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ON LAX-ALGEBRAIC (CO)NUCLEI

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Abstract. Quantic (co)nuclei provide a convenient technique for constructing quotients and subquantales of quantales. This paper shows its analogue for the lax-algebraic approach to topology of M. M. Clementino, D. Hofmann, and W. Tholen, based in a monad and a unital quantale. As a result, we get a machinery for constructing quotient categories and subcategories, which provides, in particular, several of the already defined ones by M. M. Clementino *et al.* We also get a representation theorem for the approach of M. M. Clementino *et al.*

1. Introduction

In 1970, M. Barr [2] represented the category **Top** of topological spaces and continuous maps as the category of lax algebras and lax homomorphisms for the canonical extension of the ultrafilter monad on the category **Set** of sets and maps to the category **Rel** of sets and relations. In a series of papers, M. M. Clementino, D. Hofmann, and W. Tholen [3, 4, 6, 7, 10, 11] generalized this approach to an arbitrary monad $\mathbb{T}$ on **Set** and the category V-**Rel** of sets and V-relations, where V is an arbitrary unital quantale. In particular, they showed that many of the existing categories of topological structures (e.g., preordered sets, premetric spaces in the sense of F. W. Lawvere [17], approach spaces of R. Lowen [18], probabilistic metric spaces of B. Schweizer and A. Sklar [23]) can be represented as the categories $(\mathbb{T}, V)$-**Cat** of lax algebras and lax homomorphisms with respect to a suitable monad $\mathbb{T}$ and a quantale V. Additionally, given a lax homomorphism of unital quantales $V_1 \xrightarrow{\varphi} V_2$ that is compatible with the lax extensions of the respective monad $\mathbb{T}$, [6] showed the existence of the so-called *change-of-base functor* $(\mathbb{T}, V_1)$-**Cat** $\xrightarrow{B_\varphi}$ $(\mathbb{T}, V_2)$-**Cat**. With this technique in hand, one obtains the next pairs of functors:

(1) **Ord** → **Set** → **Ord** (preordered sets and sets);
(2) **Met** → **Ord** → **Met** (premetric spaces);
(3) **ProbMet** → **Met** → **ProbMet** (probabilistic metric spaces);
(4) **App** → **Top** → **App** (approach and topological spaces);

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(5) $(\mathbb{U}, V)$-**Cat** $\to$ **Top** $\to$ $(\mathbb{U}, V)$-**Cat**, where $\mathbb{U}$ is the ultrafilter monad, and V is a completely distributive quantale with some additional properties;
(6) V-**Cls** $\to$ **Cls** $\to$ V-**Cls** (V-closure spaces and closure spaces), where V is a completely distributive quantale with some additional properties.

The concept of a *quantic (co)nucleus* [16,21] provides a convenient technique for constructing quotients and subquantales of quantales. In particular, the following important result holds (note that a quantic (co)nucleus on a quantale V is a map $V \xrightarrow{h} V$ satisfying certain conditions [16, 21]).

Proposition 1.1. *Every quantic (co)nucleus $V \xrightarrow{h} V$ provides a quantale $V_h = \{u \in V \mid h(u) = u\}$ and a quantale homomorphism $V \xrightarrow{h} V_h$ ($V_h \xrightarrow{h} V$). Every surjective (injective) quantale homomorphism can be represented in this form.*

Every quantic nucleus on a unital quantal is a lax homomorphism of unital quantales. The same holds for *unital* quantic conuclei (preserving the quantale unit). A (unital) quantic (co)nucleus h which is compatible with the lax extension of the monad $\mathbb{T}$ provides the change-of-base functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{B_h}$ $(\mathbb{T}, V)$-**Cat**. This paper presents a lax-algebraic analogue of Proposition 1.1, in which the quantale V is replaced with the category $(\mathbb{T}, V)$-**Cat**, calling a compatible quantic (co)nucleus – *lax-algebraic (co)nucleus*, and its respective quotient (subobject) – *lax-algebraic quotient (subobject)*. As a result, we arrive at a convenient technique for producing quotients and subcategories of the categories $(\mathbb{T}, V)$-**Cat**, thereby obtaining not only the above-mentioned five examples, but also new ones, which are related to the categories of H-labeled graphs and multi-ordered sets of, e.g., [7]. We also show an application of our (co)nuclei technique to (op-)canonical extensions of monads of G. Seal [24] and topological theories of D. Hofmann [10].

Additionally, we provide a lax-algebraic analogue of the following representation theorem for quantales in terms of quantic nuclei [21, Theorem 3.1.2] (note that given a set X, $\mathcal{P}(X)$ denotes the powerset of X).

Theorem 1.2 (Quantale representation theorem)**.** *If V is a (unital) quantale, then there exists a semigroup (monoid) S and a quantic nucleus j on the free quantale $\mathcal{P}(S)$ over S such that $V \cong \mathcal{P}(S)_j$.*

2. Lax-algebraic and quantic preliminaries

2.1. $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors

This section briefly outlines the setting of lax-algebraic approach to topology as it is developed in, e.g., [4, 7, 12, 13, 24]. The theory was motivated by the result of M. Barr [2], who showed that the category **Top** of topological spaces and continuous maps is isomorphic to the category of lax Eilenberg-Moore algebras with respect to the canonical lax extension of the ultrafilter monad on the category **Set** of sets and maps to the category **Rel** of sets and relations. With this result in mind, M. M. Clementino, D. Hofmann, and W. Tholen (joined later on by G. Seal) proposed the following framework for doing topology.

We begin with the necessary preliminaries on quantales [16, 21].

Definition 2.1. A $\bigvee$-*semilattice* is a partially ordered set (poset), which has an arbitrary $\bigvee$. A *quantale* V is a $\bigvee$-semilattice, equipped with an associative binary operation $V \times V \xrightarrow{\otimes} V$ (called *multiplication*) such that $u \otimes (\bigvee S) = \bigvee_{s \in S}(u \otimes s)$ and $(\bigvee S) \otimes u = \bigvee_{s \in S}(s \otimes u)$ for every $u \in V$ and every $S \subseteq V$. A quantale V is called *unital* provided that its multiplication has a unit k.

The top and the bottom element of a $\bigvee$-semilattice V will be denoted $\top_V$ and $\bot_V$, respectively. In particular, the two-element chain $2 = \{\bot_2, \top_2\}$ is a unital quantale, in which $\otimes = \wedge$ and $k = \top_2$.

Definition 2.2. Given a unital quantale V, V-**Rel** is the category whose objects are sets, and whose morphisms are V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$ which are maps $X \times Y \xrightarrow{r} V$. The composite of V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$ and $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Z$ is defined by $(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z)$. Given a set X, the identity morphism 1_X on X is provided by the V-relation

$$1_X(x, y) = \begin{cases} k, & x = y \\ \bot_V, & \text{otherwise.} \end{cases}$$

It is easy to see that the category **2**-**Rel** is isomorphic to **Rel**. Moreover, V-**Rel** is obviously a quantaloid [22], with $\bigvee$ on hom-sets given by the pointwise evaluation of maps. Additionally, every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$ has the converse V-relation $Y \overset{r^\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\,} X$ given by $r^\circ(y, x) = r(x, y)$.

Proposition 2.3. *There exists a functor* **Set** $\xrightarrow{(-)_\circ}$ V-**Rel** *that takes a map* $X \xrightarrow{f} Y$ *to the* V*-relation* $X \overset{f_\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$ *defined by*

$$f_\circ(x, y) = \begin{cases} k, & f(x) = y \\ \bot_V, & \textit{otherwise.} \end{cases}$$

If V *has at least two elements, i.e.,* $k \neq \bot_V$*, then* $(-)_\circ$ *is a non-full embedding.*

We will identify a map $X \xrightarrow{f} Y$ and its respective relation $X \overset{f_\circ}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$, employing the notation "f" for both. Then $1_X \leqslant f^\circ \cdot f$ and $f \cdot f^\circ \leqslant 1_Y$.

Definition 2.4. Given a monad $\mathbb{T} = (T, m, e)$ [19] on **Set**, a *lax extension* $\hat{\mathbb{T}} = (\hat{T}, m, e)$ of $\mathbb{T}$ to V-**Rel** is given by a correspondence V-**Rel** $\xrightarrow{\hat{T}}$ V-**Rel** that takes a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$ to a V-relation $TX \overset{\hat{T}r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} TY$, and, additionally, satisfies the four axioms below:

(1) $r \leqslant s$ implies $\hat{T}r \leqslant \hat{T}s$ for every V-relations $X \underset{s}{\overset{r}{\rightrightarrows}} Y$;

(2) $\hat{T}s \cdot \hat{T}r \leqslant \hat{T}(s \cdot r)$ for every V-relations $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Y$, $Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\,} Z$;

(3) $Tf \leqslant \hat{T}f$ and $(Tf)^\circ \leqslant \hat{T}(f^\circ)$ for every map $X \xrightarrow{f} Y$;

(4) $\hat{T}\hat{T} \xrightarrow{m} \hat{T}$ and $1_{V\text{-}\mathbf{Rel}} \xrightarrow{e} \hat{T}$ are *lax natural transformations*, i.e.,

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}\hat{T}r}\,\big\downarrow & \leqslant & \big\downarrow\,{\scriptstyle \hat{T}r} \\ TTY & \xrightarrow[m_Y]{} & TY \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ {\scriptstyle r}\,\big\downarrow & \leqslant & \big\downarrow\,{\scriptstyle \hat{T}r} \\ Y & \xrightarrow[e_Y]{} & TY \end{array}$$

for every V-relation $X \overset{r}{\relbar\!\!\!\mapstochar\!\!\!\rightarrow} Y$.

The following provides an example of lax extensions of monads.

Example 2.5. (1) The identity monad $\mathbb{I}$ on **Set** has a lax extension $\hat{\mathbb{I}}$ to V-**Rel** given by the identity monad on V-**Rel**.
(2) Every monad $\mathbb{T}$ on **Set** has a lax extension $\hat{\mathbb{T}}^{\top}$ to V-**Rel** given on a V-relation $X \overset{r}{\relbar\!\!\!\mapstochar\!\!\!\rightarrow} Y$ by $(\hat{T}^{\top}r)(\mathfrak{x},\mathfrak{y}) = \top_V$ for every $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$.

The axioms of Definition 2.4 imply that $\hat{T}(r \cdot f) = \hat{T}r \cdot Tf$ and $\hat{T}(f^{\circ} \cdot s) = (Tf)^{\circ} \cdot \hat{T}s$ for every map $X \xrightarrow{f} Y$ and every V-relations $Y \overset{r}{\relbar\!\!\!\mapstochar\!\!\!\rightarrow} Z$, $Z \overset{s}{\relbar\!\!\!\mapstochar\!\!\!\rightarrow} Y$.

Definition 2.6. Let $\hat{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on **Set** to V-**Rel**. A $(\mathbb{T},V)$*-category* is a pair (X,a), which comprises a set X and a V-relation $TX \overset{a}{\relbar\!\!\!\mapstochar\!\!\!\rightarrow} X$ such that

$$\begin{array}{ccc} TTX & \xrightarrow{m_X} & TX \\ {\scriptstyle \hat{T}a}\,\big\downarrow & \leqslant & \big\downarrow\,{\scriptstyle a} \\ TX & \xrightarrow[a]{} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{e_X} & TX \\ & {\scriptstyle 1_X}\searrow \;\leqslant & \big\downarrow\,{\scriptstyle a} \\ & & X. \end{array}$$

A $(\mathbb{T},V)$*-functor* $(X,a) \xrightarrow{f} (Y,b)$ is a map $X \xrightarrow{f} Y$ such that

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\,\big\downarrow & \leqslant & \big\downarrow\,{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

$(\mathbb{T},V)$-**Cat** stands for the construct of $(\mathbb{T},V)$-categories and $(\mathbb{T},V)$-functors. If $\mathbb{T}$ is the identity monad on **Set** extended as in Example 2.5 (1), then $(\mathbb{T},V)$-**Cat** is the category V-**Cat** of V*-categories* and V*-functors*, i.e., the category of (small) categories enriched over the monoidal-closed category V [15].

Remark 2.7. The reader may notice that every V-category (X,a) satisfies

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle a}\,\big\downarrow & \leqslant & \big\downarrow\,{\scriptstyle a} \\ X & \xrightarrow[a]{} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} X & \xrightarrow{1_X} & X \\ & {\scriptstyle 1_X}\searrow \;\leqslant & \big\downarrow\,{\scriptstyle a} \\ & & X, \end{array}$$

which are equivalent to $a \cdot a \leqslant a$, or $a(x, y) \otimes a(y, z) \leqslant a(x, z)$ for every $x, y, z \in X$, and $1_X \leqslant a$, or $k \leqslant a(x, x)$ for every $x \in X$, respectively. Moreover, every V-functor $(X, a) \xrightarrow{f} (Y, b)$ has the property

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle a}\,\big\downarrow\!\!\!\!\!\mapstochar & \leqslant & \big\downarrow\!\!\!\!\!\mapstochar\,{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y, \end{array}$$

equivalent to $f \cdot a \leqslant b \cdot f$, or $a(x, z) \leqslant b(f(x), f(z))$ for every $x, z \in X$. Fuzzy-oriented readers will see immediately that the category V-**Cat** is the category of lattice-valued preordered sets and monotone maps in the sense of, e.g., [27, 28].

Every construct $(\mathbb{T}, V)$-**Cat** is topological (in the sense of, e.g., [1]), which is shown in, e.g., [3].

2.2. Change-of-base functors

This subsection recalls a passage between the categories $(\mathbb{T}, V)$-**Cat**, based in different quantales [6].

