\beta_\mu} c_{\beta_\mu} A \subseteq c_{\beta_\mu}(c_\mu i_\mu c_\mu A \cap M_\mu) \subseteq c_{\beta_\mu}(c_\mu i_\mu c_\mu A) = c_\mu i_\mu c_\mu A$, that is, $\beta_{\beta_\mu} \subseteq \beta_\mu$. Also $\beta_\mu \subseteq \pi_{\beta_\mu} \subseteq \beta_{\beta_\mu} \subseteq \beta_\mu$ by 2.2. □

Corollary 2.8. $\alpha_{\beta_\mu} = s_{\beta_\mu} = \beta_\mu$.

Proof. $\beta_\mu \subseteq \alpha_{\beta_\mu} \subseteq s_{\beta_\mu} \subseteq \beta_{\beta_\mu} = \beta_\mu$ by 2.2 and 2.7. □

3. Extremally μ- disconnected generalized topological spaces

Definition 3.1. A GTS (X, μ) is called *extremally μ-disconnected* if $c_\mu U \cap M_\mu \in \mu$ for every $U \in \mu$.

Theorem 3.2. *A GTS (X, μ) is extremally μ-disconnected if and only if, for any disjoint μ-open sets U and V, $c_\mu U \cap c_\mu V \cap M_\mu = \emptyset$.*

Proof. Let (X,μ) be extremally μ-disconnected and U and V be disjoint μ-open sets. Thus, $c_\mu U \cap V = \emptyset$ and $U \cap c_\mu V = \emptyset$. Then, $c_\mu U \cap M_\mu \cap V = \emptyset$ and $U \cap c_\mu V \cap M_\mu = \emptyset$. Since $c_\mu U \cap M_\mu \in \mu$, it follows that $c_\mu U \cap c_\mu V \cap M_\mu = \emptyset$. Conversely, assume that $c_\mu U \cap c_\mu V \cap M_\mu = \emptyset$. Let $W \in \mu$. If $c_\mu W = X - M_\mu$ then $c_\mu W \cap M_\mu = \emptyset \in \mu$. Now W and $X - c_\mu W$ are disjoint so that $c_\mu W \cap c_\mu(X - c_\mu W) \cap M_\mu = \emptyset$. Hence, $c_\mu W \cap M_\mu \subseteq X - c_\mu(X - c_\mu W) = i_\mu c_\mu W$. Since $i_\mu c_\mu W \subseteq c_\mu W \cap M_\mu$, $c_\mu W \cap M_\mu \in \mu$. □

Theorem 3.3. *A GTS (X,μ) is extremally μ-disconnected if and only if, for each $U \in \mu$ and μ-closed set F such that $U \subseteq F$, there exist a $V_1 \in \mu$ and a μ-closed set F_1 such that $U \subseteq F_1 \cap M_\mu \subseteq V_1 \subseteq F$.*

Proof. Let (X,μ) be extremally μ-disconnected. Let $U \in \mu$ and F be a μ-closed set with $U \subseteq F$. Then, $U \cap (X - F) = \emptyset$ and, by 3.2 $c_\mu U \cap c_\mu(X - F) \cap M_\mu = \emptyset$, that is, $c_\mu U \cap M_\mu \subseteq X - c_\mu(X - F)$. Since $i_\mu F = X - c_\mu(X - F) \subseteq F$, $U \subseteq c_\mu U \cap M_\mu \subseteq i_\mu F \subseteq F$. Conversely, let U and V be disjoint μ-open sets. Then, $U \subseteq X - V$. Then, by our assumption, there exist a $V_1 \in \mu$ and a μ-closed set F such that $U \subseteq F_1 \cap M_\mu \subseteq V_1 \subseteq (X - V)$. Then, it follows that $c_\mu U \cap c_\mu V \cap M_\mu = \emptyset$. □

Theorem 3.4. *If (X,μ) is an extremally μ-disconnected GTS, then the following statements are equivalent:*

(i) (X,π_μ) *is π_μ-connected.*
(ii) (X,β_μ) *is β_μ-connected.*
(iii) (X,s_{β_μ}) *is s_{β_μ}-connected.*
(iv) (X,β_μ) *is β_μ-irreducible.*

