Math. Appl. **5** (2016), 21–38
DOI: 10.13164/ma.2016.03

GENERALIZED EULER VECTOR FIELDS ASSOCIATED TO THE WEIL BUNDLES

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Abstract. The notion of a Euler vector field is usually defined on the tangent bundle of a finite dimensional manifold M. In this paper, we generalize this notion to the Weil bundle T^AM, for any Weil algebra A and we study some properties.

1. Introduction

Let M be a smooth manifold of dimension $m \geq 1$, we denote by TM the tangent bundle of M. Usually, an *Euler vector field* is defined as a vector field on TM generated by the infinitesimal homotheties on the fibers of TM and is denoted by ξ_{TM}. In local coordinate system $(x^1, \cdots, x^m)$ of M, we denote by $(x^i, \dot{x}^i)$ the local coordinate (adapted) of TM. The local expression of ξ_{TM} is given by $\xi_{TM} = \dot{x}^i \frac{\partial}{\partial \dot{x}^i}$. The Euler vector field plays an essential role in the geometry of tangent bundle and is used in the global formulation of second order ordinary differential equation on a manifold M, it is also used to generalize to tensor fields the well known Euler's theorem on *homogeneous functions*. On the other hand, given a Weil algebra A, there is a product preserving functor T^A from the category $\mathcal{M}f$ of all smooth manifolds and all smooth maps to $\mathcal{M}f$ which generalizes the tangent functor called Weil functor associated to A (see [4]). We adopt the notations of [4] and by T^AM we denote the smooth manifold of all A-velocities of M and consider each element of T^AM in the form of an A-jet $j^A\varphi$, $\varphi \in C^\infty\left(\mathbb{R}^k, M\right)$. By $\pi^A_M : T^AM \to M$ we denote the canonical projection such that, $\pi^A_M\left(j^A\varphi\right) = \varphi(0)$ and, for any $f \in C^\infty(M, N)$, the map $T^Af \in C^\infty\left(T^AM, T^AN\right)$ is defined by $T^Af\left(j^A\varphi\right) = j^A(f \circ \varphi)$ where $\varphi \in C^\infty\left(\mathbb{R}^k, M\right)$. When A is the space of all r-jets of $\mathbb{R}^k$ into $\mathbb{R}$ with source $0 \in \mathbb{R}^k$ denoted by $J^r_0\left(\mathbb{R}^k, \mathbb{R}\right)$, the corresponding Weil functor is the functor of k-dimensional velocities of order r and denoted by T^r_k, in particular for $k = 1$, it is called a tangent functor of order r and denoted by T^r.

The aim of this paper is to generalize the concept of Euler vector field on the Weil bundle T^AM, which will be able to be one of the mains element of the generalized Lagrangian mechanic on T^AM. We will denote it by ξ_{T^AM}. Beyond these considerations, we prove that the Euler vector field obtained is such that, for any $f \in C^\infty(M, N)$, the vector fields ξ_{T^AM} and ξ_{T^AN} are T^Af-related. In particular, we have a natural transformation

$$\xi_A : T^A \to T \circ T^A \ \left(\text{over the identity on } T^A\right).$$

MSC (2010): primary 53C15, 58A20; secondary 53C75, 53D05, 58A32.
Keywords: Weil functor, Euler vector field, Dirac structures and natural transformations.

We define a natural Euler vector field associated to the Weil functor T^A as a natural transformation $\xi : T^A \to T \circ T^A$ (over the identity on T^A) and prove that there is a canonical bijective correspondence between the set of all natural Euler vector fields $T^A \to TT^A$ and the set of all derivations of A. In the particular case where $A = J_0^r(\mathbb{R}, \mathbb{R})$, we prove that any natural Euler vector field $\xi : T^r \to T \circ T^r$ is of the form

$$\sum_{\beta=1}^{r} a_\beta \xi_\beta$$

where $a_1, \cdots, a_r$ are the real numbers and ξ_β $(1 \leq \beta \leq r)$ is the natural Euler vector field generated by the derivation ϕ_β on $J_0^r(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^{r+1}$ defined below.

So, the paper is organized as follows. In Section 2, we recall briefly the main result of [3], about lifts of tensor fields to the Weil bundle. In Section 3, we define the generalized Euler vector field on $T^A M$ and establish some properties. In Section 4, the concept of homogeneous tensor fields on the manifold $T^A M$ is defined and some properties are studied. In the last Section some homogenity properties of the tangent lifts of order r of Poisson and Dirac manifolds related to Euler vector field $\xi_{T^r M}$ are established which generalize the similar results established in [8] and [9].

All manifolds and maps considered in the paper are assumed to be infinitely differentiable. We fix a Weil algebra A of height $h \geq 2$ and of width $k \geq 1$, for any $g \in C^\infty(\mathbb{R}^k, \mathbb{R})$ and any multiindex $\beta = (\beta_1, \cdots, \beta_k)$, we denote by

$$D_\beta(g)(z) = \frac{1}{\beta!} \frac{\partial^{|\beta|} g}{(\partial z_1)^{\beta_1} \cdots (\partial z_k)^{\beta_k}}(z)$$

the partial derivative with respect to the multiindex β of g.

2. Preliminaries

Let A be a Weil algebra, it is a real commutative and finite dimensional algebra with unit which is of the form $A = \mathbb{R} \cdot 1_A \oplus N_A$, where N_A is the ideal of nilpotent elements of A. For any multiindex $\alpha \neq 0$, the vector $e_\alpha = j^A(x^\alpha)$ is an element of N_A and the family $\{e_\alpha, 0 < |\alpha| \leq h\}$ generates N_A. We denote by B_A the set of all multiindices such that $(e_\alpha)_{\alpha \in B_A}$ is a basis of N_A and $\overline{B}_A$ its complementary with respect to the set $\{\gamma \in \mathbb{N}^n, 1 \leq |\gamma| \leq h\}$. We put $e_0 = 1_A$, it is clear that $(e_0, e_\alpha)_{\alpha \in B_A}$ is a basis of A. For $\beta \in \overline{B}_A$, we put $e_\beta = \lambda_\beta^\alpha e_\alpha$ and we have $|\alpha| = |\beta|$ or $\lambda_\beta^\alpha = 0$. So,

$$e_\alpha \cdot e_\beta = \begin{cases} e_{\alpha+\beta} & \text{if} \quad \alpha + \beta \in B_A \\ \lambda_{\alpha+\beta}^\gamma e_\gamma & \text{if} \quad \alpha + \beta \in \overline{B}_A \end{cases}.$$

Using this basis of A, one deduces an adapted local coordinate system of $T^A M$ in the following way: let (U, x^i) be a local coordinate system of M, an adapted local coordinate system induced by (U, x^i) over the open $T^A U$ of $T^A M$ denoted by $\left(x_0^i, x_\alpha^i\right)_{\alpha\in B_A}$ is given by

$$\begin{cases} x_0^i & = & x^i \circ \pi_M^A \\ x_\alpha^i & = & \overline{x}_\alpha^i + \sum\limits_{\beta\in\overline{B}_A} \lambda_\beta^\alpha \overline{x}_\beta^i \end{cases}, \text{where } \overline{x}_\beta^i \left(j^A g\right) = \frac{1}{\beta!} \cdot D_\beta \left(x^i \circ g\right)(z)\Big|_{z=0}.$$

In the sequel, the coordinate function x_0^i is denoted by x^i. The upper index (α) on the tensor fields $\boldsymbol{\varphi}$ on M is the α-lift of $\boldsymbol{\varphi}$ to its Weil bundle (see [2,9]). More precisely, for $f \in C^\infty(M)$ and $\alpha \in \mathbb{N}^n$ such that $0 \le |\alpha| \le h$, we define the map $f^{(\alpha)} \in C^\infty(T^A M)$ (α-lift of f) in the following way

$$f^{(\alpha)}\left(j^A\varphi\right) = \frac{1}{\alpha!} D_\alpha \left(f \circ \varphi\right)(z)\Big|_{z=0}$$

for any $j^A\varphi \in T^A M$. In the same way for $X \in \mathfrak{X}(M)$, $X^{(\alpha)} \in \mathfrak{X}(T^A M)$ denote the α-lift of X. In local coordinate $(x^1, \cdots, x^m)$ such that $X = X^i \frac{\partial}{\partial x^i}$, we have

$$X^{(\alpha)} = \sum_{\beta\in B_A} \left(\left(X^i\right)^{(\beta-\alpha)} + \sum_{\gamma\in\overline{B_A}} \lambda_\gamma^\beta \left(X^i\right)^{(\gamma-\alpha)} \right) \frac{\partial}{\partial x_\beta^i}.$$

For the measures of convenience, we put $f^{(\alpha)} = 0$ and $X^{(\alpha)} = 0$ for $|\alpha| > h$ or $\alpha \notin \mathbb{N}^k$.