Definition 2.8. A *lax homomorphism of unital quantales* $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, l)$ is a map $V \xrightarrow{\varphi} W$ such that

(1) $\bigvee \varphi(A) \leqslant \varphi(\bigvee A)$ for every $A \subseteq V$;
(2) $\varphi(a) \otimes \varphi(b) \leqslant \varphi(a \otimes b)$ for every $a, b \in V$;
(3) $l \leqslant \varphi(k)$.

If the above items are equalities, then φ is a *unital quantale homomorphism.* Skipping item (3), one gets the notion of *(lax) homomorphism of quantales.*

The first condition of Definition 2.8 is equivalent to φ being order-preserving.

Proposition 2.9. *Every lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ gives a lax functor V-**Rel** $\xrightarrow{\varphi}$ W-**Rel** defined by $\varphi(X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y) = X \overset{\varphi r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y$, where φr is the composition of the maps $X \times Y \xrightarrow{r} V$ and $V \xrightarrow{\varphi} W$.*

Proof. The proof is available in, e.g., [6]. We just recall that a *lax functor* φ should satisfy the following:

(1) $\varphi r \leqslant \varphi s$ for every V-relations $X \underset{s}{\overset{r}{\rightrightarrows}} Y$ such that $r \leqslant s$;
(2) $\varphi s \cdot \varphi r \leqslant \varphi(s \cdot r)$ for every V-relations $X \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Y$ and $Y \overset{s}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z$;
(3) $1_X \leqslant \varphi 1_X$ for every set X.

□

The next lemma (see, e.g., [6] for the proof) will be useful for us later.

Lemma 2.10. *Given a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$, maps $X \xrightarrow{f} Y$, $W \xrightarrow{g} Z$, and V-relations $Y \overset{r}{\relbar\joinrel\mapstochar\joinrel\rightarrow} Z$, $U \overset{s}{\relbar\joinrel\mapstochar\joinrel\rightarrow} X$, it follows that*

$$f \leqslant \varphi f, \quad f^\circ \leqslant \varphi(f^\circ), \quad \varphi(g^\circ \cdot r \cdot f) = g^\circ \cdot \varphi r \cdot f \quad \textit{and} \quad f \cdot \varphi s \leqslant \varphi(f \cdot s).$$

If φ is a unital quantale homomorphism, then the inequalities are equalities.

Definition 2.11. Given lax extensions $\hat{T}$ and $\check{T}$ of a functor T on **Set** to the categories V-**Rel** and W-**Rel**, respectively, a lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$ is said to be *compatible* with $\hat{T}$ and $\check{T}$ provided that $\check{T}(\varphi r) \leqslant \varphi(\hat{T} r)$ for every V-relation r, which means

$$\begin{array}{ccc} V\text{-}\mathbf{Rel} & \xrightarrow{\hat{T}} & V\text{-}\mathbf{Rel} \\ \varphi \downarrow & \leqslant & \downarrow \varphi \\ W\text{-}\mathbf{Rel} & \xrightarrow[\check{T}]{} & W\text{-}\mathbf{Rel}. \end{array}$$

φ is *strictly compatible* provided that the above inequalities are equalities.

Proposition 2.12. *Given lax extensions $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ of a monad $\mathbb{T}$ on* **Set** *to the categories V-*__Rel__ *and W-*__Rel__*, respectively, every lax homomorphism of unital quantales $V \xrightarrow{\varphi} W$, compatible with $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$, induces a functor $(\mathbb{T}, V)$-*__Cat__ *$\xrightarrow{B_\varphi}$ $(\mathbb{T}, W)$-*__Cat__*, $B_\varphi((X, a) \xrightarrow{f} (Y, b)) = (X, \varphi a) \xrightarrow{f} (Y, \varphi b)$. If φ is injective (a $\bigvee$-preserving order-embedding), then B_φ is a (full) embedding.*

Proof. To show the fullness of B_φ, note that given a $(\mathbb{T}, W)$-functor $B_\varphi(X, a) \xrightarrow{f} B_\varphi(Y, b)$, $\bigvee$-preservation of φ and Lemma 2.10 imply that $\varphi(f \cdot a) = f \cdot \varphi a \leqslant \varphi b \cdot Tf = \varphi(b \cdot Tf)$, and therefore, $f \cdot a \leqslant b \cdot Tf$. □

B_φ is called the *change-of-base functor* associated to φ [6].

Definition 2.13. Given posets $(X, \leqslant)$, $(Y, \leqslant)$ and two order-preserving maps $(X, \leqslant) \underset{g}{\overset{f}{\rightleftarrows}} (Y, \leqslant)$, g is said to be *right adjoint* to f (denoted $f \dashv g$) provided that $1_X \leqslant gf$ and $fg \leqslant 1_Y$ (pointwise).

We recall that every $\bigvee$-semilattice homomorphism $V \xrightarrow{\varphi} W$ has a right adjoint $W \xrightarrow{\varphi^{\triangleright}} V$ that is defined by $\varphi^{\triangleright}(w) = \bigvee\{v \in V \mid \varphi(v) \leqslant w\}$. Also note that Definition 2.13 is a particular instance of Definition 2.21.

Proposition 2.14. *Let $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on* **Set** *to the categories V-*__Rel__ *and W-*__Rel__*, respectively, and let $V \underset{\psi}{\overset{\varphi}{\rightleftarrows}} W$ be lax homomorphisms of unital quantales compatible with the structure of the lax extensions. If $\varphi \dashv \psi$, then $B_\varphi \dashv B_\psi$ (B_ψ is a right adjoint to B_φ).*

2.3. Algebraic functors

This subsection recalls a passage between $(\mathbb{T}, V)$-**Cat** and V-**Cat** [6].

Proposition 2.15. *Given a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T} = (T, m, e)$ on* **Set***, there exists a (concrete) functor $(\mathbb{T}, V)$-*__Cat__ *$\xrightarrow{A_e}$ V-*__Cat__ *defined by $A_e((X, a) \xrightarrow{f} (Y, b)) = (X, a \cdot e_X) \xrightarrow{f} (Y, b \cdot e_Y)$.*

A_e is called the *algebraic functor* associated with e. Note that the general theory of algebraic functors provides a passage between the categories $(\mathbb{T}, V)$-**Cat**, based in different monads (thus relying on monad morphisms). This paper employs a specific case only, i.e., the monad morphism $\mathbb{I} \xrightarrow{e} \mathbb{T}$.

Definition 2.16. A lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** to the category V-**Rel** is *strict* provided that $e_Y^\circ \cdot \hat{T}r \cdot e_X = r$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$.

Example 2.5 (1) is an instance of a strict lax extension.

Proposition 2.17. *A lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on* **Set** *to the category* V-**Rel** *is strict iff $e_Y^\circ \cdot \hat{T}r \cdot e_X \leqslant r$ for every V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$.*

Proof. For the necessity, note that, given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, $r \leqslant e_Y^\circ \cdot e_Y \cdot r \overset{(\dagger)}{\leqslant} e_Y^\circ \cdot \hat{T}r \cdot e_X$, where (†) uses Definition 2.4 (4) (right-hand side). □

Corollary 2.18 ([12])**.** *Every lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on* **Set** *to* V-**Rel** *is strict on $(\mathbb{T}, V)$-categories.*

Proof. Given a $(\mathbb{T}, V)$-category (X, a), $e_X^\circ \cdot \hat{T}a \cdot e_{TX} \overset{(\dagger)}{\leqslant} a \cdot \hat{T}a \cdot e_{TX} \overset{(\dagger\dagger)}{\leqslant} a \cdot m_X \cdot e_{TX} = a$, where (†) ((††)) uses the right-hand (left-hand) side of the definition of $(\mathbb{T}, V)$-categories (Definition 2.6). □

Remark 2.19. A lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ is not necessarily strict on V-categories. Consider the lax extension $\hat{\mathbb{I}}^\top$ (of the identity monad $\mathbb{I}$) of Example 2.5 (2) over a quantale V such that $k \neq \top_V$. A singleton set $X = \{*\}$ and the V-category $X \xrightarrow{1_X} X$ then provide $1_X(*, *) = k < \top_V = (\hat{I}^\top 1_X)(*, *)$.

Proposition 2.20. *A_e has a left adjoint V-**Cat** $\xrightarrow{A^\circ}$ $(\mathbb{T}, V)$-**Cat**, $A^\circ((X, a) \xrightarrow{f} (Y, b)) = (X, e_X^\circ \cdot \hat{T}a) \xrightarrow{f} (Y, e_Y^\circ \cdot \hat{T}b)$. The adjoint situation $A^\circ \dashv A_e$ is concrete (both its unit and co-unit are given by the identity maps), and therefore, the functor A° preserves final sinks. If the lax extension $\hat{\mathbb{T}}$ of $\mathbb{T}$ is strict on V-categories, then A° is a full embedding.*

In the setting of Remark 2.19, V-**Cat** $\xrightarrow{A^\circ}$ $(\mathbb{I}, V)$-**Cat** is not an embedding, since $A^\circ(X, 1_X) = A^\circ(X, a^\top)$, where $a^\top(*, *) = \top_V$, but $(X, 1_X) \neq (X, a^\top)$.

2.4. (Co)nuclei in ordered categories

In this subsection, we recall a category-theoretic construction (appearing, e.g., in the study of *Karoubi envelopes* [14]), a particular instance of which will provide quantic (co)nuclei of the next subsection.

Definition 2.21. Given an ordered category **C** (hom-sets are partially ordered, and composition of morphisms is order-preserving), $\mathbf{C}_\triangleright$ is a subcategory of **C**, with the same objects and with morphisms $V \xrightarrow{\varphi} W$ being such that there is a **C**-morphism $W \xrightarrow{\psi} V$ with $\varphi \dashv \psi$ in **C**, i.e., $1_V \leqslant \psi \cdot \varphi$ and $\varphi \cdot \psi \leqslant 1_W$ (cf. Definition 2.13). The right adjoint of a $\mathbf{C}_\triangleright$-morphism φ is denoted $\varphi^\triangleright$.

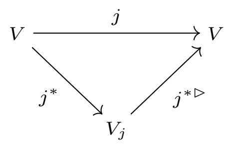
Definition 2.22. A morphism $V \xrightarrow{j} V$ of an ordered category $\mathbf{C}$ is a $\mathbf{C}$-*nucleus on* V provided that j is idempotent ($j \cdot j = j$) and expanding ($1_V \leqslant j$).

We use the notations of [21], i.e., "j" ("g") for (co)nuclei. Definition 2.22 is a variation of "closure operators" on categories (see, e.g., [8]).

Definition 2.23. An ordered category $\mathbf{C}$ *has equalizers of nuclei* provided that for every $\mathbf{C}$-nucleus $V \xrightarrow{j} V$, there exists an equalizer of the pair $(j, 1_V)$.

Now we will provide some folklore results (to make a better link to quantic nuclei of the next subsection, we show a brief sketch of their proofs).

Proposition 2.24. *Let* $\mathbf{C}$ *be an ordered category with equalizers of nuclei, and let* j *be a* $\mathbf{C}$*-nucleus on* V*. There exists a* $\mathbf{C}_{\triangleright}$*-morphism* $V \xrightarrow{j^*} V_j$ *such that the triangle*

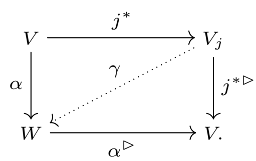


commutes, and, moreover, $j^* \cdot j^{*\triangleright} = 1_{V_j}$ *(i.e.,* j^* *is a* $\mathbf{C}$*-retraction).*

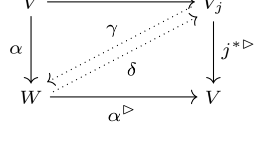
Proof. By the assumption, there exists an equalizer (V_j, φ) of $(j, 1_V)$. By the universal property of equalizers, there exists a unique $\mathbf{C}$-morphism $V \xrightarrow{j^*} V_j$ such that $V \xrightarrow{j} V = V \xrightarrow{j^*} V_j \xrightarrow{\varphi} V$. Moreover, $\varphi \cdot j^* \cdot \varphi = j \cdot \varphi = \varphi$ implies $j^* \cdot \varphi = 1_{V_j}$. From $1_V \leqslant j = \varphi \cdot j^*$, one gets the adjunction $j^* \dashv \varphi$ in $\mathbf{C}$. □

Proposition 2.25. *Let* $\mathbf{C}$ *be an ordered category with equalizers of nuclei, and let* $V \xrightarrow{\alpha} W$ *be a* $\mathbf{C}_{\triangleright}$*-morphism that is a* $\mathbf{C}$*-epimorphism (and therefore,* α *is a* $\mathbf{C}$*-retraction).*

(1) *For the adjunction* $\alpha \dashv \alpha^{\triangleright}$*,* $j := \alpha^{\triangleright} \cdot \alpha$ *is a* $\mathbf{C}$*-nucleus on* V*.*
(2) *There exists a unique* $\mathbf{C}_{\triangleright}$*-isomorphism* $V_j \xrightarrow{\gamma} W$ *that makes the next diagram commute*



Proof. We note first that, since $V \xrightarrow{\alpha} W$ is both a $\mathbf{C}_{\triangleright}$-morphism and a $\mathbf{C}$-epimorphism, $\alpha \cdot \alpha^{\triangleright} = 1_W$ by adjunction. Also, by adjunction, $j := \alpha^{\triangleright} \cdot \alpha$ is a $\mathbf{C}$-nucleus on V. By the construction from the proof of Proposition 2.24, (j^*, V_j) is a coequalizer of $(j, 1_V)$ in $\mathbf{C}$. Thus, one gets the existence of unique $\mathbf{C}$-morphisms $V_j \xrightarrow{\gamma} W$, $W \xrightarrow{\delta} V_j$ that make the next diagram commute



(recall that (j^*, V_j) is a coequalizer of $(j, 1_V)$, note that $\alpha \cdot j = \alpha \cdot \alpha^{\rhd} \cdot \alpha = \alpha$, and dualize these for $(V_j, j^{*\rhd})$). Moreover, $j^{*\rhd} \cdot \delta \cdot \gamma = \alpha^{\rhd} \cdot \gamma = j^{*\rhd}$ implies $\delta \cdot \gamma = 1_{V_j}$. Since α is an epimorphism in $\mathbf{C}$, $\gamma \cdot \delta \cdot \alpha = \gamma \cdot j^* = \alpha$ implies $\gamma \cdot \delta = 1_W$, and therefore, γ is an isomorphism in $\mathbf{C}$, and thus, a $\mathbf{C}_{\rhd}$-isomorphism. □

Dualizing the above results, one gets conuclei and their properties.

