Math. Appl. **4** (2015), 123–128
DOI: 10.13164/ma.2015.10

PROPERTIES OF CONTRA SEMI-CONTINUITY FUNCTIONS AND σ-ζ_μ-SETS IN GENERALIZED TOPOLOGICAL SPACES

RIM MESSAOUD and AMIRA MISSAOUI

Abstract. The main purpose of this paper is to introduce and study some characterizations and several properties of contra semi-continuity functions and σ-ζ_μ-sets in generalized topological spaces.

1. Introduction

In [2, 4, 6], Á. Császár founded the theory of generalized topological spaces and studied the elementary character of these classes. He especially introduced the notions of continuous functions on generalized topological spaces and investigated the characterizations of generalized continuous function. We recall some notions defined in [3]. It is well known that the concept of connectedness plays an important role in topological spaces.

General topology is important in many fields of applied sciences as well as branches of mathematics. In reality, it is used in data mining, computational topology for geometric design and molecular design, computer-aided design, computer-aided geometric design, digital topology, information system, particle physics, quantum physics, etc. The theory of generalized topological spaces, which was founded by Á. Császár, is one of the most important developments of general topology in recent years.

General topology has been associated with concepts of limit points, homeomorphisms, continuity, and related concepts of closed sets, open sets, neighborhoods, convergent sequences, connectedness, continua and manifolds. The existing general topologies use special means to introduce the topology on the basis of axioms relating to closed sets or closure, open sets, directed sets, and filters. The end in view of all such general definitions seems to be to obtain the definitions of a limit point and continuity of transformations and to find out under what conditions special topologies such as metric topologies may be obtained from those satisfying the definitions.

2. Preliminaires

We recall some basic concepts and results.

Let X be a nonempty set and let g be a family of subsets of X. g is called a generalized topology (briefly GT) on X if $\emptyset \in g$ and $G_i \in g$ for each $i \in I \neq \emptyset$

MSC (2010): primary 54A05, 54B15, 54C10, 54D15, 54C08.
Keywords: generalized topology, contra-semi-continuity, σ-ζ_μ-sets.

implies $\bigcup_{i\in I} G_i \in g$. We call the pair (X, g) a generalized topological space (briefly GTS). The elements of g are called g-open subset of X and the complements are called g-closed subset of X.

Let (X, g) be a GTS. For $A \subset X$, the closure of A and the interior of A in X are defined as follows:

$cl_g(A) = \bigcap\{F : F$ is g-closed in X and $A \subset F\}$,

$int_g(A) = \bigcup\{V : V$ is g-open in X and $V \subset A\}$.

For all A, $B \in X$ we have $A \subseteqq B$, i.e., $A \subset B$ or $A = B$.

In this paper, spaces always mean generalized topological spaces on which no separation axiom is assumed. The family of all g-open subsets of X and the family of all g-closed subsets of X are denoted by $gO(X)$ and $gC(X)$, respectively. For $x \in X$, the family of all g-open subsets containing x and the family of all g-closed subsets containing x are denoted by $gO(X, x)$ and $gC(X, x)$, respectively. We simply use cA and iA instead of $cl_g(A)$ and $int_g(A)$, respectively. Sometimes, the generalized topology on X is also denoted by g_X, i.e., $g_X = gO(X)$.

Let us denote the class of all g-semi-open sets on X by $\sigma(g_X)$ (σ for short).

Definition 2.1. ([3]) A function $f : (X, g_X) \to (Y, g_Y)$ is called (g_X, g_Y)-continuous, if $f^{-1}(V) \in gO(X)$ for every $V \in gO(Y)$.

Definition 2.2. ([7, 8]) A function $f : (X, g_X) \to (Y, g_Y)$ is called contra (g_X, g_Y)-continuous, if $f^{-1}(V) \in gC(X)$ for every $V \in gO(Y)$.

Definition 2.3. ([5, 8]) Let (X, g_X) be a GTS. A subset A of X is said to be g-semi-open if $A \subset c(iA)$.