Proposition 2.1. *For any tensor field* $\boldsymbol{t}$ *of type* $(0,p)$ *on* M*, the tensor field* $\boldsymbol{t}^{(\alpha)}$ *is the only tensor field of type* $(0,p)$ *on* $T^A M$ *satisfying*

$$\boldsymbol{t}^{(\alpha)}\left(X_1^{(\beta_1)}, \cdots, X_p^{(\beta_p)}\right) = \left(\boldsymbol{t}\left(X_1, \cdots, X_p\right)\right)^{(\alpha-\beta)}$$

where $\beta = \beta_1 + \cdots + \beta_p$ *and* $X_1, \cdots, X_p \in \mathfrak{X}(M)$.

Proof. See [3]. □

3. Main results

We recall that, for any $t \in \mathbb{R}$

$$\exp(t) = e^t = 1 + \sum_{p\ge 1} \frac{t^p}{p!}.$$

3.1. Euler vector fields on $T^A M$.

Let M be a smooth manifold of dimension $m \ge 1$. For any $\varphi \in C^\infty\left(\mathbb{R}^k, M\right)$, we consider the family of smooth maps $\{\varphi_t\}_{t\in\mathbb{R}} \subset C^\infty\left(\mathbb{R}^k, M\right)$ such that

$$\varphi_t(z) = \varphi(\exp(t) z)$$

for any $z \in \mathbb{R}^k$. We consider the smooth map

$$\begin{array}{cccc} \Psi_{A,M}: & \mathbb{R} \times T^A M & \to & T^A M \\ & \left(t, j^A\varphi\right) & \mapsto & j^A(\varphi_t) \end{array}.$$

The map $\Psi_{A,M}$ is a one parameter subgroup of a vector field, which we denote by $\xi_{T^A M}$.

Definition 3.1. The vector field $\xi_{T^A M}$ is called a generalized Euler vector field on $T^A M$.

Let (U, x^i) be a local coordinate system of M and (x^i, x^i_α) the local coordinate system of $T^A M$ induced by (U, x^i). We have $\frac{d(x^i \circ \Psi_{A,M}(t, j^A\varphi))}{dt}\Big|_{t=0} = 0$, by the same way using the equalities

$$\frac{d\left(x^i_\alpha \circ \Psi_{A,M}\left(t, j^A\varphi\right)\right)}{dt}\Big|_{t=0} = |\alpha|\, \overline{x}^i_\alpha \left(j^A\varphi\right) + \sum_{\beta \in \overline{B_A}} |\beta|\, \lambda^\alpha_\beta \overline{x}^i_\beta \left(j^A\varphi\right) = |\alpha|\, x^i_\alpha \left(j^A\varphi\right),$$

we deduce that the local expression of $\xi_{T^A M}$ is given by

$$\xi_{T^A M} = \sum_{\alpha \in B_A} |\alpha|\, x^i_\alpha \frac{\partial}{\partial x^i_\alpha}.$$

Example 3.2. (1) For $A = J^1_0\left(\mathbb{R}^k, \mathbb{R}\right)$, we have

$$\xi_{T^1_k M} = \sum_{|\alpha|=1} x^i_\alpha \frac{\partial}{\partial x^i_\alpha}.$$

In particular, when $k = 1$, we have

$$\xi_{TM} = \dot{x}^i \frac{\partial}{\partial \dot{x}^i}.$$

Therefore, ξ_{TM} is the classic Euler vector field on TM.

(2) More generally, if $A = J^r_0\left(\mathbb{R}^k, \mathbb{R}\right)$, for each manifold M, the Euler vector field on $T^r_k M$ is given by

$$\xi_{T^r_k M} = \sum_{1 \le |\alpha| \le r} |\alpha|\, x^i_\alpha \frac{\partial}{\partial x^i_\alpha}$$

In particular, when $k = 1$, we obtain

$$\xi_{T^r M} = \sum_{\alpha=1}^{r} \alpha x^i_\alpha \frac{\partial}{\partial x^i_\alpha}.$$

Proposition 3.3. *The Euler vector field $\xi_{T^A M}$ is the only vector field on $T^A M$ satisfying*

$$\xi_{T^A M}\left(g^{(\alpha)}\right) = |\alpha|\, g^{(\alpha)}$$

for any $0 \le |\alpha| \le h$ and $g \in C^\infty(M)$.

Proof. Let $j^A\varphi \in T^A M$ and $0 \le |\alpha| \le h$. Then, we have

$$\begin{aligned}\xi_{T^A M}\left(g^{(\alpha)}\right)\left(j^A\varphi\right) &= \frac{d}{dt}\left[g^{(\alpha)}\left(j^A\varphi_t\right)\right]\Big|_{t=0} = \frac{d}{dt}\left[\frac{1}{\alpha!} D_\alpha\left(g \circ \varphi_t\right)\right]\Big|_{t=0} \\ &= \frac{1}{\alpha!}\frac{d}{dt}\left[D_\alpha\left(g \circ \varphi\right)(0) \exp\left(|\alpha|\, t\right)\right]\Big|_{t=0} \\ &= |\alpha| \left(\frac{1}{\alpha!} D_\alpha\left(g \circ \varphi\right)(0)\right)\end{aligned}$$

and we deduce $\xi_{T^A M}\left(g^{(\alpha)}\right) = |\alpha|\, g^{(\alpha)}$. □

Remark 3.4. In particular, when $A=\mathbb{D}$, we have the classic formulas

$$\begin{cases} \xi_{TM}\left(g^{(0)}\right) & = & 0 \\ \xi_{TM}\left(g^{(1)}\right) & = & g^{(1)} \end{cases}$$

where $g^{(0)}=g\circ\pi_M$ and $g^{(1)}(v)=v(g)(\varsigma)$, with $v\in T_\varsigma M$.

Proposition 3.5. *Let $X\in\mathfrak{X}(M)$, we have*

$$\left[X^{(\alpha)},\xi_{T^AM}\right]=|\alpha|\,X^{(\alpha)}$$

for any $0\le|\alpha|\le h$.

Proof. By calculation. □

Proposition 3.6. *For any tensor field* $\mathbf{t}$ *of the type $(0,p)$ on M, we have*

$$\mathcal{L}_{\xi_{T^AM}}\boldsymbol{t}^{(\alpha)}=|\alpha|\ \boldsymbol{t}^{(\alpha)}$$

for any $0\le|\alpha|\le h$.