2.5. Quantic (co)nuclei

This subsection shows that the machinery of quantic (co)nuclei (see, e.g., [16, 21]) is an instance of the general technique of (co)nuclei in ordered categories.

Let **LQuant** (**LUQuant**) be the category of (unital) quantales and lax-homomorphisms of (unital) quantales. The next folklore lemma (the proof of which, however, is given here for the sake of self-completeness) shows that the category $\mathbf{LQuant}_{\rhd}$ ($\mathbf{LUQuant}_{\rhd}$) is the category **Quant** (**UQuant**) of (unital) quantales and (unital) quantale homomorphisms.

Lemma 2.26. *A lax homomorphism of (unital) quantales* $(V, \otimes, k) \xrightarrow{\varphi} (W, \otimes, l)$ *has a right adjoint* $\varphi^{\rhd}$*, which is, additionally, a lax homomorphism of (unital) quantales, iff* φ *is a (unital) quantale homomorphism.*

Proof. "⇒": Since φ has a right adjoint, φ is $\bigvee$-preserving. Given $a, b \in V$, we have to verify that $\varphi(a \otimes b) \leqslant \varphi(a) \otimes \varphi(b)$. By the properties of the adjunction $\varphi \dashv \varphi^{\rhd}$, $a \otimes b \leqslant \varphi^{\rhd}\varphi(a) \otimes \varphi^{\rhd}\varphi(b) \leqslant \varphi^{\rhd}(\varphi(a) \otimes \varphi(b))$ implies $\varphi(a \otimes b) \leqslant \varphi(a) \otimes \varphi(b)$. Additionally, by adjunction, $k \leqslant \varphi^{\rhd}(l)$ implies $\varphi(k) \leqslant l$. □

We recall now the definition of quantic (co)nucleus from, e.g., [16, 21].

Definition 2.27. A *quantic nucleus* on a quantale V is a map $V \xrightarrow{j} V$ such that for every u, $v \in V$,

(1) if $u \leqslant v$, then $j(u) \leqslant j(v)$;
(2) $u \leqslant j(u)$;
(3) $jj(u) = j(u)$;
(4) $j(u) \otimes j(v) \leqslant j(u \otimes v)$.

Definition 2.28. A *quantic conucleus* on a quantale V is a map $V \xrightarrow{g} V$, satisfying Definition 2.27 (1), (3), (4), and the condition $g(u) \leqslant u$ for every $u \in V$. A quantic conucleus g on a unital quantale $(V, \otimes, k)$ is *unital* if $k \leqslant g(k)$.

Quantic nuclei are exactly the **LQuant**-nuclei, or, equivalently, **LUQuant**-nuclei. Quantic conuclei are exactly the **LQuant**-conuclei. Every **LUQuant**-conucleus is a quantic conucleus. The converse implication does not hold. Consider, e.g., the quantale $V = ([0, 1], \wedge, 1)$ and the map $V \xrightarrow{g} V$, $g(u) = u \wedge \frac{1}{2}$. Then g is a quantic conucleus, but not an **LUQuant**-conucleus, since $g(1) = \frac{1}{2} < 1$. Unital quantic conuclei, however, are precisely the **LUQuant**-conuclei.

Proposition 2.29. *The category* **LQuant** *has equalizers of (co)nuclei. The category* **LUQuant** *has equalizers of nuclei and unital conuclei.*

Proof. Given a quantic nucleus $V \xrightarrow{j} V$, $V_j := \{u \in V \mid j(u) = u\}$ is a (unital) quantale, in which $\bigvee_j S = j(\bigvee S)$ for every $S \subseteq V_j$, and $u \otimes_j v = j(u \otimes v)$ for every $u, v \in V_j$ ($k_j = j(k)$). The inclusion $V_j \xhookrightarrow{e} V$ provides an equalizer of $(j, 1_V)$ in **LQuant** (**LUQuant**).

Given a quantic conucleus $V \xrightarrow{g} V$, $V_g := \{u \in V \mid g(u) = u\}$ is a subquantale of V. The inclusion $V_g \xhookrightarrow{e} V$ provides an equalizer of $(g, 1_V)$ in **LQuant**. If g is unital, then V_g is a unital subquantale of V, and therefore, $V_g \xhookrightarrow{e} V$ provides an equalizer of $(g, 1_V)$ in **LUQuant**. □

Proposition 2.29 ensures the validity of Propositions 2.24, 2.25 in the category **LQuant** (**LUQuant**). The (unital) quantale homomorphism $V \xrightarrow{j^*} V_j$ ($V \xrightarrow{\alpha} W$) in Proposition 2.24 (Proposition 2.25) is surjective. The respective results for conuclei can be obtained by dualization.

3. Lax-algebraic nuclei and their quotients

This section provides lax-algebraic analogues of Propositions 2.24, 2.25 in which the quantale V is replaced with the category $(\mathbb{T}, V)$-**Cat**.

Since every quantic nucleus on a unital quantale is a lax homomorphism of unital quantales, one arrives at the following concept.

Definition 3.1. For a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** to V-**Rel**, a quantic nucleus $V \xrightarrow{j} V$ is *compatible* with $\hat{\mathbb{T}}$ if $\hat{T}(jr) \leqslant j(\hat{T}r)$ for every V-relation r.

From now on, (strictly) compatible quantic nuclei will be called *(strict)* $\mathbb{T}$*-nuclei* or *(strict) lax-algebraic nuclei.* In the case of the category V-**Cat**, every quantic nucleus on V is strictly compatible.

3.1. From nuclei to quotients

This subsection shows a lax-algebraic analogue of Proposition 2.24. We begin with a procedure of constructing a lax extension of a monad from a given one. We recall (cf. Proposition 2.25) that every unital quantale homomorphism $V \xrightarrow{\varphi} W$ gives the lax homomorphism of unital quantales $W \xrightarrow{\varphi^{\triangleright}} V$ such that $\varphi^{\triangleright}\varphi$ is a quantic nucleus on V. We also use *Galois correspondences* between concrete categories of [1, Definition 6.25], which are stronger than adjunctions. If $\mathbf{A} \xrightarrow{G} \mathbf{B}$ and $\mathbf{B} \xrightarrow{F} \mathbf{A}$ are concrete functors over $\mathbf{X}$, then (F, G) is a Galois correspondence between $\mathbf{A}$ and $\mathbf{B}$ over $\mathbf{X}$ iff there exist concrete natural transformations η and ε (the components of both η and ε are identities in $\mathbf{X}$) such that $(\eta, \varepsilon) : F \dashv G : \mathbf{A} \to \mathbf{B}$ is an adjoint situation [1, Remark 19.8 (3)].

Proposition 3.2. *Let $\hat{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on* **Set** *to V-***Rel**, *and let $V \xrightarrow{\varphi} W$ be a surjective unital quantale homomorphism.*

(1) *If $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus, then W-***Rel** $\xrightarrow{\hat{T}_\varphi}$ *W-***Rel** *with $\hat{T}_\varphi(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y) = TX \overset{\varphi\hat{T}(\varphi^{\triangleright} r)}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;\;} TY$ is a lax extension $\hat{\mathbb{T}}_\varphi$ of the monad $\mathbb{T}$ to W-***Rel**.

(2) *Let $\check{\mathbb{T}}$ be a lax extension of $\mathbb{T}$ to W-***Rel**. *Then φ and $\varphi^{\triangleright}$ are compatible with $\hat{T}$ and $\check{T}$ iff $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus and $\check{T} = \hat{T}_\varphi$.*

(3) *If $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus, then $(B_{\varphi}, B_{\varphi^{\triangleright}})$ is a Galois correspondence between $(\mathbb{T}, V)$-**Cat** and $(\mathbb{T}, W)$-**Cat**, in which B_{φ} is surjective on morphisms.*

Proof. Ad (1). We check the four axioms of Definition 2.4 in a row (recall Lemma 2.10). Given W-relations $X \overset{r}{\underset{s}{\rightrightarrows}} Y$ such that $r \leqslant s$, $\varphi^{\triangleright} r \leqslant \varphi^{\triangleright} s$ implies $\hat{T}(\varphi^{\triangleright} r) \leqslant \hat{T}(\varphi^{\triangleright} s)$ implies $\varphi\hat{T}(\varphi^{\triangleright} r) \leqslant \varphi\hat{T}(\varphi^{\triangleright} s)$. For W-relations $X \overset{r}{\longrightarrow} Y$, $Y \overset{s}{\longrightarrow} Z$, $\hat{T}_{\varphi} s \cdot \hat{T}_{\varphi} r \leqslant \varphi(\hat{T}(\varphi^{\triangleright} s) \cdot \hat{T}(\varphi^{\triangleright} r)) \leqslant \varphi\hat{T}(\varphi^{\triangleright} s \cdot \varphi^{\triangleright} r) \leqslant \varphi\hat{T}(\varphi^{\triangleright}(s \cdot r)) = \hat{T}_{\varphi}(s \cdot r)$. Given a map $X \xrightarrow{f} Y$, $Tf \leqslant \hat{T}f \leqslant \hat{T}(\varphi^{\triangleright} f)$ implies $Tf = \varphi(Tf) \leqslant \varphi\hat{T}(\varphi^{\triangleright} f) = \hat{T}_{\varphi} f$, and $(Tf)^{\circ} \leqslant \hat{T}(f^{\circ}) \leqslant \hat{T}(\varphi^{\triangleright}(f^{\circ}))$ implies $(Tf)^{\circ} = \varphi(Tf)^{\circ} \leqslant \varphi\hat{T}(\varphi^{\triangleright}(f^{\circ})) = \hat{T}_{\varphi}(f^{\circ})$. Given a W-relation $X \overset{r}{\longrightarrow} Y$, $m_Y \cdot \hat{T}_{\varphi}\hat{T}_{\varphi} r = m_Y \cdot \varphi\hat{T}(\varphi^{\triangleright}\varphi\hat{T}(\varphi^{\triangleright} r)) \overset{(\dagger)}{\leqslant} m_Y \cdot \varphi\varphi^{\triangleright}\varphi(\hat{T}\hat{T}(\varphi^{\triangleright} r)) \overset{(\dagger\dagger)}{=} m_Y \cdot \varphi(\hat{T}\hat{T}(\varphi^{\triangleright} r)) = \varphi(m_Y \cdot \hat{T}\hat{T}(\varphi^{\triangleright} r)) \overset{(\dagger\dagger\dagger)}{\leqslant} \varphi(\hat{T}(\varphi^{\triangleright} r) \cdot m_X) = \varphi\hat{T}(\varphi^{\triangleright} r) \cdot m_X = \hat{T}_{\varphi} r \cdot m_X$, where $(\dagger)$ uses the assumption that $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus, $(\dagger\dagger)$ relies on the properties of Definition 2.13, and $(\dagger\dagger\dagger)$ relies on Definition 2.4 (4) for $\hat{T}$. Also, $\hat{T}_{\varphi} r \cdot e_X = \varphi\hat{T}(\varphi^{\triangleright} r) \cdot e_X = \varphi(\hat{T}(\varphi^{\triangleright} r) \cdot e_X) \overset{(\dagger)}{\geqslant} \varphi(e_Y \cdot \varphi^{\triangleright} r) = e_Y \cdot \varphi\varphi^{\triangleright} r \overset{(\dagger\dagger)}{=} e_Y \cdot r$, where $(\dagger)$ uses Definition 2.4 (4) for $\hat{T}$, and $(\dagger\dagger)$ uses the assumption that φ is surjective.

Ad (2). Since one part of "⇒" has already been considered in, e.g., [4], we show that $\check{T} = \hat{T}_{\varphi}$. Given a W-relation $X \overset{s}{\longrightarrow} Y$, on the one hand, compatibility of $\varphi^{\triangleright}$ gives $\hat{T}(\varphi^{\triangleright} s) \leqslant \varphi^{\triangleright}(\check{T}s)$, and thus, $\varphi\hat{T}(\varphi^{\triangleright} s) \leqslant \check{T}s$ by Definition 2.13; on the other hand, $\check{T}s \overset{(\dagger)}{=} \check{T}(\varphi\varphi^{\triangleright} s) \overset{(\dagger\dagger)}{\leqslant} \varphi\hat{T}(\varphi^{\triangleright} s)$, where $(\dagger)$ uses our assumption on the surjectivity of φ, and $(\dagger\dagger)$ relies on compatibility of φ.