The family of all g-semi-open sets in X is denoted by $gSO(X)$. The complement of a g-semi-open set is said to be g-semi-closed. The family of all g-semi-closed sets in X is denoted by $gSC(X)$.

Denote by:

$$gSO(X) \times gO(Y) = \{A \times B; A \in gSO(X), B \in gO(Y)\},$$

$$gSC(X) \times gC(Y) = \{A \times B; A \in gSC(X), B \in gC(Y)\}.$$

Definition 2.4. ([1]) Let g_X and g_Y be generalized topologies on X and Y, respectively.

Then, a function $f : X \to Y$ is said to be contra-(g_X, g_Y)-semi-continuous if for each g-open set U in Y, $f^{-1}(U)$ is g-semi-closed in X.

3. Properties of contra semi-continuity on GTS's

Definition 3.1. A function $f : (X, g_X) \to (Y, g_Y)$ is called contra-(g_X, g_Y)-semi-continuous at some $x \in X$, if, for $V \in gC(Y, f(x))$, there exists $U \in gSO(X, x)$ such that $f(U) \subset V$.

Theorem 3.2. *Let $f : (X, g_X) \to (Y, g_Y)$ be a function. Then, the following statements are equivalent:*

(1) *f is contra-(g_X, g_Y)-semi-continuous.*
(2) *$f^{-1}(F) \in gSO(X)$ for any $F \in gC(Y)$.*
(3) *For each $x \in X$ and each $V \in gO(Y)$ with $f(x) \notin V$, there exists $U \in gSC(X)$ such that $x \notin U$ and $f^{-1}(V) \subset U$.*

(4) *f is contra-(g_X, g_Y)-semi-continuous at any $x \in X$.*

Proof. (1) $\Rightarrow$ (2). Let $F \in gC(Y)$. Then, $Y - F \in gO(Y)$. By (1),
$$f^{-1}(Y - F) = X - f^{-1}(F) \in gSC(X).$$
Thus, $f^{-1}(F) \in gSO(X)$.

(2) $\Rightarrow$ (1). Let $V \in gO(Y)$, implies $Y - V \in gC(Y)$. By (2) $f^{-1}(Y - V) \in gSO(X)$. Hence, $f^{-1}(V) \in gSC(X)$. Then, f is contra-(g_X, g_Y)-semi-continuous.

(1) $\Rightarrow$ (3). Let $x \in X$ and $V \in gO(Y)$ with $f(x) \notin V$. Then, $x \notin f^{-1}(V)$. By (1), $f^{-1}(V) \in gSC(X)$. Put $U = f^{-1}(V)$. Then, $f^{-1}(V) \subset U$ and $x \notin U$.

(3) $\Rightarrow$ (1). Let $V \in gO(Y)$. For each $x \in f^{-1}(Y - V)$, $f(x) \notin V$. By (3), there exists $U_x \in gSC(X)$ such that $x \notin U_x$ and $f^{-1}(V) \subset U_x$. Then,
$$x \in X - U_x \subset X - f^{-1}(V) = f^{-1}(Y - V).$$
We have
$$\bigcup_{x \in f^{-1}(Y-V)} \{x\} \subset \bigcup_{x \in f^{-1}(Y-V)} (X - U_x) \subset f^{-1}(Y - V).$$
Thus,
$$f^{-1}(Y - V) = \bigcup_{x \in f^{-1}(Y-V)} (X - U_x) \in gSO(X).$$
This implies $f^{-1}(V) \in gSC(X)$. Hence, f is contra-(g_X, g_Y)-semi-continuous.

(2) $\Rightarrow$ (4). Let $x \in X$ and $V \in gC(Y, f(x))$. By (2), $f^{-1}(V) \in gSO(X)$. Put $U = f^{-1}(V)$. We have $U \in gSO(X, x)$ and $f(U) \subset V$.