Proof. For any $X_1,\cdots,X_p\in\mathfrak{X}(M)$ and multiindices $\beta_1,\cdots,\beta_p$ we have

$$\begin{aligned} &\mathcal{L}_{\xi_{T^AM}}\boldsymbol{t}^{(\alpha)}\left(X_1^{(\beta_1)},\cdots,X_p^{(\beta_p)}\right)\\ =&\mathcal{L}_{\xi_{T^AM}}\left(\boldsymbol{t}\left(X_1,\cdots,X_p\right)\right)^{(\alpha-\beta)}-\sum_{i=1}^{p}\boldsymbol{t}^{(\alpha)}\left(X_1^{(\beta_1)},\cdots,\mathcal{L}_{\xi_{T^AM}}X_i^{(\beta_i)},\cdots,X_p^{(\beta_p)}\right)\\ =&|\alpha-\beta|\left(\boldsymbol{t}\left(X_1,\cdots,X_p\right)\right)^{(\alpha-\beta)}+|\beta|\left(\boldsymbol{t}\left(X_1,\cdots,X_p\right)\right)^{(\alpha-\beta)}\\ =&|\alpha|\,\boldsymbol{t}^{(\alpha)}\left(X_1^{(\beta_1)},\cdots,X_p^{(\beta_p)}\right). \end{aligned}$$

So, $\mathcal{L}_{\xi_{T^AM}}\boldsymbol{t}^{(\alpha)}=|\alpha|\ \boldsymbol{t}^{(\alpha)}$. □

3.2. Natural Euler vector fields

Let M and N be smooth manifolds, we begin this subsection by the fundamental property.

Proposition 3.7. *For any $f\in C^\infty(M,N)$, the Euler vector fields ξ_{T^AM} on T^AM and ξ_{T^AN} on T^AN are T^Af-related.*

Proof. Let $j^A\varphi\in T^AM$. Then we have

$$\begin{aligned} TT^Af\circ\xi_{T^AM}\left(j^A\varphi\right) &= TT^Af\left(\frac{d}{dt}\Psi_{A,M}\left(t,j^A\varphi\right)\Big|_{t=0}\right)\\ &= \frac{d}{dt}\left[T^Af\left(j^A\varphi_t\right)\right]\Big|_{t=0}=\frac{d}{dt}\left(j^A\left(f\circ\varphi_t\right)\right)\Big|_{t=0}\\ &= \frac{d}{dt}\Psi_{A,M}\left(t,j^A\left(f\circ\varphi\right)\right)\Big|_{t=0}=\xi_{T^AN}\circ T^Af\left(j^A\varphi\right). \end{aligned}$$

Therefore, $TT^Af\circ\xi_{T^AM}=\xi_{T^AN}\circ T^Af$. □

Definition 3.8. We call natural Euler vector fields associated to T^A any natural transformation $T^A\to T\circ T^A$ over the identity of T^A.

Example 3.9. By Proposition 3.7, it follows that the family $\{\xi_{T^AM}\}$ is a natural transformation $T^A\to T\circ T^A$ over the identity of T^A. So, it is a natural Euler vector field associated to T^A which we denote $\xi_A:T^A\to T\circ T^A$.

Given two Weil algebras A and B, all natural transformations $T^A \to T^B$ correspond exactly to the algebra homomorphism from A to B. In fact, for a natural transformation $\varphi^{A,B} : T^A \to T^B$, the algebra homomorphism associated is given by the linear map

$$\varphi_{\mathbb{R}}^{A,B} : A = T^A\mathbb{R} \to T^B\mathbb{R} = B.$$

On the other hand, the functor $T \circ T^A$ corresponds to Weil algebra $\mathbb{D} \otimes A$ which is identified with the Weil algebra $A^2 = A \times A$ endowed by the following structure: for any $(a,b), (a',b') \in A^2$,

$$(a,b) \cdot (a',b') \quad = \quad (aa', ab' + a'b).$$

Proposition 3.10. *Let $\varphi^{A,B} : T^A \to T^B$, the algebra homomorphism associated to Weil algebras A and B. We have*

$$T\left(\varphi_M^{A,B}\right) \circ \xi_{T^A M} = \xi_{T^B M} \circ \varphi_M^{A,B}$$

for any m-dimensional manifold M. In other words, the vector fields $\xi_{T^A M}$ and $\xi_{T^B M}$ are $\varphi_M^{A,B}$-related.

Proof. Let $j^A g \in T^A M$, we put $F_t = \Psi_{A,M}(t,\cdot)$. We have

$$\begin{aligned} T\left(\varphi_M^{A,B}\right) \circ \xi_{T^A M}\left(j^A g\right) &= T\left(\varphi_M^{A,B}\right)\left(\frac{d}{dt}F_t\left(j^A g\right)\Big|_{t=0}\right) \\ &= \frac{d}{dt}\left(\varphi_M^{A,B} \circ F_t\left(j^A g\right)\right)\Big|_{t=0} \\ &= \frac{d}{dt}\left(F_t \circ \varphi_M^{A,B}\left(j^A g\right)\right)\Big|_{t=0} \\ &= \xi_{T^B M}\left(\varphi_M^{A,B}\left(j^A g\right)\right). \end{aligned}$$

Thus, we get $T\left(\varphi_M^{A,B}\right) \circ \xi_{T^A M}\left(j^A g\right) = \xi_{T^B M}\left(\varphi_M^{A,B}\left(j^A g\right)\right)$ for any $j^A g \in T^A M$. □

Remark 3.11. We assume that A is a (k,r)-algebra. The surjective algebra homomorphism $\varrho : J_0^r(\mathbb{R}^k, \mathbb{R}) \to A$ determines a natural transformation $\varrho : T_k^r \to T^A$. So, for any manifold M the vector fields $\xi_{T_k^r M}$ and $\xi_{T^A M}$ are ϱ_M-related.

Applying the theory of Weil functors, we characterize all natural Euler vector fields on Weil functors (bundles) as follows.

Theorem 3.12. *There is a bijective correspondence between the set of natural Euler vector fields $T^A \to T \circ T^A$ and the set of the derivations of A.*

Proof. Let $\varphi_A : T^A \to T \circ T^A$ the natural Euler vector field, the map $\varphi_{A,\mathbb{R}} : A \to A \times A$ is an algebra homomorphism over A. It has the form

$$\varphi_{A,\mathbb{R}}(a) = (a, D(a))$$

where $D : A \to A$ is a linear map. On the other hand, for any $a, b \in A$,

$$\varphi_{A,\mathbb{R}}(ab) = \varphi_{A,\mathbb{R}}(a) \cdot \varphi_{A,\mathbb{R}}(b).$$

Since

$$\begin{cases} \varphi_{A,\mathbb{R}}(a) \cdot \varphi_{A,\mathbb{R}}(b) & = & (ab, aD(b) + bD(a)) \\ \varphi_{A,\mathbb{R}}(ab) & = & (ab, D(ab)) \end{cases},$$

we obtain $D(ab) = aD(b) + bD(a)$. So, D is a derivation of A.

Inversely, consider $D : A \to A$ the derivation, the map

$$\begin{array}{rccc} \varphi_D^A : & A & \to & A \times A \\ & a & \mapsto & (a, D(a)) \end{array}$$

is a morphism of Weil algebras. It induces a natural transformation $\overline{\varphi}_D^A : T^A \to T \circ T^A$. It is clear that $\overline{\varphi}_{D,\mathbb{R}}^A = \varphi_{A,\mathbb{R}}$. The rest of the proof is similar to Theorem (35.13) in [4]. □

Remark 3.13. Let $D : A \to A$ be a derivation, we consider the Euler vector field ξ_{D,T^AM} induced by D on a m-dimensional manifold M. In local coordinate $(x_1, \cdots, x_m)$, we have

$$\xi_{D,T^AM} = \sum_{\alpha,\beta \in B_A} \left(e_\beta^* \circ D(e_\alpha)\right) x_\alpha^i \frac{\partial}{\partial x_\alpha^i}$$

where $(e_\alpha)_{\alpha \in B_A}$ is the basis of N_A, $(e_\alpha^*)_{\alpha \in B_A}$ its dual and $D(e_0) = 0$.