"⇐" can be verified as follows. For compatibility of φ, we notice that given a V-relation $X \overset{r}{\longrightarrow} Y$, $\check{T}(\varphi r) = \varphi\hat{T}(\varphi^{\triangleright}\varphi r) \overset{(\dagger)}{\leqslant} \varphi\varphi^{\triangleright}\varphi\hat{T}(r) = \varphi(\hat{T}r)$, where $(\dagger)$ uses the assumption that $\varphi^{\triangleright}\varphi$ is a $\mathbb{T}$-nucleus. For compatibility of $\varphi^{\triangleright}$, we notice that given a W-relation $X \overset{s}{\longrightarrow} Y$, $\hat{T}(\varphi^{\triangleright} s) = \hat{T}(\varphi^{\triangleright}\varphi\varphi^{\triangleright} s) \overset{(\dagger)}{\leqslant} \varphi^{\triangleright}\varphi\hat{T}(\varphi^{\triangleright} s) = \varphi^{\triangleright}(\check{T}s)$, where $(\dagger)$ relies on $\varphi^{\triangleright}\varphi$ being a $\mathbb{T}$-nucleus.

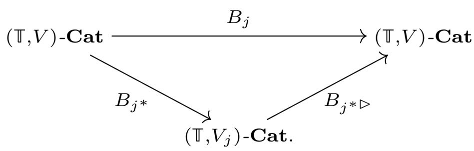
Ad (3). The first part of the claim follows from [1, Theorem 21.24]. For the second part (stronger than fullness, implying, e.g., that B_{φ} is surjective on objects), note that, given a $(\mathbb{T}, W)$-functor $(X, a) \xrightarrow{f} (Y, b)$, $B_{\varphi}B_{\varphi^{\triangleright}}((X, a) \xrightarrow{f} (Y, b)) = (X, \varphi\varphi^{\triangleright} a) \xrightarrow{f} (Y, \varphi\varphi^{\triangleright} b) = (X, a) \xrightarrow{f} (Y, b)$, since φ is surjective. □

The next result is an immediate consequence of Proposition 3.2.

Theorem 3.3. *Let j be a $\mathbb{T}$-nucleus on a unital quantale V.*

(1) *V_j-**Rel** $\xrightarrow{\hat{T}_j}$ V_j-**Rel** with $\hat{T}_j(X \overset{r}{\longrightarrow} Y) = TX \xrightarrow{j^*\hat{T}(j^{*\triangleright} r)} TY$ is a lax extension $\hat{\mathbb{T}}_j$ of the monad $\mathbb{T}$ to V_j-**Rel**.*

(2) *Both* $V \xrightarrow{j^*} V_j$ *and* $V_j \xrightarrow{j^{*\triangleright}} V$ *are compatible lax homomorphisms of unital quantales, and therefore, there exists a factorization*



(3) $(B_{j^*}, B_{j^{*\triangleright}})$ *is a Galois correspondence between* $(\mathbb{T}, V)$-**Cat**, $(\mathbb{T}, V_j)$-**Cat**, *in which* B_{j^*} *is surjective on morphisms.*

From now on, given a lax extension $\hat{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** to V-**Rel**, and a $\mathbb{T}$-nucleus j, the pair $(B_{j^*}, (\mathbb{T}, V_j)\text{-}\mathbf{Cat})$ will be called the $\mathbb{T}$-*quotient* or *lax-algebraic quotient* of $(\mathbb{T}, V)$-**Cat** with respect to j.

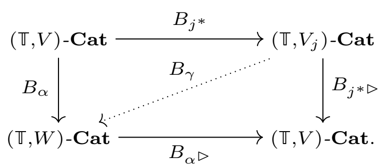
3.2. From quotients to nuclei

This subsection shows a lax-algebraic analogue of Proposition 2.25.

Proposition 3.4. *Let* $\hat{\mathbb{T}}$ *and* $\check{\mathbb{T}}$ *be lax extensions of a monad* $\mathbb{T}$ *on* **Set** *to the categories* V-**Rel** *and* W-**Rel**, *respectively, let* $V \xrightarrow{\alpha} W$ *be a surjective unital quantale homomorphism, and let* $\alpha \dashv \alpha^{\triangleright}$ *be the corresponding adjunction, in which both* α *and* $\alpha^{\triangleright}$ *are compatible with the lax extensions.*

(1) $j := \alpha^{\triangleright}\alpha$ *is a* $\mathbb{T}$*-nucleus on* V.

(2) *There exists a unique unital quantale isomorphism* $V_j \xrightarrow{\gamma} W$, *which makes the next diagram commute*



Proof. Ad (1). To show that j is a $\mathbb{T}$-nucleus, note that, given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$, $\hat{T}(jr) = \hat{T}(\alpha^{\triangleright}\alpha r) \leqslant \alpha^{\triangleright}(\check{T}\alpha r) \leqslant \alpha^{\triangleright}\alpha(\hat{T}r) = j(\hat{T}r)$, since both α and $\alpha^{\triangleright}$ are compatible.

Ad (2). For compatibility of γ, note that, for a V_j-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!+\;\;} Y$, $\check{T}(\gamma r) = \check{T}(\gamma j r) = \check{T}(\gamma j^* j^{*\triangleright} r) = \check{T}(\alpha j^{*\triangleright} r) \overset{(\dagger)}{\leqslant} \alpha(\hat{T} j^{*\triangleright} r) = \gamma j^*(\hat{T} j^{*\triangleright} r) = \gamma(\hat{T}_j r)$, where $(\dagger)$ uses compatibility of α. □

In the rest of the subsection, we will try to clarify, which concrete functors $(\mathbb{T}, V)$-**Cat** $\to$ $(\mathbb{T}, W)$-**Cat** can be expressed as $\mathbb{T}$-quotients as in Proposition 3.4.

Definition 3.5. For a quantale V and $u \in V$, there is the adjunction $u \otimes (-) \dashv u \multimap (-)$, providing a map $V \times V \xrightarrow{\multimap} V$, $v \multimap u = \bigvee\{w \in V \mid v \otimes w \leqslant u\}$.

We recall a convenient property of the map $\multimap$ of Definition 3.5.

Proposition 3.6. *Given a unital quantale* V, $(V, \multimap)$ *is a* V*-category.*

Proof. By Remark 2.7, the conditions of Definition 2.6 in case of V-categories amount to $\multimap \cdot \multimap \leqslant \multimap$ and $1_V \leqslant \multimap$. For the latter, note that $k \leqslant u \multimap u$ for every $u \in V$. For the former, one employs the fact that given $u, v, w \in V$, $u \otimes (u \multimap v) \otimes (v \multimap w) \leqslant v \otimes (v \multimap w) \leqslant w$, i.e., $(u \multimap v) \otimes (v \multimap w) \leqslant u \multimap w$. □

An analogue of Proposition 3.6 for a *commutative* quantale V ($u \otimes v = v \otimes u$ for every $u, v \in V$) can be found in [11]. Proposition 3.6 appears in a slightly more restrictive form in [20, Proposition 1] and, in a more general form, in [26]. For a quantale V, every $u \in V$ gives another adjunction $(-) \otimes u \dashv (-) \circ\!\!- u$. These two adjunctions coincide for every $u \in V$ iff V is commutative.

We fix now a concrete functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{F}$ $(\mathbb{T}, W)$-**Cat**. Since the functor F is concrete, given a $(\mathbb{T}, V)$-category (X, a), we denote $F(X, a)$ by $(X, \mathcal{F}a)$. With the help of the algebraic functors of Propositions 2.15, 2.20 (whose notations will employ now the respective underlying quantale, e.g., A_e^W and A_V°), one gets that $A_e^W F A_V^\circ(V, \multimap) = (V, \mathfrak{a} := \mathcal{F}(e_V^\circ \cdot \hat{T}(\multimap)) \cdot e_V)$ is a W-category, providing a map $V \xrightarrow{\mathfrak{a}} W$, $\mathfrak{a}(u) = \mathfrak{a}(k, u)$.

Below is the motivation for the concept of "strictness" of Definition 2.16.

Proposition 3.7. *Let $\hat{\mathbb{T}}$ be a lax extension of a monad $\mathbb{T}$ on* **Set** *to V-***Rel***, which is strict on V-categories. If $F(X, a) = (X, \alpha a)$ for a lax homomorphism of unital quantales $V \xrightarrow{\alpha} W$, then $\alpha = \mathfrak{a}$.*

Proof. $\mathfrak{a} = \mathfrak{a}(k, -) = (\alpha(e_V^\circ \cdot \hat{T}(\multimap)) \cdot e_V)(k, -) = (\alpha(e_V^\circ \cdot \hat{T}(\multimap) \cdot e_V))(k, -) \overset{(\dagger)}{=} (\alpha(\multimap))(k, -) = \alpha(k \multimap (-)) = \alpha$, where (†) relies on strictness. □

Since $(V, \multimap)$ is not a $(\mathbb{T}, V)$-category (unless $\mathbb{T}$ is the identity monad), one cannot employ Corollary 2.18 instead of the strictness assumption. More precisely, in the setting of Remark 2.19, $\multimap (k, k) = k < \top_V = (\hat{I}^\top \multimap)(k, k)$.

To continue, we need an additional result on compatibility of poset adjunctions (recall Definition 2.11).

Proposition 3.8. *Let $\hat{\mathbb{T}}$ and $\check{\mathbb{T}}$ be lax extensions of a monad $\mathbb{T}$ on* **Set** *to V-***Rel** *and W-***Rel***, respectively, and let $V \xrightarrow{\varphi} W$ be a $\bigvee$-preserving compatible lax homomorphism of unital quantales. Then $\varphi^\triangleright$ is a compatible lax homomorphism of unital quantales iff φ is a strictly compatible unital quantale homomorphism.*

Proof. "⇒: Given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,} Y$, since φ is compatible, we show that $\varphi(\hat{T}r) \leqslant \check{T}(\varphi r)$. Indeed, $\hat{T}r \leqslant \hat{T}(\varphi^\triangleright \varphi r) \leqslant \varphi^\triangleright \check{T}(\varphi r)$ by compatibility of $\varphi^\triangleright$, and thus, $\varphi(\hat{T}r) \leqslant \check{T}(\varphi r)$ by adjunction. The rest follows from Lemma 2.26.

"⇐": For compatibility of $\varphi^\triangleright$, note that, for a W-relation $X \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\,} Y$, $\varphi(\hat{T}(\varphi^\triangleright s)) \overset{(\dagger)}{=} \check{T}(\varphi\varphi^\triangleright s) \leqslant \check{T}s$, where (†) uses the strict compatibility of φ. Thus, $\hat{T}(\varphi^\triangleright s) \leqslant \varphi^\triangleright(\check{T}s)$ by adjunction. □

We are now ready to provide the second lax-algebraic analogue of Proposition 2.25, which is a consequence of Theorem 3.3 and Propositions 3.4, 3.7, 3.8.

Proposition 3.9. *Let $\check{\mathbb{T}}$ ($\hat{\mathbb{T}}$) be a (strict) lax extension of a monad $\mathbb{T}$ on* **Set** *to W-***Rel** *(V-***Rel***). A concrete functor $(\mathbb{T}, V)$-**Cat** $\xrightarrow{F}$ $(\mathbb{T}, W)$-**Cat** can be represented in the form of a $\mathbb{T}$-quotient iff*

(1) $V \xrightarrow{\alpha} W$ *is a surjective unital quantale homomorphism;*
(2) $\check{T}(\alpha r) = \alpha(\hat{T}r)$ *for every* V*-relation* r*;*
(3) $\mathcal{F}a = \alpha a$ *for every* $(\mathbb{T}, V)$*-category* (X, a)*.*

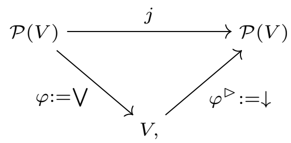
The necessity of conditions (1)–(3) of Proposition 3.9 is a consequence of our definition of $\mathbb{T}$-quotients.

3.3. Representation theorem for the categories $(\mathbb{T}, V)$-Cat

With the technique of lax-algebraic nuclei in hand, this subsection provides a lax-algebraic analogue of the quantale representation theorem (Theorem 1.2).

First, we recall from, e.g., [16, 21] that given a semigroup $(S, \otimes)$ (monoid $(S, \otimes, k)$), the powerset $\mathcal{P}(S)$ is the free (unital) quantale over S, in which $\bigvee$ are given by the set-theoretic unions, and $A \otimes B = \{a \otimes b \mid a \in A,\, b \in B\}$ for every $A, B \in \mathcal{P}(S)$ (the unit is given by the singleton $\{k\}$).

Second, given a (unital) quantale V, one has the underlying semigroup (monoid) of V. The quantic nucleus of Theorem 1.2 is then given by the following commutative diagram



in which $\bigvee \dashv \downarrow$ is the well-known adjunction provided by arbitrary joins and lower sets (i.e., sets $A \subseteq V$ such that $b \leqslant a \in A$ implies $b \in A$), and $\bigvee$ is a surjective (unital) quantale homomorphism.

To lift the above triangle to a given category $(\mathbb{T}, V)$-**Cat**, we begin with an analogue of Proposition 3.2.