(4) $\Rightarrow$ (2). Let $F \in gC(Y)$. By (4), there exists $U_x \in gSO(X, x)$ such that $f(U_x) \subset F$. Then,
$$x \in U_x \subset f^{-1}(F).$$
We have
$$\bigcup_{x \in f^{-1}(F)} \{x\} \subset \bigcup_{x \in f^{-1}(F)} U_x \subset f^{-1}(F).$$
Thus,
$$f^{-1}(F) = \bigcup_{x \in f^{-1}(F)} U_x.$$
This implies $f^{-1}(F) \in gSO(X)$. □

Definition 3.3. Let $f : (X, g_X) \to (Y, g_Y)$ be a function. The graph
$$G(f) = \{(x, f(x)) : x \in X\}$$
of f is called contra-(g_X, g_Y)-semi-graph, if for each $(x, y) \in X \times Y - G(f)$, there are $A \in gSO(X, x)$, $B \in gC(Y, y)$ such that $(A \times B) \cap G(f) = \emptyset$.

Theorem 3.4. *Let $f : (X, g_X) \to (Y, g_Y)$ be a function and let $G(f)$ be the graph of f. Then, the following are equivalent.*

(1) *$G(f)$ is contra-(g_X, g_Y)-semi-graph,*
(2) *For each $(x, y) \in X \times Y - G(f)$, there are $A \in gSO(X, x)$, $B \in gC(Y, y)$ such that $f(A) \cap B = \emptyset$.*

Proof. The proof follows from the fact that $(A \times B) \cap G(f) = \emptyset$ if and only if $f(A) \cap B = \emptyset$ for any $A \subset X$ and $B \subset Y$. □

Definition 3.5. Let (X, g_X) be a GTS.

(1) ([7]) X is called g-Urysohn, if for each pair $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gO(X, x_1)$, $V \in gO(X, x_2)$ such that $cU \cap cV = \emptyset$.
(2) X is called g-σ-T_2, if for each pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gSO(X, x_1)$, $V \in gSO(X, x_2)$ such that $U \cap V = \emptyset$.
(3) X is called g-σ-T_1, if for each pair of $x_1, x_2 \in X$ with $x_1 \neq x_2$, there are $U \in gSO(X, x_1)$, $V \in gSO(X, x_2)$ such that $x_2 \notin U$ and $x_1 \notin V$.

Theorem 3.6. *Let $f : (X, g_X) \to (Y, g_Y)$ be contra-(g_X, g_Y)-semi-continuous. If Y is g-Urysohn, then $G(f)$ is a contra-(g_X, g_Y)-semi-graph and $G(f) \in gSC(X) \times gC(Y)$.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then, $f(x) \neq y$. Since Y is g-Urysohn, there are $U \in gO(Y, f(x))$, $V \in gO(Y, y)$ such that $cU \cap cV = \emptyset$.

Note that $cU \in gC(Y, f(x))$. Since f is contra-(g_X, g_Y)-semi-continuous, by Theorem 3.2, there are $A \in gSO(X, x)$ such that $f(A) \subset cU$. Put $B = cV$. Obviously, $B \in gC(Y, y)$ and $f(A) \cap B = \emptyset$. By Theorem 3.4, $G(f)$ is a contra-(g_X, g_Y)-semi-graph.

Note that $f(x) \in U$ and $cU \cap cV = \emptyset$. Then, $f(x) \notin cV$. Thus, $x \in X - f^{-1}(cV)$. So

$$(x, y) \in (X - f^{-1}(cV)) \times V \in gSO(X) \times gO(Y).$$

We claim that $((X - f^{-1}(cV)) \times V) \cap G(f) = \emptyset$. Otherwise, pick

$$(a, b) \in (X - f^{-1}(cV) \times V) \cap G(f).$$

For every pair, $(a, b) \in G(f)$ implies $b = f(a)$. For every pair, $(a, b) \in (X - f^{-1}(cV)) \times V$ implies $f(a) \notin cV$. Then, $f(a) \in V \subset cV$, a contradiction. Thus