Let $\varphi_1 : T^A \to T \circ T^A$ and $\varphi_2 : T^A \to T \circ T^A$ be two natural Euler vector fields. For any real number b, we define the natural Euler vector fields $\varphi_1 + \varphi_2$ and $b\varphi_1$ by

$$\begin{array}{rcl} (\varphi_1 + \varphi_2)_M & = & \varphi_{1,M} + \varphi_{2,M} \\ (b\varphi_1)_M & = & b\varphi_{1,M} \end{array}$$

for any manifold M. Note that $\varphi_{1,M} + \varphi_{2,M}$ is the sum of the vector fields $\varphi_{1,M}$ and $\varphi_{2,M}$, while $b\varphi_{1,M}$ is the product of vector field $\varphi_{1,M}$ by the scalar b. We denote by $\mathbf{Der}(A)$ the vector space of all the derivations of A and $\mathbf{Nev}(A)$ the space of all natural Euler vector fields. By the theorem above, we have a map

$$\begin{array}{rccc} \Phi : & \mathbf{Der}(A) & \to & \mathbf{Nev}(A) \\ & D & \mapsto & \overline{\varphi}_D^A \end{array}.$$

So, we have

Corollary 3.14. *The map $\Phi : \mathbf{Der}(A) \to \mathbf{Nev}(A)$ is an isomorphism of vector spaces.*

Example 3.15. We consider the classical Euler vector field ξ_{TM} defined on TM. The Euler vector field ξ_{TM} is obtained with the help of the derivation d defined on $\mathbb{D} \simeq \mathbb{R}^2$ such that,

$$d(x, y) = (0, y)$$

for any $(x, y) \in \mathbb{R}^2$.

Example 3.16. The natural Euler vector field $\xi_A : T^A \to T \circ T^A$ (Example 3.9) is determined by the derivation $D_A : A \to A$ defined by

$$D_A(e_\alpha) = |\alpha| e_\alpha$$

for any $0 \leq |\alpha| \leq h$.

Example 3.17. All natural Euler vector fields associated to the tangent functor T are of the form

$$b\xi_{TM}$$

where b is a real parameter. In fact, the structure of Weil algebra $\mathbb{D}=\mathbb{R}^2$ is given by

$$(x_1,y_1)\cdot(x_2,y_2)=(x_1x_2,x_1y_2+x_2y_1).$$

Let $\varphi_M : TM \to TTM$ be the Euler vector field on TM, it is associated to a derivation $d:\mathbb{R}^2\to\mathbb{R}^2$. It has the form

$$d(x,y)=(0,by)$$

with $b\in\mathbb{R}$. It follows that the Euler vector field associated to φ_M is $b\xi_{TM}$.

In the next subsection, we generalize this result of the previous example to a tangent bundle of a higher order.

3.3. The natural Euler vector fields $T^r \to T\circ T^r$

Using the identification $\mathbb{R}^{r+1}\simeq J_0^r(\mathbb{R},\mathbb{R})$ with the canonical basis $(e_\alpha)_{0\leq\alpha\leq r}$ such that

$$e_\alpha\cdot e_\beta=\frac{(\alpha+\beta)!}{\alpha!\beta!}e_{\alpha+\beta},$$

we have

Lemma 3.18. *For any $0<\beta\leq r$, the linear map $\phi_\beta : J_0^r(\mathbb{R},\mathbb{R})\to J_0^r(\mathbb{R},\mathbb{R})$ defined by*

$$\begin{cases}\phi_\beta(e_0) & = & 0\\ \phi_\beta(e_{\alpha+1}) & = & \frac{(\alpha+\beta)!}{\alpha!\beta!}e_{\alpha+\beta}\end{cases}$$

is a derivation.

Proof. By calculation. □

Remark 3.19. For any $\beta=1,\cdots,r$, we denote by $\xi_\beta : T^r \to T\circ T^r$ the natural Euler vector field related to ϕ_β. For any manifold M of dimension $m\geq 1$, we have locally

$$\xi_{\beta,T^rM}=\sum_{\alpha=0}^{r-\beta}\frac{(\alpha+\beta)!}{\alpha!\beta!}x^i_{\alpha+1}\frac{\partial}{\partial x^i_{\alpha+1}}.$$

Lemma 3.20. *Any derivation $\phi:\mathbb{R}^{r+1}\to\mathbb{R}^{r+1}$ is of the form*

$$\phi=\sum_{\beta=1}^{r}a_\beta\cdot\phi_\beta$$

where $a_1,\cdots,a_r$ are the real numbers.

Proof. For any $\alpha=0,\cdots,r$, we have $e_0\cdot e_\alpha=e_\alpha$, therefore $\phi(e_\alpha)\cdot e_0+\phi(e_0)\cdot e_\alpha=\phi(e_\alpha)$. It follows that

$$\phi(e_0)\cdot e_\alpha=0,\quad \forall\alpha=0,\cdots,r.$$

Thus, $\phi(e_0) = 0$. We put

$$\phi(e_1) = \sum_{\beta=0}^{r} a_\beta e_\beta$$

with $a_0, a_1, \cdots, a_r$ being real numbers. Using the relation $e_1 \cdot e_1 = 2e_2$, we have

$$\phi(e_2) = \phi(e_1) \cdot e_1 = \sum_{\beta=0}^{r-1} (\beta+1) a_\beta e_{\beta+1}.$$

In the same way, $e_2 \cdot e_1 = 3e_3$, it follows that, $3\phi(e_3) = \phi(e_2) \cdot e_1 + \phi(e_1) \cdot e_2$. Now

$$\begin{aligned} \phi(e_2) \cdot e_1 &= \sum_{\beta=0}^{r-2} (\beta+1)(\beta+2) a_\beta e_{\beta+2}, \\ \phi(e_1) \cdot e_2 &= \sum_{\beta=0}^{r-2} \frac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2}. \end{aligned}$$

We deduce that,

$$\phi(e_2) \cdot e_1 + \phi(e_1) \cdot e_2 = \sum_{\beta=0}^{r-2} 3 \frac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2}.$$

So,

$$\phi(e_3) = \sum_{\beta=0}^{n-2} \frac{(\beta+1)(\beta+2)}{2} a_\beta e_{\beta+2}.$$

Looking the at expressions of $\phi(e_1)$, $\phi(e_2)$ and $\phi(e_3)$ we put

$$\phi(e_\alpha) = \sum_{\beta=0}^{r-\alpha+1} \frac{(\alpha+\beta-1)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta-1}.$$

By induction, using the relation $e_\alpha \cdot e_1 = (\alpha+1) e_{\alpha+1}$, we obtain

$$(\alpha+1)\phi(e_{\alpha+1}) = \phi(e_\alpha) \cdot e_1 + \phi(e_1) \cdot e_\alpha.$$

Now,

$$\begin{aligned} \phi(e_\alpha) \cdot e_1 &= \sum_{\beta=0}^{r-\alpha+1} \frac{(\alpha+\beta-1)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta-1} \cdot e_1 = \sum_{\beta=0}^{r-\alpha} \frac{(\alpha+\beta)!}{(\beta-1)!\alpha!} a_\beta e_{\alpha+\beta}, \\ \phi(e_1) \cdot e_\alpha &= \sum_{\beta=0}^{r} a_\beta e_\beta \cdot e_\alpha = \sum_{\beta=0}^{r-\alpha} \frac{(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}. \end{aligned}$$

We deduce that

$$\phi(e_\alpha) \cdot e_1 + \phi(e_1) \cdot e_\alpha = \sum_{\beta=0}^{r-\alpha} \frac{(\alpha+1)(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}.$$

Thus,

$$\phi(e_{\alpha+1}) = \sum_{\beta=0}^{r-\alpha} \frac{(\alpha+\beta)!}{\beta!\alpha!} a_\beta e_{\alpha+\beta}.$$

On the other hand, $\phi(e_r) = a_0 e_{r-1} + a_1 e_r$ and $e_r \cdot e_1 = 0$. Thus, $\phi(e_r) \cdot e_1 + \phi(e_1) \cdot e_r = 0$. As

$$\begin{array}{rcl} \phi(e_r) \cdot e_1 & = & r a_0 e_r, \\ \phi(e_1) \cdot e_r & = & a_0 e_r, \end{array}$$

it follows that $a_0 = 0$. So, for any $\alpha = 0, \cdots, r-1$, we have

$$\phi(e_{\alpha+1}) = \sum_{\beta=1}^{r-\alpha} a_\beta \frac{(\alpha+\beta)!}{\beta!\alpha!} e_{\alpha+\beta} = \sum_{\beta=1}^{r-\alpha} a_\beta \phi_\beta(e_{\alpha+1}).$$

This yields the result. □

Theorem 3.21. *All natural Euler vector fields $T^r \to T \circ T^r$ are of the form*

$$\sum_{\beta=1}^{r} a_\beta \xi_{\beta,\cdot}$$

where $a_1, \cdots, a_r$ are real numbers.