Proof. (i)⇒(ii) Suppose that (X,β_μ) is not β_μ-connected. Then, by 2.6, there are disjoint non-empty β_μ-open sets U and V such that $M_{\beta_\mu} = U \cup V$. Then, $U \subseteq c_\mu i_\mu c_\mu U \cap M_\mu$ and $V \subseteq c_\mu i_\mu c_\mu V \cap M_\mu$. Since (X,μ) is extremally μ-disconnected, $i_\mu(c_\mu i_\mu c_\mu U \cap M_\mu) = c_\mu i_\mu c_\mu U \cap M_\mu$ and $i_\mu(c_\mu i_\mu c_\mu V \cap M_\mu) = c_\mu i_\mu c_\mu V \cap M_\mu$. Therefore, $U \subseteq i_\mu c_\mu i_\mu c_\mu U = i_\mu c_\mu U$. Similarly, $V \subseteq i_\mu c_\mu U$. So U and V are π_μ- open, which means that (X,π_μ) is not π_μ-connected.
(ii)⇔(iii) holds because $s_{\beta_\mu} = \beta_\mu$ by 2.8.
(ii)⇒(iv) Suppose that (X,β_μ) is not β_μ-irreducible. Then, there are non-empty disjoint β_μ-open sets U and V. Let $P = c_\mu U \cap M_\mu$ and $Q = X - c_\mu U$. Since $U \subseteq c_\mu i_\mu c_\mu U \cap M_\mu$, $c_\mu U \subseteq c_\mu(c_\mu i_\mu c_\mu U \cap M_\mu) = c_\mu c_\mu i_\mu c_\mu U = c_\mu i_\mu c_\mu U = c_\mu i_\mu c_\mu(c_\mu U) = c_\mu i_\mu c_\mu(c_\mu U \cap M_\mu)$ by 2.1. Therefore, $c_\mu U \cap M_\mu \subseteq c_\mu i_\mu c_\mu(c_\mu U \cap M_\mu) \cap M_\mu$. Consequently, $P \in \beta_\mu$. Since $\mu \subseteq \beta_\mu$, $Q \in \beta_\mu$. Since $M_\mu = M_{\beta_\mu}$,$M_{\beta_\mu} = P \cup Q$. So, (X,β_μ) is not β_μ-connected.
(iv)⇒(i) holds because $\pi_\mu \subseteq \beta_\mu$ by 2.2. □

Theorem 3.5. *In an extremally μ-disconnected GTS (X,μ), the following statements hold:*

(i) $\beta_\mu = \pi_\mu$.
(ii) $s_\mu = \alpha_\mu$.
(iii) $s_\mu \subseteq \pi_\mu$.

Proof. (i) $\pi_\mu \subseteq \beta_\mu$ by 2.2. Let $A \in \beta_\mu$. Then, $A \subseteq c_\mu i_\mu c_\mu A \cap M_\mu$. Since X is extremally μ-disconnected, $c_\mu i_\mu c_\mu A \cap M_\mu \in \mu$. Therefore, $i_\mu(c_\mu i_\mu c_\mu A \cap M_\mu) = c_\mu i_\mu c_\mu A \cap M_\mu$. Thus, $A \subseteq i_\mu c_\mu i_\mu c_\mu A = i_\mu c_\mu A$.
(ii) $\alpha_\mu \subseteq s_\mu$ by 2.2. So, we will prove that $s_\mu \subseteq \alpha_\mu$. $A \in s_\mu$ implies $A \subseteq c_\mu i_\mu A \cap M_\mu \in \mu$. Therefore, $A \subseteq i_\mu(c_\mu i_\mu A \cap M_\mu) \subseteq i_\mu c_\mu i_\mu A$.
(iii) holds by (i) and 2.2. □

Corollary 3.6. *For an extremally μ-disconnected GTS (X,μ), the following three conditions are equivalent:*

(i) $(X, \pi_{\mu)}$ *is irreducible,*
(ii) $(X, \beta_{\mu)}$ *is β_μ-irreducible,*
(iii) $(X, \beta_{\mu)}$ *is β_μ-connected.*

A point $x \in X$ of a GTS (X,μ) is a *point of extremal μ-disconnectedness* of X if there are no disjoint μ-open sets U and V such that $x \in c_\mu U \cap c_\mu V \cap M_\mu$. Observe that every point in $X - M_\mu$ is a point of extremal μ-disconnectedness.

Theorem 3.7. *A GTS (X,μ) is extremally μ-disconnected space if and only if every point of X is a point of extremal μ-disconnectedness.*

Theorem 3.8. *An extremally μ-disconnected GTS (X,μ) is μ-connected if and only if $M, N \in \mu, M \neq \emptyset, N \neq \emptyset$ imply $M \cap N \neq \emptyset$.*

Proof. The condition is clearly sufficient in view of 2.6 and the property of extremal μ-disconnectedness is not required here. Conversely, we assume that $U, V \in \mu$, $U \neq \emptyset$, $V \neq \emptyset$ imply $U \cap V = \emptyset$. Then, $\emptyset \neq U \subseteq c_\mu U \cap M_\mu \subseteq (X - V) \cap M_\mu \neq M_\mu$ and $c_\mu U \cap M_\mu \in \mu$ because (X,μ) is extremally μ-disconnected GTS. So that $M_\mu = (c_\mu U \cap M_\mu) \cup (X - c_\mu U)$ which contradicts the μ-connectedness of (X,μ). □