$$(x, y) \in (X - f^{-1}(cV)) \times V \subset X \times Y - G(f).$$

This implies $X \times Y - G(f) \in gSO(X) \times gO(Y)$. Hence,

$$G(f) \in gSC(X) \times gC(Y).$$

□

Theorem 3.7. *Let $f : (X, g_X) \to (Y, g_Y)$ be contra-(g_X, g_Y)-semi-continuous injection. If Y is g-Urysohn, then X is g-σ-T_2.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Put $y_1 = f(x_1)$, $y_2 = f(x_2)$. Then, $y_1 \neq y_2$. Since Y is g-Urysohn, there are $V_1 \in gO(Y, y_1)$, $V_2 \in gO(Y, y_2)$ such that $cV_1 \cap cV_2 = \emptyset$. Since f is contra-(g_X, g_Y)-semi-continuous, we have

$$f^{-1}(cV_1),\ f^{-1}(cV_2) \in gSO(X).$$

Put $U_1 = f^{-1}(cV_1)$, $U_2 = f^{-1}(cV_2)$. Then, $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$. Thus, X is g-σ-T_2. □

Finally, we give Theorem 3.8 in order to compare the results of Theorem 3.6 with those of Theorem 3.7.

Theorem 3.8. *Let $f : (X, g_X) \to (Y, g_Y)$ be a contra-(g_X, g_Y)-semi-continuous injection. If $G(f)$ is contra-(g_X, g_Y)-semi-graph, then X is g-σ-T_1.*

Proof. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since f is injective, we have $f(x_1) \neq f(x_2)$. We have $(x_1, f(x_2)) \in X \times Y - G(f)$. By Theorem 3.6, there are $U \in gSO(X, x_1)$ and $B \in gC(Y, f(x_2))$ such that $f(U) \cap B = \emptyset$. Since f is contra-(g_X, g_Y)-semi-continuous, we have

$$f^{-1}(B) \in gSO(X, x_2).$$

Put $f^{-1}(B) = V$. Then, $x_1 \in U$ and $f(U) \cap B = \emptyset$ imply $f(x_1) \notin B$. Thus, $x_1 \notin V$. Then, $f(U) \cap B = \emptyset$ implies $U \cap f^{-1}(B) = \emptyset$. By $f(x_2) \in B$, $x_2 \notin U$. Thus, X is g-σ-T_1. □

4. σ-ζ_μ-sets in generalized topological spaces

Definition 4.1. Let A be a subset of a generalized topological space (X, μ). The σ-$\zeta_\mu(A)$ is defined as follows:

$$\sigma\text{-}\zeta_\mu(A) = \cap\{U : A \subseteqq U, U \in gSO(X)\}.$$

Proposition 4.2. *For subsets, A, B and C_γ ($\gamma \in \Gamma$) of a generalized topological space (X, μ), the following properties hold:*

(1) $A \subseteqq \sigma$-$\zeta_\mu(A)$;
(2) σ-$\zeta_\mu(\sigma$-$\zeta_\mu(A)) = \sigma$-$\zeta_\mu(A)$;
(3) *If* $A \subseteqq B$, *then* σ-$\zeta_\mu(A) \subseteqq \sigma$-$\zeta_\mu(B)$;
(4) σ-$\zeta_\mu(\cap\{C_\gamma : \gamma \in \Gamma\}) \subseteqq \cap\{\sigma$-$\zeta_\mu(C_\gamma) : \gamma \in \Gamma\}$;
(5) σ-$\zeta_\mu(\cup\{C_\gamma : \gamma \in \Gamma\}) = \cup\{\sigma$-$\zeta_\mu(C_\gamma) : \gamma \in \Gamma\}$.

Proof. (1) This is obvious from the definition.

(2) By (1), we have σ-$\zeta_\mu(\sigma$-$\zeta_\mu(A)) \supseteqq \sigma$-$\zeta_\mu(A)$. Suppose that $x \notin \sigma$-$\zeta_\mu(A)$. Then, there exists $U \in gSO(X)$ such that $A \subseteqq U$ and $x \notin U$. Since $A \subseteqq \sigma$-$\zeta_\mu(A) \subseteqq U$, we have $x \notin \sigma$-$\zeta_\mu(\sigma$-$\zeta_\mu(A))$ and, hence, σ-$\zeta_\mu(\sigma$-$\zeta_\mu(A)) \subseteqq \sigma$-$\zeta_\mu(A)$.