Proposition 3.10. *Let* $\check{\mathbb{T}}$ *be a lax extension of a monad* $\mathbb{T}$ *on* **Set** *to* W-**Rel***, and let* $V \xrightarrow{\varphi} W$ *be a surjective unital quantale homomorphism.*

(1) V-**Rel** $\xrightarrow{\check{T}^{\varphi}}$ V-**Rel** *with* $\check{T}^{\varphi}(X \overset{r}{\longrightarrow\!\!\!\!\!+\!\!\!\!\!\longrightarrow} Y) = TX \xrightarrow{\varphi^{\triangleright}\check{T}(\varphi r)} TY$ *is a lax extension* $\check{\mathbb{T}}^{\varphi}$ *of the monad* $\mathbb{T}$ *to* V-**Rel***.*
(2) *Both* φ *and* $\varphi^{\triangleright}$ *are strictly compatible with* $\check{T}^{\varphi}$ *and* $\check{T}$*.*
(3) $\varphi^{\triangleright}\varphi$ *is a strict* $\check{\mathbb{T}}^{\varphi}$*-nucleus.*

Proof. Ad (1). We check the four axioms of Definition 2.4 in a row (recall Lemma 2.10). The first two of them can be shown similarly to the proof of Proposition 3.2. Given a map $X \xrightarrow{f} Y$, $Tf \leqslant \check{T}f = \check{T}(\varphi f)$ implies $Tf \leqslant \varphi^{\triangleright}(Tf) \leqslant \varphi^{\triangleright}\check{T}(\varphi f) = \check{T}^{\varphi} f$, and also $(Tf)^{\circ} \leqslant \check{T}(f^{\circ}) = \check{T}(\varphi(f^{\circ}))$ implies $(Tf)^{\circ} \leqslant \varphi^{\triangleright}(Tf)^{\circ} = \varphi^{\triangleright}\check{T}(\varphi(f^{\circ})) = \check{T}^{\varphi}(f^{\circ})$. Given a W-relation $X \overset{r}{\longrightarrow\!\!\!\!\!+\!\!\!\!\!\longrightarrow} Y$, $m_Y \cdot \check{T}^{\varphi}\check{T}^{\varphi} r = m_Y \cdot \varphi^{\triangleright}\check{T}(\varphi\varphi^{\triangleright}\check{T}(\varphi r)) \overset{(\dagger)}{=} m_Y \cdot \varphi^{\triangleright}(\check{T}\check{T}(\varphi r)) \leqslant \varphi^{\triangleright}(m_Y \cdot \check{T}\check{T}(\varphi r)) \overset{(\dagger\dagger)}{\leqslant} \varphi^{\triangleright}(\check{T}(\varphi r) \cdot m_X) = \varphi^{\triangleright}\check{T}(\varphi r) \cdot m_X = \check{T}^{\varphi} r \cdot m_X$, where (†) uses surjectivity of φ, and (††) relies on Definition 2.4 (4) for $\check{T}$. Additionally, $\check{T}^{\varphi} r \cdot e_X = \varphi^{\triangleright}\check{T}(\varphi r) \cdot e_X = \varphi^{\triangleright}(\check{T}(\varphi r) \cdot e_X) \overset{(\dagger)}{\geqslant} \varphi^{\triangleright}(e_Y \cdot \varphi r) \geqslant e_Y \cdot \varphi^{\triangleright}\varphi r \overset{(\dagger\dagger)}{\geqslant} e_Y \cdot r$, where (†) relies on Definition 2.4 (4) for $\check{T}$, and (††) employs Definition 2.13.

Ad (2). For strict compatibility of $\varphi^{\triangleright}$, note that, given a W-relation $X \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, $\check{T}^{\varphi}(\varphi^{\triangleright} s) = \varphi^{\triangleright}\check{T}(\varphi\varphi^{\triangleright} s) \overset{(\dagger)}{=} \varphi^{\triangleright}(\check{T}s)$, where $(\dagger)$ relies on φ being surjective. The rest follows from Proposition 3.8.

Ad (3). Immediate from the previous item. □

Propositions 3.4, 3.10 and Theorem 3.3 imply the next representation theorem for the categories $(\mathbb{T}, V)$-**Cat**.

Theorem 3.11 (Representation Theorem). *Given a category* $(\mathbb{T}, V)$-**Cat**, *there exist a monoid* S, *a lax extension of* $\mathbb{T}$ *to* $\mathcal{P}(S)$-**Rel**, *and a strict* $\mathbb{T}$-*nucleus* j *on* $\mathcal{P}(S)$ *such that* $(\mathbb{T}, V)$-**Cat** $\cong (\mathbb{T}, \mathcal{P}(S)_j)$-**Cat**.

Theorem 3.11 says that every category $(\mathbb{T}, V)$-**Cat** can be represented as a lax-algebraic quotient of a category of the form $(\mathbb{T}, \mathcal{P}(S))$-**Cat** for some monoid S. Thus, given a monad $\mathbb{T}$ on **Set**, the categories of the form $(\mathbb{T}, \mathcal{P}(S))$-**Cat** provide a kind of "generating class" for the categories of the form $(\mathbb{T}, V)$-**Cat**.

4. Lax-algebraic conuclei and their subobjects

The results of this section dualize those of the previous one.

Since every unital quantic conucleus is a lax homomorphism of unital quantales, similarly to Definition 3.1, one has the concept of $\mathbb{T}$-*conucleus*.

4.1. From conuclei to subobjects

We begin by constructing a lax extension of a monad from a given one. Every unital quantale homomorphism $V \overset{\varphi}{\to} W$ gives the lax homomorphism of unital quantales $W \overset{\varphi^{\triangleright}}{\longrightarrow} V$ such that $\varphi\varphi^{\triangleright}$ is a unital quantic conucleus on W.

Proposition 4.1. *Let* $\check{\mathbb{T}}$ *be a lax extension of a monad* $\mathbb{T}$ *on* **Set** *to* W-**Rel**, *and let* $V \overset{\varphi}{\to} W$ *be an injective unital quantale homomorphism.*

(1) *If* $\varphi\varphi^{\triangleright}$ *is a* $\mathbb{T}$-*conucleus, then* V-**Rel** $\overset{\check{T}_{\varphi}}{\longrightarrow}$ V-**Rel** *with* $\check{T}_{\varphi}(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y) = TX \overset{\varphi^{\triangleright}\check{T}(\varphi r)}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TY$ *is a lax extension* $\check{\mathbb{T}}_{\varphi}$ *of the monad* $\mathbb{T}$ *to* V-**Rel**.

(2) *Let* $\hat{\mathbb{T}}$ *be a lax extension of* $\mathbb{T}$ *to* V-**Rel**. *Then* φ *and* $\varphi^{\triangleright}$ *are compatible with* $\hat{T}$ *and* $\check{T}$ *iff* $\varphi\varphi^{\triangleright}$ *is a* $\mathbb{T}$-*conucleus and* $\hat{T} = \check{T}_{\varphi}$.

(3) *If* $\varphi\varphi^{\triangleright}$ *is a* $\mathbb{T}$-*nucleus, then* $(B_{\varphi}, B_{\varphi^{\triangleright}})$ *is a Galois correspondence between* $(\mathbb{T}, V)$-**Cat** *and* $(\mathbb{T}, W)$-**Cat**, *in which* B_{φ} *is a full embedding.*

Proof. Ad (3). The second part of the claim follows from Proposition 2.12 (we recall that $\varphi^{\triangleright}\varphi = 1_V$). □

The next result is a consequence of Proposition 4.1.

Theorem 4.2. *Let* g *be a* $\mathbb{T}$-*conucleus on a unital quantale* V.

(1) V_g-**Rel** $\overset{\check{T}_g}{\longrightarrow}$ V_g-**Rel** *with* $\check{T}_g(X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y) = TX \overset{g^{*\triangleright}\check{T}(g^{*}r)}{\longrightarrow\!\!\!\!\!\!\!+\;\;} TY$ *is a lax extension* $\check{\mathbb{T}}_j$ *of the monad* $\mathbb{T}$ *to* V_g-**Rel**.

(2) *Both* $V_g \xrightarrow{g^*} V$ *and* $V \xrightarrow{g^{*\triangleright}} V_g$ *are compatible lax homomorphisms of unital quantales, and therefore, there exists a factorization*

$$\begin{array}{ccc} (\mathbb{T},V)\text{-}\mathbf{Cat} & \xrightarrow{\quad B_g \quad} & (\mathbb{T},V)\text{-}\mathbf{Cat} \\ & {\scriptstyle B_{g^{*\triangleright}}}\searrow \quad \nearrow{\scriptstyle B_{g^*}} & \\ & (\mathbb{T},V_g)\text{-}\mathbf{Cat}. & \end{array}$$

(3) $(B_{g^*}, B_{g^{*\triangleright}})$ *is a Galois correspondence between* $(\mathbb{T},V)$-**Cat**, $(\mathbb{T},V_g)$-**Cat**, *and* B_{g^*} *is a full embedding.*

For a lax extension $\check{\mathbb{T}}$ of a monad $\mathbb{T}$ on **Set** to V-**Rel**, and a $\mathbb{T}$-conucleus g, the pair $((\mathbb{T},V_g)\text{-}\mathbf{Cat}, B_{g^*})$ will be called the $\mathbb{T}$-*subobject* or *lax-algebraic subobject* of $(\mathbb{T},V)$-**Cat** with respect to g.

4.2. From subobjects to conuclei

Proposition 4.3. *Let* $\hat{\mathbb{T}}$ *and* $\check{\mathbb{T}}$ *be lax extensions of a monad* $\mathbb{T}$ *on* **Set** *to the categories* V-**Rel** *and* W-**Rel**, *respectively, let* $V \xrightarrow{\alpha} W$ *be an injective unital quantale homomorphism, and let* $\alpha \dashv \alpha^{\triangleright}$ *be the corresponding adjunction, in which both* α *and* $\alpha^{\triangleright}$ *are compatible with the lax extensions.*

(1) $g := \alpha\alpha^{\triangleright}$ *is a* $\mathbb{T}$*-conucleus on* W.

(2) *There exists a unique unital quantale isomorphism* $W_g \xrightarrow{\gamma} V$ *that makes the next diagram commute*

$$\begin{array}{ccc} (\mathbb{T},W)\text{-}\mathbf{Cat} & \xrightarrow{\;B_{g^{*\triangleright}}\;} & (\mathbb{T},W_g)\text{-}\mathbf{Cat} \\ {\scriptstyle B_{\alpha^{\triangleright}}}\downarrow & \quad {\scriptstyle B_\gamma}\swarrow & \downarrow{\scriptstyle B_{g^*}} \\ (\mathbb{T},V)\text{-}\mathbf{Cat} & \xrightarrow[\;B_\alpha\;]{} & (\mathbb{T},W)\text{-}\mathbf{Cat}. \end{array}$$

The second approach follows from Theorem 4.2, and Propositions 3.7, 3.8, 4.3.

Proposition 4.4. *Let* $\check{\mathbb{T}}$ *(*$\hat{\mathbb{T}}$*) be a (strict) lax extension of a monad* $\mathbb{T}$ *on* **Set** *to* W-**Rel** *(*V-**Rel***). A concrete functor* $(\mathbb{T},V)\text{-}\mathbf{Cat} \xrightarrow{F} (\mathbb{T},W)\text{-}\mathbf{Cat}$ *can be represented in the form of a* $\mathbb{T}$*-subobject iff*

(1) $V \xrightarrow{\alpha} W$ *is an injective unital quantale homomorphism;*
(2) $\check{T}(\alpha r) = \alpha(\hat{T}r)$ *for every* V*-relation* r*;*
(3) $\mathcal{F}a = \alpha a$ *for every* $(\mathbb{T},V)$*-category* (X,a).

5. Applications to (op-)canonical extensions

In [24] (see also [25]), G. Seal introduced two lax extensions of monads on **Set** to V-**Rel** called *canonical* and *op-canonical* extensions. In this section, we show an application of our (co)nuclei technique to those extensions.

We begin with some necessary preliminaries.

Definition 5.1 ([9])**.** A complete lattice V is *completely distributive* provided that for every family $\{S_i \,|\, i \in I\}$ of subsets of V, it follows that $\bigwedge_{i\in I} \bigvee S_i = \bigvee_{f\in F} \bigwedge_{i\in I} f(i)$, where F is the set of choice maps $I \xrightarrow{f} \bigcup_{i\in I} S_i$ with $f(i) \in S_i$.

Definition 5.2 ([24]). A functor $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$ is *taut* provided that it preserves pullbacks of monomorphisms along arbitrary maps (it follows that if $A \overset{\iota}{\hookrightarrow} X$ is an injection, then $TA \overset{T\iota}{\hookrightarrow} TX$ is also an injection). A monad $\mathbb{T} = (T, m, e)$ is *taut* provided that T is taut, and both e and m are *taut*, i.e., $e_X(y) \in TA$ iff $y \in A$, and $m_X(\mathfrak{X}) \in TA$ iff $\mathfrak{X} \in TTA$ for every set X and every $A \subseteq X$.

Given a completely distributive unital quantale V and a taut monad $\mathbb{T}$, there exist two lax extensions of $\mathbb{T}$ to V-**Rel**.

Canonical extension: given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, for every $A \subseteq X$ and every $u \in V$, define $r_u[A] = \{y \in Y \mid \text{there is an } x \in X \text{ with } u \leqslant r(x, y)\}$ and set $(\hat{T}r)(\mathfrak{x}, \mathfrak{y}) = \bigvee\{u \in V \mid \mathfrak{y} \in T(r_u[A]) \text{ for every } A \subseteq X \text{ with } \mathfrak{x} \in TA\}$.

Op-canonical extension: given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, for every $A \subseteq X$ and every $u \in V$, define $r^{\circ}_u[A] = \{x \in X \mid \text{there is a } y \in B \text{ with } u \leqslant r(x, y)\}$ and set $(\hat{T}'r)(\mathfrak{x}, \mathfrak{y}) = \bigvee\{u \in V \mid \mathfrak{x} \in T(r^{\circ}_u[B]) \text{ for every } B \subseteq Y \text{ with } \mathfrak{y} \in TB\}$.