4. (μ,τ)-UPPER (LOWER) SEMI-CONTINUOUS FUNCTIONS

Definition 4.1. Let (X,μ) be a GTS and R be the set of real numbers with GT $\tau[a,b]$, $a, b \in R$, generated by the generalized basis $\mathcal{B}$ consisting of left open rays $[a,c)$ and right open rays $(c,b]$, $a < c < b$. A function $f : (X,\mu) \to (R, \tau[a,b])$ is called *(μ,τ)-upper semi-continuous* (*(μ,τ)-lower semi-continuous*) if $f^{-1}([a,c))(f^{-1}((c,b])$ is μ-open for every c with $a < c < b$.

Let (X,μ) and (Y,τ) be GTSs. A mapping $f : X \to Y$ is said to be (μ,τ)-*continuous* ([2]) if $f^{-1}(U)$ is μ-open for each τ-open set $U \subseteq Y$.

Theorem 4.2. *A function $f : (X,\mu) \to (R,\tau[a,b])$ is (μ,τ)-continuous if and only if f is both (μ,τ)-upper semi-continuous and (μ,τ)-lower semi-continuous.*

Proof. Let G be τ-open. Then, G is a union of μ-open sets in $\mathcal{B}$. Since f is both (μ,τ)-upper semi-continuous and (μ,τ)-lower semi-continuous, the inverses of these μ-open sets in $\mathcal{B}$ are μ-open. Hence, $f^{-1}(G)$ is μ-open and so f is (μ,τ)-continuous. The converse is obvious. □

Theorem 4.3. *A function $f : (X,\mu) \to (R,\tau[a,b])$ is (μ,τ)-upper semi-continuous if and only if $cf : (X,\mu) \to (R,\tau[ca,cb])$ ($cf : (X,\mu) \to (R,\tau[cb,ca])$) is (μ,τ)-upper (lower) semi-continuous for every $c > 0$ ($c < 0$).*

Theorem 4.4. *A function $f : (X, \mu) \to (R, \tau[a, b])$ is (μ, τ)- lower semi-continuous if and only if $cf : (X, \mu) \to (R, \tau[ca, cb])$ ($cf : (X, \mu) \to (R, \tau[cb, ca])$) is (μ, τ)-lower (upper) semi-continuous for every $c > 0$ ($c < 0$).*

Theorem 4.5. *A function $f : (X, \mu) \to (R, \tau)[a, b]$ is (μ, τ)- upper (lower) semi-continuous if and only if $f + t : (X, \mu) \to (R, \tau[a + t, b + t])$ is (μ, τ)-upper (lower) semi-continuous for every $t \in \mathbb{R}$.*

In the following theorem, the condition of strong GTS is not required, thereby, improving Theorem 2.5 [5].

Theorem 4.6. *Let (X, μ) be GTS. Then, a subset $A \subseteq X$ is μ-open (μ-closed) if and only if its characteristic function $\chi_A : (X, \mu) \to (R, \tau[0, 1])$ is (μ, τ)-lower semi-continuous ((μ, τ)-upper semi-continuous).*

Theorem 4.7. *Let (X, μ) be an extremally μ-disconnected GTS. Let U and V be two disjoint μ-open sets. Then, there exists a (μ, τ)-upper semi-continuous function $f : (X, \mu) \to (R, \tau[0, 1])$ such that $f(U) = \{0\}$ and $f(V) = \{1\}$.*

Proof. Since $U \subseteq X - V$, by 3.3, there exist a $G_{1/2} \in \mu$ and a μ-closed set $F_{1/2}$ such that $U \subseteq F_{1/2} \cap M_\mu \subseteq G_{1/2} \subseteq X - V$
Again, since $U \subseteq F_{1/2}$ and $G_{1/2} \subseteq X - V$, there exist μ-open sets $G_{1/4}, G_{3/4}$ and μ-closed sets $F_{1/4}, F_{3/4}$ such that

$$U \subseteq F_{1/4} \cap M_\mu \subseteq G_{1/4} \subseteq F_{1/2} \text{ and } G_{1/2} \subseteq F_{3/4} \cap M_\mu \subseteq G_{3/4} \subseteq X - V$$

Thus,

$$U \subseteq F_{1/4} \cap M_\mu \subseteq G_{1/4} \subseteq F_{1/2} \cap M_\mu \subseteq G_{1/2} \subseteq F_{3/4} \cap M_\mu \subseteq G_{3/4} \subseteq X - V$$