Proof. Let $E : T^r \to T \circ T^r$ be a natural Euler vector field, it induces a derivation $\phi : \mathbb{R}^{r+1} \to \mathbb{R}^{r+1}$, where the structure of Weil algebra is defined above. So, by the previous lemma, there are the real numbers $a_1, \cdots, a_r$ such that

$$\phi = \sum_{\beta=1}^{r} a_\beta \phi_\beta$$

On the other hand, for any manifold M, locally we have

$$\begin{array}{rcl} E_{T^r M} & = & \sum_{\alpha,\gamma=1}^{r} \left(e_\gamma^* \circ \phi(e_\alpha)\right) x_\alpha^i \frac{\partial}{\partial x_\alpha^i} \\ & = & \sum_{\beta=1}^{r} \sum_{\alpha,\gamma=1}^{r} a_\beta \left(e_\gamma^* \circ \phi_\beta(e_\alpha)\right) x_\alpha^i \frac{\partial}{\partial x_\alpha^i} \\ & = & \sum_{\beta=1}^{r} a_\beta \xi_{\beta, T^r M}. \end{array}$$

Thus, we obtain $E = \sum_{\beta=1}^{r} a_\beta \xi_{\beta,\cdot}$. □

3.4. Absolute operators seen as natural Euler vector fields

Let F be a bundle functor on the category $\mathcal{M}f$. We denote by 0_M the zero vector field on M.

Definition 3.22. ([4]) A natural operator $R : T \rightsquigarrow T \circ F$ is said to be an absolute operator if $R_M X = R_M 0_M$ for every vector field X of M.

Let D be a derivation of A, for any real number t, $\phi_t = \exp(tD) \in \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ is the group of all automorphisms of A. It is a Lie subgroup of Lie group $GL(A)$. The map $\phi_t : A \to A$ is an automorphism of A inducing a natural transformation $\phi_{t,M} : T^A M \to T^A M$. Consider the map $D(M) : \mathbb{R} \times T^A M \to T^A M$ such that

$$D(M)(t, \xi) = \phi_{t,M}(\xi)$$

for any $(t,\xi) \in \mathbb{R} \times T^A M$. It is one parameter subgroup of the vector field $X_{D(M)} : T^A M \to TT^A M$. On the other hand, for any $f \in C^\infty(M,N)$ we have $T^A f \circ \phi_{t,M} = \phi_{t,N} \circ T^A f$ for every t. It follows that $X_{D(M)}$ and $X_{D(N)}$ are $T^A f$-related. We get a natural Euler vector field $X_D : T^A \to T \circ T^A$ associated to D.

Remark 3.23. The constant map $X \mapsto X_{D(M)}$ for all $X \in \mathfrak{X}(TM)$ forms an absolute operator, $\mathrm{Op}(D) : T \rightsquigarrow T \circ T^A$, which is said to be generated by D. In [4], it is shown that every absolute operator $R : T \rightsquigarrow T \circ T^A$ is of the form $R = \mathrm{Op}(D)$.

Corollary 3.24. *There is bijective correspondence between the set of absolute operators and the set of natural Euler vector fields associated to T^A.*

The main result on the prolongations of vector fields related to Weil bundle is given by I. Kolář ([4]). In fact, it proves that all natural operators $T \rightsquigarrow T \circ T^A$ are of the form

$$\mathrm{af}\,(c) \circ \mathcal{T}^A + \mathrm{op}\,(D)$$

where $\mathrm{af}\,(c)$ is the natural affinor determined by $c \in A$, $\mathcal{T}^A$ the flow operator and $\mathrm{op}\,(D)$ the absolute operator determined by the derivation D.

Corollary 3.25. *Let $X \in \mathfrak{X}(M)$, any prolongation of X from M to $T^A M$ is of the form*

$$\sum_{0 \le |\alpha| \le h} a_\alpha X^{(\alpha)} + E$$

where E is an Euler vector on $T^A M$ induced by some derivation of A and a_0, a_α are the real numbers.

Proof. Let $X \in \mathfrak{X}(M)$ and $\widetilde{X}$ be a prolongation of X on $T^A M$. We have $\widetilde{X} = \mathrm{af}\,(a) \circ \mathcal{T}^A X + \mathrm{Op}\,(D)\, 0_M$, for a some derivation D and $a \in A$. As $a = \sum_{0 \le |\alpha| \le h} a_\alpha e_\alpha$ we obtain

$$\widetilde{X} = \sum_{0 \le |\alpha| \le h} a_\alpha X^{(\alpha)} + E$$

where E is the Euler vector field induced by D. □

Corollary 3.26. *All prolongations of the vector field X from M to $T^r M$ are of the form*

$$\sum_{\alpha=0}^{r} a_\alpha X^{(\alpha)} + \sum_{\beta=1}^{r} b_\beta \xi_{\beta,M}$$

where a_α, b_β are real numbers.

4. Homogeneous tensor fields on the Weil bundles

The notion of homogenity for functions on $\mathbb{R}^n$ can be extended in an obvious way for functions, vector fields, differential forms, multivector fields on the Weil bundle $T^A M$ of a manifold M of dimension $m > 0$. In this subsection, we generalize the results of [2] while replacing the tangent bundle of higher order by any Weil bundle.

4.1. Homogeneous tensor fields

Let M be a smooth manifold of dimension $m > 0$. We denote by $\xi_{T^A M}$ the Euler vector field on the Weil bundle $T^A M$. The global flow of $\xi_{T^A M}$ is given by the map

$$F_t : T^A M \to T^A M, \quad j^A g \mapsto j^A (g_t)$$

for any real number t.

Definition 4.1. A tensor $\boldsymbol{\varphi}$ on $T^A M$ is said to be homogeneous of degree $|\alpha|$ $\left(\alpha \in \mathbb{N}^k\right)$ if

$$F_t^* \boldsymbol{\varphi} = e^{|\alpha| t} \boldsymbol{\varphi}$$

for any real number t.

Proposition 4.2. *A tensor $\boldsymbol{\varphi}$ on $T^A M$ is homogeneous of degree $|\alpha|$ if and only if*

$$\mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi} = |\alpha| \, \boldsymbol{\varphi}.$$

Proof. Supposing that $\boldsymbol{\varphi}$ is homogeneous tensor fields of degree $|\alpha|$, we have:

$$\mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi} = \lim_{t \to 0} \left(\frac{F_t^* \boldsymbol{\varphi} - \boldsymbol{\varphi}}{t} \right) = \lim_{t \to 0} \left(\frac{e^{|\alpha| t} - 1}{t} \right) \boldsymbol{\varphi} = |\alpha| \, \boldsymbol{\varphi}.$$

Inversely, supposing that $\mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi} = |\alpha| \, \boldsymbol{\varphi}$, for any $z \in T^A M$ the function $X : t \mapsto F_t^* \boldsymbol{\varphi}(z)$ is the solution of a differential equation $\frac{du}{dt} = |\alpha| \, u$ with initial condition $u(0) = \boldsymbol{\varphi}(z)$. Indeed, $X(0) = \boldsymbol{\varphi}(z)$ and $\frac{d}{dt}\left(F_t^* \boldsymbol{\varphi}\right) = F_t^* \mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi} = |\alpha| \, F_t^* \boldsymbol{\varphi}$, it follows that $F_t^* \boldsymbol{\varphi}(z) = e^{|\alpha| t} \boldsymbol{\varphi}(z)$. □

Example 4.3.
1. Let φ be a tensor field of the type $(0, p)$ on a manifold M, for any $|\alpha| \leq h$, the tensor $\varphi^{(\alpha)}$ (α-prolongation of φ on $T^A M$) is a homogeneous tensor field of degree $|\alpha|$.
2. Let X be a vector field on a manifold M. The vector field $X^{(\alpha)}$ (α-prolongation of X on $T^A M$) is a homogeneous vector field of degree $-|\alpha|$.
3. If f_1 and f_2 are homogeneous functions of degree $|\alpha_1|$ and $|\alpha_2|$ respectively on $T^A M$. Then $f_1 \cdot f_2$ is a homogeneous function of degree $|\alpha_1| + |\alpha_2|$.