We apply the technique of lax-algebraic (co)nuclei to (op-)canonical extensions, studying canonical extensions only (op-canonical ones being similar).

Proposition 5.3. *The canonical extension $\hat{T}$ is strict.*

Proof. Given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, for every $x_0 \in X$, $y_0 \in Y$, $(\hat{T}r)(e_X(x_0), e_Y(y_0)) = \bigvee\{u \in V \mid e_Y(y_0) \in T(r_u[A]) \text{ for every } A \subseteq X \text{ with } e_X(x_0) \in TA\} = \bigvee_{i \in I} u_i$. For every $i \in I$, $x_0 \in \{x_0\}$ implies (e is taut) $e_X(x_0) \in T\{x_0\}$ implies $e_Y(y_0) \in T(r_{u_i}[\{x_0\}])$ implies (e is taut) $y_0 \in r_{u_i}[\{x_0\}]$ implies $u_i \leqslant r(x_0, y_0)$. The claimed result now follows from Proposition 2.17. □

5.1. Canonical quotients

We assume that the quantale V also satisfies the next two properties.

$$u \otimes v = \bot_V \text{ implies } u = \bot_V \text{ or } v = \bot_V, \text{ for every } u, v \in V \tag{A}$$

$$\bot_V < \bigwedge(V \backslash \{\bot_V\}) \tag{B}$$

The map $V \xrightarrow{j} V$ defined by

$$j(u) = \begin{cases} \bot_V, & u = \bot_V \\ \top_V, & \text{otherwise} \end{cases}$$

is a quantic nucleus on V (one needs (A), to prove Definition 2.27 (4)).

Proposition 5.4. *The quantic nucleus j is compatible with $\hat{T}$.*

Proof. Given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$, we show that $\hat{T}(jr) \leqslant j(\hat{T}r)$. Fix $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, and let $p := (\hat{T}(jr))(\mathfrak{x}, \mathfrak{y}) = \bigvee\{u \in V \mid \mathfrak{y} \in T((jr)_u[A]) \text{ for every } A \subseteq X \text{ with } \mathfrak{x} \in TA\}$ and $q := j(\hat{T}r)(\mathfrak{x}, \mathfrak{y}) = j(\bigvee\{u \in V \mid \mathfrak{y} \in T(r_u[A]) \text{ for every } A \subseteq X \text{ with } \mathfrak{x} \in TA\})$. If $p = \top_V > \bot_V$ (otherwise, the claim is clear), then $p = \bigvee_{i \in I} u_i$, with $I \neq \varnothing$, and $u_i > \bot_V$ for every $i \in I$. If now $q = \bot_V$, then, for every $u \in V$ with $u > \bot_V$, there is an $A \subseteq X$ with $\mathfrak{x} \in TA$ and $\mathfrak{y} \notin T(r_u[A])$. Denote $u_0 = \bigwedge(V \backslash \{\bot_V\})$ noticing that $u_0 > \bot_V$ (by (B)). Then, there is an $i_0 \in I$ with $u_0 \leqslant u_{i_0}$. For every $u \in V$ with $u > \bot_V$, $(jr)_u[A] = \{y \in Y \mid \text{there is an } x \in A \text{ with } u \leqslant jr(x,y)\} = \{y \in Y \mid \text{there is an } x \in A \text{ with } \bot_V < r(x,y)\}$, and therefore, $(jr)_{u_0}[A] = (jr)_{u_{i_0}}[A]$. Thus, we suppose that $u_0 = u_{i_0}$, and therefore, there is an $A_0 \subseteq X$ with $\mathfrak{x} \in TA_0$, $\mathfrak{y} \in T((jr)_{u_0}[A_0])$, and $\mathfrak{y} \notin T(r_{u_0}[A_0])$. Since T is taut, it follows that $\mathfrak{y} \in T((jr)_{u_0}[A_0] \backslash r_{u_0}[A_0])$. Since $(jr)_{u_0}[A_0] \backslash r_{u_0}[A_0] = \{y \in Y \mid (\text{there is an } x \in A_0 \text{ with } \bot_V < r(x,y)) \text{ and } (u_0 \nleqslant r(x,y) \text{ for every } x \in A_0)\} = \{y \in Y \mid (\text{there is an } x \in A_0 \text{ with } \bot_V < r(x,y)) \text{ and } (r(x,y) = \bot_V \text{ for every } x \in A_0)\} = \varnothing$ and $\varnothing \subseteq r_{u_0}[A_0]$, it follows that $\mathfrak{y} \in T\varnothing \subseteq T(r_{u_0}[A_0])$, which contradicts the earlier $\mathfrak{y} \notin T(r_{u_0}[A_0])$. □

With Proposition 5.4 in hand, one gets the commutative triangle

$$\begin{array}{ccc} (\mathbb{T},V)\text{-}\mathbf{Cat} & \xrightarrow{\quad B_j \quad} & (\mathbb{T},V)\text{-}\mathbf{Cat} \\ & {\scriptstyle B_{j^*}} \searrow \quad \nearrow {\scriptstyle B_{j^*\triangleright}} & \\ & (\mathbb{T},2)\text{-}\mathbf{Cat}. & \end{array}$$

Following the terminology of [24], we call the pair $(B_{j^*}, (\mathbb{T},2)\text{-}\mathbf{Cat})$ the *canonical* $\mathbb{T}$*-quotient* of $(\mathbb{T},V)$-**Cat** with respect to j.

5.2. Canonical subobjects

We define a map $V \xrightarrow{g} V$ by

$$g(u) = \begin{cases} \top_V, & u = \top_V \\ \bot_V, & \text{otherwise}, \end{cases}$$

and recall that a quantale V is *strictly two-sided* [21] provided that $k = \top_V$.

Lemma 5.5. *The map g is a unital conucleus on V iff V is strictly two-sided.*

Proof. To prove Definition 2.27 (4), one uses condition $\top_V \otimes \top_V = \top_V$, which is a consequence of $k = \top_V$. We show that g is unital iff $k = \top_V$. The sufficiency is clear, and for the necessity, one gets from $k < \top_V$ that $k \leqslant g(k) = \bot_V$. Then $k = \bot_V$ implies that V is a singleton, i.e., $k = \top_V$, contradicting $k < \top_V$. □

We assume that V is strictly two-sided (and completely distributive).

Proposition 5.6. *The quantic conucleus g is compatible with $\hat{T}$.*

Proof. Given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\,} Y$, we show that $\hat{T}(gr) \leqslant g(\hat{T}r)$. Fix $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, and let $p := (\hat{T}(gr))(\mathfrak{x},\mathfrak{y}) = \bigvee\{u \in V \mid \mathfrak{y} \in T((gr)_u[A]) \text{ for every } A \subseteq X \text{ with } \mathfrak{x} \in TA\}$ and $q := g(\hat{T}r)(\mathfrak{x},\mathfrak{y}) = g(\bigvee\{u \in V \mid \mathfrak{y} \in T(r_u[A]) \text{ for every } A \subseteq X \text{ with } \mathfrak{x} \in TA\})$. If $p = \top_V > \bot_V$ (otherwise, the claim is clear), then $p = \bigvee_{i \in I} u_i$, with $I \neq \varnothing$, and $u_i > \bot_V$ for every $i \in I$. If $q = \bot_V$, then for every $u \in V$ such that $\mathfrak{y} \in T(r_u[A])$ for every $A \subseteq X$ with $\mathfrak{x} \in TA$, it follows that $u < \top_V$. For $i \in I$, $(gr)_{u_i}[A] = \{y \in Y \mid \text{there is an } x \in X \text{ with } u_i \leqslant gr(x,y)\} = \{y \in Y \mid \text{there is an } x \in X \text{ with } r(x,y) = \top_V\} = (gr)_{\top_V}[A]$. Since $I \neq \varnothing$, we assume that there is an

$i_0 \in I$ with $u_{i_0} = \top_V$. Moreover, for every $A \subseteq X$, $r_{u_{i_0}}[A] = (gr)_{u_{i_0}}[A]$. Since for every $A \subseteq X$, $\mathfrak{x} \in TA$ implies $\mathfrak{y} \in T((gr)_{u_{i_0}}[A]) = T(r_{u_{i_0}}[A])$, we get $u_{i_0} < \top_V$, contradicting $u_{i_0} = \top_V$. □

With Proposition 5.6 in hand, one gets the commutative triangle

$$\begin{array}{ccc} (\mathbb{T},V)\text{-}\mathbf{Cat} & \xrightarrow{\quad B_g \quad} & (\mathbb{T},V)\text{-}\mathbf{Cat} \\ & {\scriptstyle B_{g^*\rhd}} \searrow \quad \nearrow {\scriptstyle B_{g^*}} & \\ & (\mathbb{T},2)\text{-}\mathbf{Cat}. & \end{array}$$

Following the terminology of [24], we call the pair $((\mathbb{T},2)\text{-}\mathbf{Cat}, B_{g^*})$ the *canonical* $\mathbb{T}$*-subobject* of $(\mathbb{T},V)$-**Cat** with respect to g. We also emphasize that, unlike the previous subsection, V does need not satisfy conditions (A), (B).

6. Applications to topological theories of D. Hofmann

In [10], D. Hofmann introduced the notion of a *topological theory* as a tool to conveniently construct lax extensions of monads on **Set**. This section shows an application of our (co)nuclei technique to those lax extensions.

We begin with some necessary preliminaries.

Definition 6.1 ([10])**.** A *topological theory* is a triple $\mathscr{T} = (\mathbb{T}, V, \xi)$, where $\mathbb{T} = (T, m, e)$ is a monad on **Set**, V is a unital quantale, and $TV \xrightarrow{\xi} V$ is a map such that the following conditions hold:

(1) $k\cdot !_{T1} \leqslant \xi \cdot Tk$ and $\otimes \cdot \langle \xi \cdot T\pi_1, \xi \cdot T\pi_2 \rangle \leqslant \xi \cdot T\otimes$, which means

$$\begin{array}{ccc} T1 & \xrightarrow{Tk} & TV \\ {\scriptstyle !_{T1}}\downarrow & \leqslant & \downarrow{\scriptstyle \xi} \\ 1 & \xrightarrow[k]{} & V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T(V\times V) & \xrightarrow{T\otimes} & TV \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \leqslant & \downarrow{\scriptstyle \xi} \\ V\times V & \xrightarrow[\otimes]{} & V, \end{array}$$

where 1 is a singleton set, $1 \xrightarrow{k} X$ is the map with value k, and $T1 \xrightarrow{!_{T1}} 1$ is the unique possible map;

(2) $1_V \leqslant \xi \cdot e_V$ and $\xi \cdot T\xi \leqslant \xi \cdot m_V$, which means

$$\begin{array}{ccc} V & \xrightarrow{e_V} & TV \\ & {\scriptstyle 1_V}\searrow \ \leqslant & \downarrow{\scriptstyle \xi} \\ & & V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} TTV & \xrightarrow{m_V} & TV \\ {\scriptstyle T\xi}\downarrow & \leqslant & \downarrow{\scriptstyle \xi} \\ TV & \xrightarrow[\xi]{} & V \end{array}$$

(3) for a set X, the map $\mathbf{Set}(X,V) \xrightarrow{\xi_X} \mathbf{Set}(TX,V)$, $\xi_X(\alpha) = \xi \cdot T\alpha$ is monotone, where for $\alpha, \beta \in \mathbf{Set}(Z,V)$, $\alpha \leqslant \beta$ iff $\alpha(z) \leqslant \beta(z)$ for every $z \in Z$;

(4) $P_V \xrightarrow{(\xi_X)_{X\in Ob(\mathbf{Set})}} P_V T$ is a natural transformation, where $\mathbf{Set} \xrightarrow{P_V} \mathbf{Set}$ is the V*-powerset functor* defined by $P_V(X \xrightarrow{f} Y) = \mathbf{Set}(X,V) \xrightarrow{P_V f} \mathbf{Set}(Y,V)$, $(P_V f(\alpha))(y) = \bigvee_{f(x)=y} \alpha(x)$.

Definition 6.2 ([12])**.** A commutative square

$$\begin{array}{ccc} W & \xrightarrow{h_2} & Y \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in **Set** is a *Beck-Chevalley square* (or *BC-square*) provided that $h_2 \cdot h_1^\circ = g^\circ \cdot f$ (or, equivalently, $h_1 \cdot h_2^\circ = f^\circ \cdot g$). A functor $\mathbf{Set} \xrightarrow{T} \mathbf{Set}$ *satisfies the Beck-Chevalley condition* (or *BC*) if it sends BC-squares to BC-squares.

Proposition 6.3 ([10])**.** *Given a topological theory $\mathscr{T}$ with T satisfying BC, there exists a lax extension of the monad $\mathbb{T}$ to V-**Rel** given on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} Y$ by $(\hat{T}r)(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\xi(T\overrightarrow{r}(\mathfrak{z})) \mid \mathfrak{z} \in T(X \times Y), T\pi_1(\mathfrak{z}) = \mathfrak{x}, T\pi_2(\mathfrak{z}) = \mathfrak{y}\}$, where $\overrightarrow{r}$ denotes the map $X \times Y \to V$ determined by r.*

Definition 6.4 ([10])**.** Given topological theories $\mathscr{T}$ and $\mathscr{T}'$, a *morphism of topological theories* $\mathscr{T} \xrightarrow{(\eta,\varphi)} \mathscr{T}'$ is given by a monad morphism $\mathbb{T}' \xrightarrow{\eta} \mathbb{T}$ and a lax homomorphism of unital quantales $V \xrightarrow{\varphi} V'$ with $\xi' \cdot T'\varphi \leqslant \varphi \cdot \xi \cdot \eta_V$, i.e.,

$$\begin{array}{ccc} T'V & \xrightarrow{\eta_V} & TV \\ {\scriptstyle T'\varphi}\downarrow & & \\ T'V' & \leqslant & \downarrow{\scriptstyle \xi} \\ {\scriptstyle \xi'}\downarrow & & \\ V' & \xleftarrow[\varphi]{} & V. \end{array}$$

A morphism (η, φ) is *strict* provided that the above diagram commutes.