By induction, for each dyadic rational number of the form $t = \frac{m}{2^n}, n = 1, 2, \ldots$ and $m = 1, 2, \ldots 2^n - 1$, we may show that, for $t_1 < t_2$, there are μ-open sets G_{t_1} and G_{t_2} and μ-closed sets F_{t_1} and F_{t_2} such that

$$U \subseteq F_{t_1} \cap M_\mu \subseteq G_{t_1} \subseteq F_{t_2} \cap M_\mu \subseteq G_{t_2} \subseteq X - V$$

Now we define a function$f : (X, \mu) \to (R, \tau[0, 1])$ as follows:

$$f(x) = \begin{cases} 0, & \text{if } x \in G_t \text{ for all } t, \\ \sup\,\{t : x \notin G_t\}, & \text{if } x \in M_\mu - \cap_t G_t, \\ \alpha \in R - [0, 1], & \text{if } x \in X - M_\mu. \end{cases}$$

Then, $f(U) = 0$ and$f(V) = 1$. We will show that f is (μ, τ)-upper semi-continuous, that is, $f^{-1}([0, a)), 0<a<1$, is μ- open. Now $x \in f^{-1}([0, a))$ implies $f(x)<a$. So, there must be a dyadic rational $t<a$ such that $x \in G_t$. Thus, $f^{-1}([0, a)) \subseteq \cup_{t<a} G_t$. On the other hand, if $x \in \cup_{t<a} G_t$, then $x \in G_{t_0}$ for some $t_0<a, f(x) \leq t_0<a$. So, $x \in f^{-1}([0, a))$. Therefore, f is (μ, τ)-upper semi-continuous. □

Corollary 4.8. *Let (X, μ) be an extremally μ-disconnected GTS. Let U and V be two disjoint μ-open sets. Then, there exists a (μ, τ)-lower semi-continuous function $f : (X, \mu) \to (R, \tau[0, 1])$ such that $f(U) = \{0\}$ and $f(V) = \{1\}$.*

Proof. By Theorem 4.7, there exists a (μ, τ)-upper semi-continuous function f such that $f(U) = \{1\}$ and $f(V) = \{0\}$. Thus $-f+1$ is (μ, τ)-lower semi-continuous and $(-f + 1)(U) = \{0\}$ and $(-f + 1)(V) = \{1\}$. □

5. Examples of Extremally μ-disconnected generalized topological spaces

For every set X, let us denote by Γ_X the collection of all *monotone mappings*, i.e., mappings $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $A \subseteq B$ implies $\gamma A \subseteq \gamma B$. A mapping $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be *enlarging* if $A \subseteq \gamma A$ for each A in $\mathcal{P}(X)$. If $\gamma \in \Gamma_X$, the collection of the sets A satisfying $A \subseteq \gamma A$ constitutes a GT on X denoted by λ_γ in [1]. We denote this GT just by γ itself. Any set $A \in \gamma$ is called γ-*open* and its complement is called γ-*closed*. For γ and γ' in Γ_X, we denote the composition $\gamma \circ \gamma'$, which is in Γ_X, by $\gamma\gamma'$. Thus, $\gamma\gamma' = \{A \subseteq X : A \subseteq \gamma\gamma' A\}$ is a GT on X.

For the purpose of the next three results, we denote, for a GTS (X, μ), the interior i_μ and the closure c_μ by i and k, respectively. Of course, both i and k are elements of Γ_X and $ki\gamma \in \Gamma_X$ is enlarging whenever $\gamma : \mathcal{P}(X) \to \mathcal{P}(X)$ is enlarging.

Lemma 5.1. *If (X, μ) is a GTS and $\gamma \in \Gamma$ is enlarging, then $\mu \subseteq \nu = ki\gamma$.*

Proof. See 2.1 of [3]. □

Theorem 5.2. *For an arbitrary GTS (X, μ), if $\gamma \in \Gamma$ is enlarging, then (X, ν) is extremally ν-disconnected GTS for $\nu = ki\gamma$.*

Proof. If A is ν-open, then $A \subseteq ki\gamma A$ and the set $ki\gamma A$ is ν-closed by 5.1. Hence, $c_\nu A \cap M_\mu \subseteq ki\gamma A \subseteq ki\gamma(c_\nu A \cap M_\nu)$ since $A \subseteq c_\nu A \cap M_\nu$. □

Theorem 5.3. *For a GTS (X, μ)*

(i) *(X, s_μ) is extremally s_μ-disconnected.*
(ii) *(X, β_μ) is extremally β_μ-disconnected.*

Proof. Apply 5.2 to $\gamma = id$, the identity mapping and $\gamma = k$, respectively. □

A GTS (X, μ) is said to be μ-*discrete* if $x \in M_\mu$ implies that $\{x\}$ is μ-open.

Theorem 5.4. *If (X, μ) is μ-discrete, then (X, μ) is extremally μ-disconnected.*

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