Proposition 4.4. *If $\boldsymbol{\varphi_1}$ and $\boldsymbol{\varphi_2}$ are homogeneous tensor fields on $T^A M$ of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then, $\boldsymbol{\varphi_1} \otimes \boldsymbol{\varphi_2}$ is homogeneous tensor field on $T^A M$ of degree $|\alpha_1| + |\alpha_2|$.*

Proof. Given $\boldsymbol{\varphi_1}$ and $\boldsymbol{\varphi_2}$ as in the statement, we have:

$$\mathcal{L}_{\xi_{T^A M}} (\boldsymbol{\varphi_1} \otimes \boldsymbol{\varphi_2}) = (\mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi_1}) \otimes \boldsymbol{\varphi_2} + \boldsymbol{\varphi_1} \otimes (\mathcal{L}_{\xi_{T^A M}} \boldsymbol{\varphi_2}) = |\alpha_1| \, \boldsymbol{\varphi_1} \otimes \boldsymbol{\varphi_2} + |\alpha_2| \, \boldsymbol{\varphi_1} \otimes \boldsymbol{\varphi_2}.$$

It follows that $\boldsymbol{\varphi_1} \otimes \boldsymbol{\varphi_2}$ is a homogeneous tensor field of degree $|\alpha_1| + |\alpha_2|$. □

Corollary 4.5. *If X_1 and X_2 are homogeneous vector fields on $T^A M$ of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then $[X_1, X_2]$ is homogeneous vector field on $T^A M$ of degree $|\alpha_1| + |\alpha_2|$.*

Proof. Let X_1 and X_2 be homogeneous vector fields on $T^A M$ of degree $|\alpha_1|$ and $|\alpha_2|$. Using the Jacobi identity, we have

$$\mathcal{L}_{\xi_{T^A M}} [X_1, X_2] = \left[X_1, \mathcal{L}_{\xi_{T^A M}} X_2\right] + \left[\mathcal{L}_{\xi_{T^A M}} X_1, X_2\right] = (|\alpha_1| + |\alpha_2|) \, [X_1, X_2] \, .$$

Therefore, $[X_1, X_2]$ is a homogeneous vector field of degree $|\alpha_1| + |\alpha_2|$. □

Corollary 4.6. *Let X be a homogeneous vector field of degree $|\alpha|$ and f a homogeneous function of degree $|\beta|$. Then $X(f)$ is a homogeneous function of degree $|\alpha| - |\beta|$.*

Proof. We have

$$\mathcal{L}_{\xi_{T^AM}} X(f) = (\mathcal{L}_{\xi_{T^AM}} X)(f) - |\beta|\, X(f) = |\alpha|\, X(f) - |\beta|\, X(f)$$

and, therefore, $X(f)$ is a homogeneous function of degree $|\alpha| - |\beta|$. □

4.2. Particular case of the differential forms

Proposition 4.7. (1) *Let ω_1 and ω_2 be homogeneous forms of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then $\omega_1 \wedge \omega_2$ is homogeneous form of degree $|\alpha_1| + |\alpha_2|$.*

(2) *Let ω be a homogeneous p-form of degree $|\alpha|$ and $X_1, \cdots, X_p$, p homogeneous vector fields of degree $|\alpha_1|, \cdots, |\alpha_p|$. Then $\omega(X_1, \cdots, X_p)$ is a homogeneous function of degree $|\alpha| + |\alpha_1| + \cdots + |\alpha_p|$.*

Proof. We know that

$$\mathcal{L}_{\xi_{T^AM}}(\omega(X_1, \cdots, X_p)) = \mathcal{L}_{\xi_{T^AM}}\omega(X_1, \cdots, X_p) + \sum_{i=1}^{n} \omega(X_1, \cdots, \mathcal{L}_{\xi_{T^AM}} X_i, \cdots, X_p).$$

As $\mathcal{L}_{\xi_{T^AM}}\omega = |\alpha|\,\omega$ and $\mathcal{L}_{\xi_{T^AM}} X_i = |\alpha_i|\, X_i$ for any $i \leq p$, we have

$$= |\alpha|\,\omega(X_1, \cdots, X_p) + (|\alpha_1| + \cdots + |\alpha_p|)\omega(X_1, \cdots, X_p).$$

So, we obtain the result. □

Remark 4.8. Let $\left(x^i, x^i_\alpha\right)_{\alpha\in B_A}$ be an adapted local coordinate system of T^AM. The local expression of the Pfaff form ω on T^AM is given by

$$\omega = a^i_0 dx^i + \sum_{\alpha\in B_A} a^\alpha_i dx^i_\alpha$$

and we have

$$\mathcal{L}_{\xi_{T^AM}}\omega = \sum_{\alpha\in B_A} \xi_{T^AM}(a^\alpha_i)\, dx^i_\alpha + a^\alpha_i \xi_{T^AM}\left(dx^i_\alpha\right).$$

Now, $\xi_{T^AM} = \sum_{\beta\in B_A} |\beta|\, x^j_\beta \frac{\partial}{\partial x^j_\beta}$ and $\xi_{T^AM}\left(dx^i_\alpha\right) = |\alpha|\, dx^i_\alpha$ and we have

$$\mathcal{L}_{\xi_{T^AM}}\omega = \sum_{\alpha,\beta\in B_A} |\beta|\, x^j_\beta \frac{\partial a^\alpha_i}{\partial x^j_\beta} dx^i_\alpha + |\alpha|\, a^\alpha_i dx^i_\alpha = \sum_{\alpha\in B_A} (\sum_{\beta\in B_A} |\beta|\, x^j_\beta \frac{\partial a^\alpha_i}{\partial x^j_\beta} + |\alpha|\, a^\alpha_i) dx^i_\alpha.$$

If ω is homogeneous of degree $|\gamma|$, then we have $a^i_0 = 0$ and

$$\sum_{\beta\in B_A} |\beta|\, x^j_\beta \frac{\partial a^\alpha_i}{\partial x^j_\beta} + |\alpha|\, a^\alpha_i = |\gamma|\, a^\alpha_i.$$

Thus, we obtain

$$\sum_{\beta\in B_A} |\beta|\, x_\beta^j \frac{\partial a_i^\alpha}{\partial x_\beta^j} = (|\gamma| - |\alpha|)\, a_i^\alpha .$$

It follows that, for each $0 < i \leq m$ and $\alpha \in B_A$, the function a_i^α is homogeneous of degree $|\gamma| - |\alpha|$.