We now show a condition on a monad $\mathbb{T}$, which ensures that the lax extension of Proposition 6.3 is strict.

Given $x \in X$, $y \in Y$ and $\mathfrak{z} \in T(X \times Y)$,

$$T\pi_1(\mathfrak{z}) = e_X(x) \text{ and } T\pi_2(\mathfrak{z}) = e_Y(y) \text{ imply } \mathfrak{z} = e_{X\times Y}(x, y). \tag{C}$$

$$\xi \cdot e_V = 1_V \tag{D}$$

Proposition 6.5. *If conditions* (C), (D) *hold, then the extension $\hat{T}$ is strict. If $\hat{T}$ is strict, then* (D) *holds.*

Proof. For the first statement, note that, given a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} Y$,

$$(\hat{T}r)(e_X(x), e_Y(y)) \overset{(C)}{=} \xi \cdot (T\overrightarrow{r}) \cdot e_{X\times Y}(x, y) = \xi \cdot e_V \cdot \overrightarrow{r}(x, y) \overset{(D)}{=} r(x, y).$$

For the second statement, we fix $u \in V$. Taking a singleton set $1 = \{*\}$, we define a V-relation $1 \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} 1$ by $r(*, *) = u$. Then,

$$\xi \cdot e_V(u) = \xi \cdot e_V \cdot \overrightarrow{r}(*, *) = \xi \cdot (T\overrightarrow{r}) \cdot e_{1\times 1}(*, *)$$

$$\leqslant (\hat{T}r)(e_1(*), e_1(*)) \leqslant r(*,*) = u.$$

The converse inequality follows from the left-hand side of Definition 6.1 (2). □

Remark 6.6. The powerset monad, the ultrafilter monad, the free H-act-monad $\mathbb{H}$ of Subsections 7.7, and the free-monoid monad $\mathbb{L}$ of Subsection 7.8 satisfy (C). The monad $\mathbb{T}_1$ given by the constant functor $\mathbf{Set} \xrightarrow{T_1} \mathbf{Set}$, $T(X \xrightarrow{f} Y) = 1 \xrightarrow{1_1} 1$ satisfies (C). The double powerset monad does not satisfy (C).

The next result will help us check, if a (co)nucleus is a lax-algebraic one.

Proposition 6.7. *Given topological theories $\mathscr{T} = (\mathbb{T}, V, \xi)$, $\mathscr{T}' = (\mathbb{T}, V', \xi')$ and a lax homomorphism of unital quantales $V \xrightarrow{\varphi} V'$, $\mathscr{T} \xrightarrow{(1_{\mathbb{T}},\varphi)} \mathscr{T}'$ is a morphism of topological theories iff φ is compatible with $\hat{T}$, $\hat{T}'$. If φ is $\bigvee$-preserving, then $(1_{\mathbb{T}}, \varphi)$ is strict iff φ is strictly compatible with $\hat{T}$, $\hat{T}'$.*

Proof. "$\Longrightarrow$": First we notice that, in our case, the topological theory morphism condition of Definition 6.2 reduces to the diagram

$$\begin{array}{ccc} TV & \xrightarrow{\xi} & V \\ {\scriptstyle T\varphi}\downarrow & \leqslant & \downarrow{\scriptstyle \varphi} \\ TV' & \xrightarrow[\xi']{} & V'. \end{array}$$

For a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} Y$ and $\mathfrak{x} \in X$, $\mathfrak{y} \in TY$, we let $S = \{\mathfrak{z} \in T(X \times Y) \mid T\pi_1(\mathfrak{z}) = \mathfrak{x}, T\pi_2(\mathfrak{z}) = \mathfrak{y}\}$. Then $p := (\hat{T}'(\varphi r))(\mathfrak{x}, \mathfrak{y}) = \bigvee\{\xi'(T(\varphi \cdot \overrightarrow{r})(\mathfrak{z})) \mid \mathfrak{z} \in S\}$ and $q := \varphi(\hat{T}r)(\mathfrak{x}, \mathfrak{y}) = \varphi(\bigvee\{\xi(T\overrightarrow{r}(\mathfrak{z})) \mid \mathfrak{z} \in S\})$. Given $\mathfrak{z} \in S$, $\xi'(T(\varphi \cdot \overrightarrow{r})(\mathfrak{z})) = \xi' \cdot T\varphi \cdot T\overrightarrow{r}(\mathfrak{z}) \overset{(\dagger)}{\leqslant} \varphi \cdot \xi \cdot T\overrightarrow{r}(\mathfrak{z})$ where (†) uses the diagram above. Thus, $p \leqslant \bigvee\{\varphi \cdot \xi \cdot T\overrightarrow{r}(\mathfrak{z}) \mid \mathfrak{z} \in S\} \overset{(\dagger\dagger)}{\leqslant} \varphi(\bigvee\{\xi(T\overrightarrow{r}(\mathfrak{z})) \mid \mathfrak{z} \in S\}) = q$, where (††) relies on Definition 2.8 (1). If φ is $\bigvee$-preserving, and $(1_{\mathbb{T}}, \varphi)$ is strict, then $p = q$.

"$\Longleftarrow$": We define a V-relation $1 \overset{r}{\longrightarrow\!\!\!\!\!\!\!\!\mapsto} V$ by $r(*, u) = u$. We can assume that $1 \times V \xrightarrow{\overrightarrow{r}} V$ is the identity map $V \xrightarrow{1_V} V$. Moreover, $(1 \xleftarrow{!_V} V \xrightarrow{1_V} V)$ is a product of 1 and V. Given $\mathfrak{u} \in TV$, $\{\mathfrak{z} \in TV \mid T!_V(\mathfrak{z}) = e_1(*), T1_V(\mathfrak{z}) = \mathfrak{u}\} = \{\mathfrak{u}\}$. Thus, $(\hat{T}'(\varphi r))(e_1(*), \mathfrak{u}) = \xi'(T(\varphi \cdot \overrightarrow{r})(\mathfrak{u})) = \xi' \cdot T\varphi \cdot T\overrightarrow{r}(\mathfrak{u}) = \xi' \cdot T\varphi(\mathfrak{u})$ and $\varphi(\hat{T}r)(e_1(*), \mathfrak{u}) = \varphi \cdot \xi \cdot T\overrightarrow{r}(\mathfrak{u}) = \varphi \cdot \xi(\mathfrak{u})$, which implies $\xi' \cdot T\varphi(\mathfrak{u}) \leqslant \varphi \cdot \xi(\mathfrak{u})$ $(\xi' \cdot T\varphi(\mathfrak{u}) = \varphi \cdot \xi(\mathfrak{u}))$ by (strict) compatibility of φ. Note that this part of the proof does not require the $\bigvee$-preservation of φ for strictness. □

Corollary 6.8. *Given a topological theory $\mathscr{T}$, a ($\bigvee$-preserving) quantic (co)nucleus h on V is a (strict) $\mathbb{T}$-(co)nucleus iff $\xi \cdot Th \leqslant h \cdot \xi$ $(\xi \cdot Th = h \cdot \xi)$.*

Given a topological theory $\mathscr{T}$, we call $\mathbb{T}$-(co)nuclei by *$\mathscr{T}$-(co)nuclei*, and $\mathbb{T}$-quotients (-subobjects) by *$\mathscr{T}$-quotients* (*-subobjects*).

6.1. $\mathscr{T}$-quotients

Throughout this subsection, we will use the quantic nucleus j of Subsection 5.1 (V satisfies (A)). Corollary 6.8 then provides the following result.

Proposition 6.9. *Given a topological theory $\mathscr{T}$, j is a $\mathscr{T}$-nucleus iff for every $\mathfrak{u} \in TV$, $\xi(\mathfrak{u}) = \bot_V$ implies $\xi(Tj(\mathfrak{u})) = \bot_V$.*

We consider now three particular examples of topological theories.

Example 6.10 ([10])**.** Given the ultrafilter monad $\mathbb{U}$ and a completely distributive unital quantale V, the map $UV \xrightarrow{\xi} V$ with $\xi(\mathfrak{u}) = \bigvee_{A\in\mathfrak{u}} \bigwedge_{u\in A} u$ provides a topological theory $\mathscr{T}_U = (\mathbb{U}, V, \xi)$.

Proposition 6.11. *If V satisfies Subsection 5.1* (B)*, then j is a $\mathscr{T}_U$-nucleus.*

Proof. For $\mathfrak{u} \in UV$ with $\xi(\mathfrak{u}) = \bot_V$, $\bot_V \in A$ for every $A \in \mathfrak{u}$ (by (B)). Since $Uj(\mathfrak{u}) = \{A \subseteq V \mid j^{-1}(A) \in \mathfrak{u}\}$, given $A \in Uj(\mathfrak{u})$, $\bot_V \in A$ (by the definition of j), and thus, $\bigwedge_{u\in A} u = \bot_V$. Thus, $\xi(Uj(\mathfrak{u})) = \bigvee_{A\in Uj(\mathfrak{u})} \bigwedge_{u\in A} u = \bot_V$. □

Example 6.12 ([5])**.** Given the free-monoid monad $\mathbb{L}$ of Subsection 7.8 and a unital quantale V, the map $LV \xrightarrow{\xi} V$ with

$$\xi(\mathfrak{u}_n) = \begin{cases} u_1 \otimes \ldots \otimes u_n, & 1 \leqslant n \\ k, & \text{otherwise} \end{cases}$$

provides a topological theory $\mathscr{T}_L = (\mathbb{L}, V, \xi)$.

In general, one cannot replace "$\otimes$" ("k") with "$\wedge$" ("$\top_V$") in Example 6.12, since Definition 6.1 (4) requires distributivity of $\otimes$ over $\bigvee$.

Proposition 6.13. *j is a $\mathscr{T}_L$-nucleus.*

Proof. Given $\mathfrak{u}_n \in LV$ with $\xi(\mathfrak{u}_n) = \bot_V$, it follows that (we assume $\bot_V < k$) $1 \leqslant n$ and $u_1 \otimes \ldots \otimes u_n = \bot_V$. By (A), there exists i_0 such that $u_{i_0} = \bot_V$. Then $\xi(Lj(\mathfrak{u}_n)) = j(u_1) \otimes \ldots \otimes j(u_{i_0}) \otimes \ldots \otimes j(u_n) = \bot_V$. □

Remark 6.14. By [5, Example 2.2.5], for the free H-act-monad $\mathbb{H}$ of Subsection 7.7, every unital quantale V with the map $H \times V \xrightarrow{\pi_2} V$ gives a topological theory $\mathscr{T}_H$. By Corollary 6.8, *every* ($\bigvee$-preserving) quantic nucleus on V is a (strict) $\mathscr{T}_H$-nucleus. Since j is $\bigvee$-preserving, j is a strict $\mathscr{T}_H$-nucleus.

6.2. $\mathscr{T}$-subobjects

Throughout this subsection, we will use the quantic conucleus g of Subsection 5.2 (V is strictly two-sided). By Corollary 6.8, one obtains the next result.

Proposition 6.15. *Given a topological theory $\mathscr{T}$, g is a $\mathscr{T}$-conucleus iff for every $\mathfrak{u} \in TV$, $\xi(\mathfrak{u}) < \top_V$ implies $\xi(Tg(\mathfrak{u})) = \bot_V$.*

Consider the topological theory $\mathscr{T}_U$ of Example 6.10.

Proposition 6.16. *g is a $\mathscr{T}_U$-conucleus.*

Proof. For $\mathfrak{u} \in UV$ with $\xi(\mathfrak{u}) < \top_V$, $\bigwedge_{u\in A} u < \top_V$ for every $A \in \mathfrak{u}$. If $\xi(Ug(\mathfrak{u})) = \bigvee_{A\in Ug(\mathfrak{u})} \bigwedge_{u\in A} u > \bot_V$, then there exists $A_0 \in Ug(\mathfrak{u})$ such that $\bigwedge_{u\in A_0} u > \bot_V$, which implies $g^{-1}(A_0) \in \mathfrak{u}$ and $\bot_V \notin A_0$. Since $g^{-1}(A_0) \in \mathfrak{u}$, $g^{-1}(A_0) \neq \varnothing$, and therefore, $\top_V \in A_0$. Then, $B_0 := g^{-1}(A_0) = \{\top_V\} \in \mathfrak{u}$ and $\bigwedge B_0 = \top_V$, which contradicts the earlier $\bigwedge B_0 < \top_V$. □

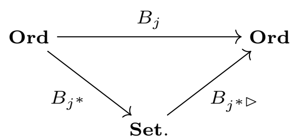
For the theory $\mathscr{T}_H$ of Remark 6.14, note that, (by Corollary 6.8), *every* ($\bigvee$-preserving) unital quantic conucleus on V is a (strict) $\mathscr{T}_H$-conucleus. The conucleus g from Subsection 5.2 though is not $\bigvee$-preserving.