Proposition 4.9. *Let ω be a homogeneous p-form on $T^A M$ of degree $|\alpha|$. Then $d\omega$, $i_{\xi_{T^A M}}\omega$ are homogeneous of degree $|\alpha|$.*

Proof. In the first case,

$$\mathcal{L}_{\xi_{T^A M}} (d\omega) = d(\mathcal{L}_{\xi_{T^A M}}\omega) = |\alpha|\, d\omega.$$

By the same argument,

$$\mathcal{L}_{\xi_{T^A M}} (i_{\xi_{T^A M}}\omega) = i_{\xi_{T^A M}} d(i_{\xi_{T^A M}}\omega) = i_{\xi_{T^A M}} (\mathcal{L}_{\xi_{T^A M}}\omega) = |\alpha|\, (i_{\xi_{T^A M}}\omega).$$

Therefore $i_{\xi_{T^A M}}\omega$ and $d\omega$ are homogeneous of degree $|\alpha|$. □

4.3. Case of multivector fields

Proposition 4.10. (1) *Let π_1 and π_2 be homogeneous multivector fields on $T^A M$ of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then, $\pi_1 \wedge \pi_2$ is homogeneous multivector field of degree $|\alpha_1| + |\alpha_2|$.*

(2) *If $\pi = X_1 \wedge \cdots \wedge X_p$ is a simple multivector field on $T^A M$, where $X_1, \cdots, X_p$ are p homogeneous vector fields of degree $|\alpha_1|, \cdots, |\alpha_p|$. Then, π is a homogeneous multivector field on $T^A M$ of degree $|\alpha_1| + \cdots + |\alpha_p|$.*

(3) *Let π_1 and π_2 be homogeneous multivector fields on $T^A M$ of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then, $[\pi_1, \pi_2]$ is a homogeneous multivector field of degree $|\alpha_1| + |\alpha_2|$.*

Proof. Let $\pi_1 \in \mathfrak{X}^p\left(T^A M\right)$ and $\pi_2 \in \mathfrak{X}^q\left(T^A M\right)$, we have:

$$\begin{aligned}\mathcal{L}_{\xi_{T^A M}} [\pi_1, \pi_2] &= -(-1)^{(p-1)(q-1)} \left[\pi_2, \mathcal{L}_{\xi_{T^A M}}\pi_1\right] + \left[\pi_1, \mathcal{L}_{\xi_{T^A M}}\pi_2\right] \\ &= -(-1)^{(p-1)(q-1)} |\alpha_1| [\pi_2, \pi_1] + |\alpha_2| [\pi_1, \pi_2] .\end{aligned}$$

We deduce that $\mathcal{L}_{\xi_{T^A M}} [\pi_1, \pi_2] = (|\alpha_1| + |\alpha_2|) [\pi_1, \pi_2]$. □

Proposition 4.11. *Let $\pi \in \mathfrak{X}^p\left(T^A M\right)$ be a homogeneous multivector field of degree $|\alpha|$. For any p homogeneous functions $f_1, \cdots, f_p$ on $T^A M$ of degree $|\alpha_1|, \cdots, |\alpha_p|$ the function $\pi(df_1, \cdots, df_p)$ is homogeneous of degree $|\alpha| + |\alpha_1| + \cdots + |\alpha_p|$.*

Proof. We have,

$$\begin{aligned}&\mathcal{L}_{\xi_{T^A M}} (\pi (df_1, \cdots, df_p)) \\ =&\mathcal{L}_{\xi_{T^A M}} \pi (df_1, \cdots, df_p) + \sum_{i=1}^{p} \pi(df_1, \cdots \mathcal{L}_{\xi_{T^A M}} df_i, \cdots, df_p) \\ =&\, |\alpha|\, \pi (df_1, \cdots, df_p) + \sum_{i=1}^{n} |\alpha_i|\, \pi (df_1, \cdots, df_p) .\end{aligned}$$

We deduce that

$$\mathcal{L}_{\xi_{T^AM}}\left(\pi\left(df_1,\cdots,df_p\right)\right)=\left(|\alpha|+|\alpha_1|+\cdots+|\alpha_p|\right)\pi\left(df_1,\cdots,df_p\right).$$

□

Let M be a smooth m-dimensional manifold. A Poisson structure on M is a $\mathbb{R}$-bilinear Lie bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibnitz rule

$$\{f,gh\}=\{f,g\}h+g\{f,h\},\quad \forall f,g,h\in C^\infty(M). \tag{4.1}$$

It follows from (4.1) that there exists a bivector field $w\in\mathfrak{X}^2(M)$ such that

$$\{f,g\}_w=w\left(df,dg\right).$$

The Jacobi identity for $\{\cdot,\cdot\}_w$ is equivalent to the Poisson condition $[w,w]=0$, where $[\cdot,\cdot]$ is the Schouten-Nijenhuis bracket. In this case, one says that the bivector field w defines the Poisson structure on M.

Proposition 4.12. *Let π be a Poisson bivector on T^AM homogeneous of degree $|\alpha|$. For any homogeneous function f of degree $|\beta|$, the hamiltonian vector field X_f is a homogeneous vector field of degree $|\alpha|+|\beta|$.*

Proof. Let $g\in C^\infty(M)$ and $|\gamma|\le h$, we have:

$$\mathcal{L}_{\xi_{T^AM}}X_f\left(g^{(\gamma)}\right)=\left[\xi_{T^AM},X_f\right]\left(g^{(\gamma)}\right)=\xi_{T^AM}\left(\pi\left(df,dg^{(\gamma)}\right)\right)-|\gamma|\,X_f\left(g^{(\gamma)}\right)$$
$$=\left(|\alpha|+|\beta|+|\gamma|\right)\pi\left(df,dg^{(\gamma)}\right)-|\gamma|\,X_f\left(g^{(\gamma)}\right)=\left(|\alpha|+|\beta|\right)X_f\left(g^{(\gamma)}\right).$$

Therefore, $\mathcal{L}_{\xi_{T^AM}}X_f=\left(|\alpha|+|\beta|\right)X_f$. □

Corollary 4.13. *Let π be a Poisson bivector on T^AM homogeneous of degree $|\alpha|$. If f_1 and f_2 are homogeneous functions of degree $|\alpha_1|$ and $|\alpha_2|$ respectively. Then, the function $\{f_1,f_2\}$ is homogeneous of degree $|\alpha|+|\alpha_1|+|\alpha_2|$.*

5. Homogeneous properties of Euler vector fields on some geometric structures

In the sequel, by $\langle\cdot,\cdot\rangle_M$ we denote the canonical pairing $TM\times_M T^*M\to\mathbb{R}$.

5.1. Case of the tangent lifts of Poisson manifolds

For any manifold M of dimension $m\ge 1$, there is a canonical diffeomorphism (see [1,3,5])

$$\kappa_M^r:T^rTM\to TT^rM$$

which is an isomorphism of vector bundles from

$$T^r(\pi_M):T^rTM\to T^rM\quad\text{to}\quad \pi_{T^rM}:TT^rM\to T^rM.$$

It is called the canonical isomorphism of flow associated to the bundle functor T^r. Consider the linear form τ_r on $J_0^r(\mathbb{R},\mathbb{R})$ defined by $\tau_r\left(j_0^rg\right)=\frac{1}{r!}\frac{d^rg}{dt^r}(t)\Big|_{t=0}$ and the canonical map

$$\alpha_M^r:T^*T^rM\to T^rT^*M$$

which is an isomorphism of vector bundles

$$\pi_{T^rM}^*:T^*T^rM\to T^rM\quad\text{and}\quad T^r\left(\pi_M^*\right):T^rT^*M\to T^rM$$

such that, for any $(u,u^*)\in T^rTM\oplus T^*T^rM$,

$$\langle \kappa_M^r(u), u^*\rangle_{T^rM} = \langle u, \alpha_M^r(u^*)\rangle'_{T^rM}$$

where $\langle\cdot,\cdot\rangle'_{T^rM} = \tau_r\circ T^r(\langle\cdot,\cdot\rangle_M)$ (see [1]). We denote by ε_M^r the inverse of α_M^r.

Let $(x^1,\cdots,x^m)$ be a local coordinate system of M, we introduce the coordinates (x^i,p_j) in T^*M, $(x^i,p_j,x^i_\beta,p_j^\beta)$ in T^rT^*M and $(x^i,x^i_\beta,\pi_j,\pi_j^\beta)$ in T^*T^rM. We have:

$$\alpha_M^r(x^i,\pi_j,x^i_\beta,\pi_j^\beta) = (x^i,x^i_\beta,p_j,p_j^\beta),\ \text{with}\ \begin{cases} p_j &= \pi_j^r \\ p_j^\beta &= \pi_j^{r-\beta}\end{cases}.$$

Let (M,w) be a Poisson manifold. The complete lift of higher order of w in the sense of [6] and denoted by $w^{(c)}$ is a Poisson bivector field on T^rM since the Poisson condition $\left[w^{(c)},w^{(c)}\right]=0$, is satisfied. Denoting by $\sharp_w : T^*M\to TM$ the anchor map induced by w, we have

$$\sharp_{w^{(c)}} = \alpha_M^r\circ T^r(\sharp_w)\circ\kappa_M^r.$$

Let $\left(x^1,\cdots,x^m\right)$ be a local coordinate system of M such that $w = w^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$, we have

$$w^{(c)} = \left(w^{ij}\right)^{(\alpha+\beta-r)}\frac{\partial}{\partial x^i_\alpha}\wedge\frac{\partial}{\partial x^j_\beta}.$$

In [6], we have shown that, for any $f,g\in C^\infty(M)$, we have:

$$\left\{f^{(\alpha)},g^{(\beta)}\right\}_{w^{(c)}} = \{f,g\}_w^{(\alpha+\beta-r)}$$

with $0\le\alpha,\beta\le r$.

Definition 5.1. The Poisson manifold (M,w) is said homogeneous related to a vector field $X\in\mathfrak{X}(M)$ if $\mathcal{L}_Xw=-w$.

Theorem 5.2. *The Poisson manifold* $\left(T^rM,w^{(c)}\right)$ *is a homogeneous Poisson manifold related to* $\frac{1}{r}\cdot\xi_{T^rM}$. *More precisely,*

$$\left[\xi_{T^rM},w^{(c)}\right] = -rw^{(c)}.$$

Proof. For any $f,g\in C^\infty(M)$, we have

$$\begin{aligned}\left[\xi_{T^rM},w^{(c)}\right]\left(df^{(\alpha)}\wedge dg^{(\beta)}\right) &= \left\langle\xi_{T^rM}, d\left(i_{w^{(c)}}\left(df^{(\alpha)}\wedge dg^{(\beta)}\right)\right)\right\rangle_{T^rM} \\ &\quad -\left\langle w^{(c)}, d\left(\left(i_{\xi_{T^rM}}df^{(\alpha)}\right)\wedge dg^{(\beta)}\right)\right\rangle_{T^rM}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left[\xi_{T^rM},w^{(c)}\right]\left(df^{(\alpha)}\wedge dg^{(\beta)}\right)\\ &= -\left\langle w^{(c)},\alpha df^{(\alpha)}\wedge dg^{(\beta)}+\beta df^{(\alpha)}\wedge dg^{(\beta)}\right\rangle_{T^rM} + \xi_{T^rM}\left(\{f,g\}_w^{(\alpha+\beta-r)}\right)\\ &= (\alpha+\beta-r)\{f,g\}_w^{(\alpha+\beta-r)} - \left\langle w^{(c)},(\alpha+\beta)\,df^{(\alpha)}\wedge dg^{(\beta)}\right\rangle\\ &= (\alpha+\beta-r)\{f,g\}_w^{(\alpha+\beta-r)} - (\alpha+\beta)\{f,g\}_w^{(\alpha+\beta-r)}.\end{aligned}$$

We deduce that

$$\left[\xi_{T^rM},w^{(c)}\right]\left(df^{(\alpha)}\wedge dg^{(\beta)}\right) = -r\{f,g\}_w^{(\alpha+\beta-r)} = -rw^{(c)}\left(df^{(\alpha)}\wedge dg^{(\beta)}\right).$$

Thus, $\left[\xi_{T^rM},w^{(c)}\right] = -rw^{(c)}$. □

Remark 5.3. For $r=1$, we obtain the result established by I. Vaisman in [9].

5.2. Tangent Dirac structures of higher order

Let M be a smooth manifold of dimension $m \geq 1$. We recall that, an almost Dirac structure on M is a subbundle of $L \subset TM \oplus T^*M$ of rank m which is maximally isotrope related to the canonical pairing on $TM \oplus T^*M$ defined by

$$\langle X \oplus \omega, Y \oplus \varpi \rangle_+ = \frac{1}{2}\left(\langle Y, \omega\rangle_M + \langle X, \varpi\rangle_M\right).$$

We put

$$\langle X \oplus \omega, Y \oplus \varpi \rangle_- = \frac{1}{2}\left(\langle Y, \omega\rangle_M - \langle X, \varpi\rangle_M\right).$$

If the space of local sections of L denoted by $\Gamma(L)$ is closed under the bracket,

$$[X \oplus \omega, Y \oplus \varpi] = [X, Y] \oplus \left(\mathcal{L}_X \varpi - \mathcal{L}_Y \omega + d\left(\langle X \oplus \omega, Y \oplus \varpi \rangle_-\right)\right)$$

we say that L is a Dirac structure on M.

Definition 5.4. A Dirac structure L on M is called a homogeneous Dirac structure related to a vector field Z on M if, for any $(X, \omega) \in \Gamma(L)$, $([Z, X] + X, \mathcal{L}_Z \omega) \in \Gamma(L)$.

Let $L \subset TM \oplus T^*M$ be a Dirac structure, we put

$$\mathcal{T}^r L = (\kappa_M^r \oplus \varepsilon_M^r)(T^r L) \subset TT^r M \oplus T^*T^r M.$$

The subbundle $\mathcal{T}^r L \subset TT^r M \oplus T^*T^r M$ is a Dirac structure on $T^r M$ (see [5]). It is called tangent Dirac structure of higher order.

Lemma 5.5. *Let $L \subset TM \oplus T^*M$ be a Dirac structure on M. We have:*

$$X^{(\beta)} \oplus w^{(r-\beta)} \in \Gamma(\mathcal{T}^r L)$$

for any $X \oplus w \in \Gamma(L)$ and $\beta = 0, \cdots, r$.

Proof. See [7]. □

Theorem 5.6. *The Dirac structure $\mathcal{T}^r L$ on $T^r M$ is a homogeneous Dirac structure related to Euler vector field $\frac{\xi_{T^r M}}{r}$.*

Proof. We recall that the space of sections of $\mathcal{T}^r L$ is generated by the space

$$\left\{X^{(\alpha)} \oplus w^{(r-\alpha)}, X \oplus w \in \Gamma(L) \text{ and } \alpha = 0, \cdots, r\right\}.$$

For each section (X_1, w_1) of L, using the equalities

$$\begin{cases} \left[\frac{\xi_{T^r M}}{r}, X_1^{(\alpha)}\right] = -\frac{\alpha}{r} X_1^{(\alpha)} \\ \mathcal{L}_{\frac{\xi_{T^r M}}{r}} w_1^{(r-\alpha)} = \left(1 - \frac{\alpha}{r}\right) w_1^{(r-\alpha)} \end{cases}$$

we have

$$\left(\left[\frac{\xi_{T^r M}}{r}, X_1^{(\alpha)}\right] + X_1^{(\alpha)}\right) \oplus \mathcal{L}_{\frac{\xi_{T^r M}}{r}} w_1^{(r-\alpha)} = X_1^{(\alpha)} \oplus w_1^{(r-\alpha)} - \frac{\alpha}{r}\left(X_1^{(\alpha)} \oplus w_1^{(r-\alpha)}\right)$$

$X_1^{(\alpha)} \oplus w_1^{(r-\alpha)}$ is a section of $\mathcal{T}^r L$, which means that

$$\left(\left[\frac{\xi_{T^r M}}{r}, X_1^{(\alpha)}\right] + X_1^{(\alpha)}\right) \oplus \mathcal{L}_{\frac{\xi_{T^r M}}{r}} w_1^{(r-\alpha)} = \left(1 - \frac{\alpha}{r}\right)\left(X_1^{(\alpha)} \oplus w_1^{(r-\alpha)}\right)$$

is a section of $\mathcal{T}^r L$. Thus, $\mathcal{T}^r L$ is a homogeneous Dirac structure related to the vector field $\frac{\xi_{T^r M}}{r}$. □

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