7. Examples

This section shows some examples of lax-algebraic quotients, thereby providing a common framework for several already defined (as well as not yet defined) categories of the form $(\mathbb{T}, V)$-**Cat**.

7.1. Preordered sets

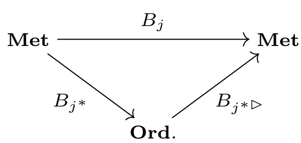
$\mathsf{2}$-**Cat** (recall the quantale $\mathsf{2} = \{\bot_\mathsf{2}, \top_\mathsf{2}\}$) is the category **Ord** of *preordered sets* (sets X, with a relation $\leqslant\, \subseteq X \times X$ such that $x \leqslant x$, and $x \leqslant y$, $y \leqslant z$ imply $x \leqslant z$, for every $x, y, z \in X$) and preorder-preserving maps [7]. The quantic nucleus $\mathsf{2} \xrightarrow{j} \mathsf{2}$, $j(\bot_\mathsf{2}) = j(\top_\mathsf{2}) = \top_\mathsf{2}$, gives the commutative triangle



Ord in the above can be replaced by an arbitrary category V-**Cat**.

7.2. Premetric spaces

Let P_+ be the unital quantale, which is given by the extended real half-line $[0, \infty]$ with the dual partial order, $\otimes = +$ and $k = 0$. P_+-**Cat** is the category **Met** of *premetric spaces* (in the sense of F. W. Lawvere [17]) and non-expansive maps [4,6]. The quantic nucleus $\mathsf{P}_+ \xrightarrow{j} \mathsf{P}_+$ of Subsection 5.1 (P_+ satisfies condition (A)) provides the commutative triangle



For a premetric space (X, a), $B_{j^*}(X, a) = (X, \leqslant)$, with $x \leqslant y$ iff $a(x, y) < \infty$.

7.3. Probabilistic metric spaces

In [11], D. Hofmann and C. D. Reis represented the category **ProbMet** of *probabilistic metric spaces* of [23] as the category V-**Cat** for a unital quantale V. They also constructed a pair of functors **ProbMet** $\to$ **Met** $\to$ **ProbMet**, which serve as an instance of lax-algebraic quotients as is shown below.

The quantale V in question is given by the set $\Delta = \{[0, \infty] \xrightarrow{f} [0, 1] \mid f$ is order-preserving and $f(x) = \bigvee_{y<x} f(y)\}$ equipped with the pointwise order. The quantale operation is defined by $(f \otimes g)(x) = \bigvee_{r+s \leqslant x} f(r) \odot g(s)$, where $\odot$ is the standard multiplication of real numbers. The quantale unit is provided by the

map $[0,\infty] \xrightarrow{\varepsilon} [0,1]$, where

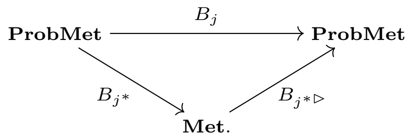
$$\varepsilon(x) = \begin{cases} 0, & x = 0 \\ 1, & \text{otherwise.} \end{cases}$$

The category **Δ-Cat** is then isomorphic to the category **ProbMet** [11].

The quantic nucleus $\Delta \xrightarrow{j} \Delta$ given by

$$(j(f))(x) = \begin{cases} 0, & x \leqslant \sup\{y \in [0,\infty] \mid f(y) = 0\} \\ 1, & \sup\{y \in [0,\infty] \mid f(y) = 0\} < x \end{cases}$$

provides the commutative triangle

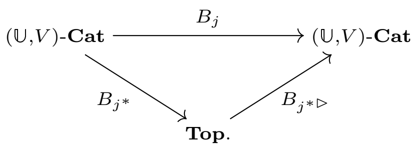


Note that, given a probabilistic metric space (X, a), $B_{j^*}(X, a) = (X, b)$ in which $b(x, y) = \sup\{z \in [0,\infty] \mid (a(x, y))(z) = 0\}$.

7.4. Generalized approach spaces

According to [6], given the ultrafilter monad $\mathbb{U}$ on **Set**, every completely distributive unital quantale V provides the canonical extension (in the sense of Section 5) $\hat{\mathbb{U}}$ of $\mathbb{U}$ to V-**Rel**, defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ by $(\hat{U}r)(\mathfrak{x}, \mathfrak{y}) = \bigwedge_{A\in\mathfrak{x}, B\in\mathfrak{y}} \bigvee_{x\in A, y\in B} r(x, y)$. In particular, $(\mathbb{U}, 2)$-**Cat** is then isomorphic to the category **Top** of topological spaces, and $(\mathbb{U}, \mathsf{P}_+)$-**Cat** is isomorphic to the category **App** of *approach spaces* of R. Lowen [18] (see, e.g., [3]).

With Proposition 5.4 in hand, one gets the commutative triangle

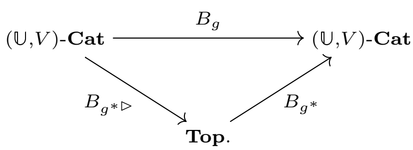


Since P_+ does not satisfy condition (B) of Subsection 5.1, the factorization is not applicable to $(\mathbb{U}, \mathsf{P}_+)$-**Cat**, i.e., to the category **App** of approach spaces.

7.5. Approach spaces

The previous subsection does not make it possible to represent the category **Top** of topological spaces as a canonical quotient of the category **App** of approach spaces. We show that **Top** can be represented as a canonical subobject of **App**.

With Proposition 5.6 in hand, one gets the commutative triangle



Since the quantale P_+ is strictly two-sided, **Top** is a canonical $\mathbb{U}$-subobject of **App** in which, given a topological space (X, a), $B_{g^*}(X, a) = (X, \delta)$ (recall from, e.g., [18] that $X \times \mathcal{P}(X) \xrightarrow{\delta} \mathsf{P}_+$ is an *approach distance* on X), where

$$\delta(x, A) = \begin{cases} 0, & x \in c(A) \\ \infty, & \text{otherwise,} \end{cases}$$

in which $c(A)$ is the closure of A in the space (X, a). Notice that, for $V = \mathsf{P}_+$, the full embedding B_{g^*} and its right adjoint $B_{g^{*\triangleright}}$ can be found in, e.g., [6,12].

7.6. V-closure spaces

In [24,25], G. Seal considered V-*closure spaces* over a quantale V as follows. Given the powerset monad $\mathbb{P}$ on **Set**, every completely distributive unital quantale V provides the canonical extension $\hat{\mathbb{P}}$ of $\mathbb{P}$ to V-**Rel**, defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ by $(\hat{P}r)(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} r(x, y)$. In particular, $(\mathbb{P}, 2)$-**Cat** is isomorphic to the category **Cls** of *closure spaces* (we recall that a *closure space* is a pair (X, c), where X is a set and $PX \xrightarrow{c} PX$ is a monotone map w.r.t. the inclusion order such that $1_X \leqslant c$ and $cc \leqslant c$; a map $(X, c_X) \xrightarrow{f} (Y, c_Y)$ is *continuous* provided that $f(c_X(A)) \subseteq c_Y(f(A))$ for every $A \in PX$), and $(\mathbb{P}, V)$-**Cat** is the category V-**Cls** of V-*closure spaces* of [25].

With Proposition 5.4 in hand, one gets the commutative triangle

$$\begin{array}{ccc} V\text{-}\mathbf{Cls} & \xrightarrow{\quad B_j \quad} & V\text{-}\mathbf{Cls} \\ & {\scriptstyle B_{j^*}} \searrow \quad \nearrow {\scriptstyle B_{j^{*\triangleright}}} & \\ & \mathbf{Cls}. & \end{array}$$

Since P_+ does not satisfy Subsection 5.1 (B), the factorization is not applicable to $(\mathbb{P}, \mathsf{P}_+)$-**Cat**, i.e., to the category **Clns** of *closeness spaces* of [24] (whose objects are the metric counterpart of closure spaces, in the same way that approach spaces are the metric counterpart of topological spaces).

We notice though that if V is a strictly two-sided completely distributive quantale, then, with the help of the quantic conucleus g of Subsection 5.2, one obtains the commutative diagram

$$\begin{array}{ccc} V\text{-}\mathbf{Cls} & \xrightarrow{\quad B_g \quad} & V\text{-}\mathbf{Cls} \\ & {\scriptstyle B_{g^{*\triangleright}}} \searrow \quad \nearrow {\scriptstyle B_{g^*}} & \\ & \mathbf{Cls}, & \end{array}$$

which (in particular) represents **Cls** as a canonical $\mathbb{P}$-subobject of **Clns**.

7.7. V-weighted H-labeled graphs

Every monoid $H = (H, \star, \ell)$ induces the free H-act-monad $\mathbb{H} = (H \times -, m, e)$ on **Set**, where $X \xrightarrow{e_X} H \times X$, $e_X(x) = (\ell, x)$ and $H \times (H \times X) \xrightarrow{m_X} H \times X$, $m_X((\alpha, (\beta, x))) = (\alpha \star \beta, x)$, for every set X.

For a unital quantale V, Remark 6.14 provides the topological theory $\mathscr{T}_H$ and the strict (Remark 6.6) lax extension $\hat{\mathbb{H}}$ to V-**Rel** defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ by $\hat{H}r((\alpha,x),(\beta,y)) = 1_H(\alpha,\beta) \otimes r(x,y)$, getting the category $(\mathbb{H},V)$-**Cat**.

Given an $(\mathbb{H},V)$-category (X,a), we write "$x \xrightarrow{(\alpha,v)} y$" for "$a((\alpha,x),y) = v$" (cf. [7, Example 2.1 (6)]). In such a way, (X,a) can be considered as a V*-weighted* H*-labeled graph* with the following two properties:

(1) $x \xrightarrow{(\ell,v)} x$ with $k \leqslant v$;

(2) $x \xrightarrow{(\alpha,u)} y$ and $y \xrightarrow{(\beta,v)} z$ implies $x \xrightarrow{(\beta\star\alpha,w)} z$ with $u \otimes v \leqslant w$.

An $(\mathbb{H},V)$-functor $(X,a) \xrightarrow{f} (Y,b)$ is a map $X \xrightarrow{f} Y$, satisfying the condition

$$x_1 \xrightarrow{(\alpha,u)} x_2 \text{ implies } f(x_1) \xrightarrow{(\alpha,v)} f(x_2) \text{ with } u \leqslant v.$$

The category $(\mathbb{H},2)$-**Cat** is isomorphic to the category of H-labeled graphs of [7, Example 2.1 (6)]. By Remark 6.14, the quantic nucleus of Subsection 5.1 provides the commutative triangle

$$\begin{array}{ccc} (\mathbb{H},V)\text{-}\mathbf{Cat} & \xrightarrow{\quad B_j \quad} & (\mathbb{H},V)\text{-}\mathbf{Cat} \\ & {\scriptstyle B_{j^*}} \searrow \quad \nearrow {\scriptstyle B_{j^*\rhd}} & \\ & (\mathbb{H},2)\text{-}\mathbf{Cat}. & \end{array}$$

For a V-weighted H-labeled graph (X,a), $B_{j^*}(X,a) = (X,b)$ is an H-labeled graph [7, Example 2.1 (6)], in which $x \xrightarrow{\alpha} y$ iff $x \xrightarrow{(\alpha,u)} y$ with $u \neq \bot_V$.

7.8. V-enriched multi-ordered sets

The *free-monoid monad* $\mathbb{L} = (L,m,e)$ on **Set** is defined by

- $LX = \{\mathfrak{x}_n \mid \mathfrak{x}_n = (x_1,\ldots,x_n),\ x_i \in X,\ n \in \mathbb{N}\bigcup\{0\}\}$, where $\mathbb{N}$ is the set of natural numbers;
- $X \xrightarrow{e_X} LX$, $e_X(x) = (x)$;
- $LLX \xrightarrow{m_X} LX$, $m_X((x_1^1,\ldots,x_{n_1}^1),\ldots,(x_1^m,\ldots,x_{n_m}^m)) = (x_1^1,\ldots,x_{n_m}^m)$.

By Example 6.12, we get the topological theory $\mathscr{T}_L$ and the strict (Remark 6.6) lax extension $\hat{\mathbb{L}}$ to V-**Rel** defined on a V-relation $X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y$ by $(\hat{L}r)(\mathfrak{x}_n,\mathfrak{y}_m) = 1_{\mathbb{N}}(n,m) \otimes (\bigotimes_{i=1}^{\min\{n,m\}} r(x_i,y_i))$ with $\bigotimes_{i=1}^{0} u_i = k$. Thus, one gets the category $(\mathbb{L},V)$-**Cat** of V*-enriched multi-ordered sets* where $(\mathbb{L},2)$-**Cat** is isomorphic to the category of multi-ordered sets of [7, Example 2.1 (5)].

By Proposition 6.13 (V satisfies (A)), the quantic nucleus of Subsection 5.1 provides the commutative triangle

$$\begin{array}{ccc} (\mathbb{L},V)\text{-}\mathbf{Cat} & \xrightarrow{\quad B_j \quad} & (\mathbb{L},V)\text{-}\mathbf{Cat} \\ & {\scriptstyle B_{j^*}} \searrow \quad \nearrow {\scriptstyle B_{j^*\rhd}} & \\ & (\mathbb{L},2)\text{-}\mathbf{Cat}. & \end{array}$$

Note that, given a V-enriched multi-ordered set (X,a), $B_{j^*}(X,a) = (X,b)$ where $\mathfrak{x}_n\, b\, x$ (recall that b is a relation) iff $a(\mathfrak{x}_n,x) \neq \bot_V$.

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