(3) Suppose that $x \notin \sigma$-$\zeta_\mu(B)$. Then, there exists $U \in gSO(X)$ such that $B \subseteqq U$ and $x \notin U$. Since $A \subseteqq B$, we have $x \notin \sigma$-$\zeta_\mu(A)$ and, hence, σ-$\zeta_\mu(A) \subseteqq \sigma$-$\zeta_\mu(B)$.

(4) Suppose that $x \notin \cap\{\sigma$-$\zeta_\mu(C_\gamma) : \gamma \in \Gamma\}$. There exists $\gamma_0 \in \Gamma$ such that $x \notin \sigma$-$\zeta_\mu(C_{\gamma_0})$ and there exists $U \in gSO(X)$ such that $x \notin U$ and $C_{\gamma_0} \subseteqq U$. Since $\cap_{\gamma\in\Gamma} C_\gamma \subseteqq C_{\gamma_0}$, we have $x \notin \sigma$-$\zeta_\mu(\cap\{C_\gamma : \gamma \in \Gamma\})$ and, hence, σ-$\zeta_\mu(\cap\{C_\gamma : \gamma \in \Gamma\}) \subseteqq \cap\{\sigma$-$\zeta_\mu(C_\gamma) : \gamma \in \Gamma\}$.

(5) Since $C_\gamma \subseteqq \cup_{\gamma\in\Gamma} C_\gamma$, by 3, we have σ-$\zeta_\mu(C_\gamma) \subseteqq \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} C_\gamma)$ and $\cup_{\gamma\in\Gamma}\sigma$-$\zeta_\mu(C_\gamma) \subseteqq \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} C_\gamma)$. Conversely, suppose that $x \notin \cup_{\gamma\in\Gamma}\sigma$-$\zeta_\mu(C_\gamma)$. Then, $x \notin \sigma$-$\zeta_\mu(C_\gamma)$ for each $\gamma \in \Gamma$ and, hence, there exists $U_\gamma \in gSO(X)$ such that $C_\gamma \subseteqq U_\gamma$ and $x \notin U_\gamma$ for each $\gamma \in \Gamma$. Therefore, we obtain $\cup_{\gamma\in\Gamma} C_\gamma) \subseteqq \cup_{\gamma\in\Gamma} U_\gamma)$ and $\cup_{\gamma\in\Gamma} U_\gamma)$ is a semi-open set not containing x. Thus, $x \notin \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} C_\gamma)$. This shows that $\cup_{\gamma\in\Gamma} C_\gamma) \supseteqq \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} C_\gamma)$. □

Definition 4.3. A subset A of a generalized topological space (X, μ) is called a σ-ζ_μ-set if $A = \sigma$-$\zeta_\mu(A)$. The family of all σ-ζ_μ-sets of (X, μ) is denoted by σ-$\zeta_\mu(X, \mu)$.

Corollary 4.4. *For subsets A and B_γ ($\gamma \in \Gamma$) of a generalized topological space (X, μ), the following properties hold:*

(1) *σ-$\zeta_\mu(A)$ is a σ-ζ_μ-set.*
(2) *If $A \in gSO(X)$, then A is a σ-ζ_μ-set.*
(3) *If B_γ is a σ-ζ_μ-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is a σ-ζ_μ-set.*
(4) *If B_γ is a σ-ζ_μ-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} B_\gamma$ is a σ-ζ_μ-set.*

Proof. (1) and (2) are obvious.

(3) Let $B_\gamma \in \sigma$-$\zeta_\mu(X,\mu)$ for each $\gamma \in \Gamma$, then, by Proposition 4.2(4) we have $\cap_{\gamma\in\Gamma} B_\gamma \supseteqq \cap_{\gamma\in\Gamma} \sigma$-$\zeta_\mu(B_\gamma) \supseteqq \sigma$-$\zeta_\mu(\cap_{\gamma\in\Gamma} B_\gamma) \supseteqq \cap_{\gamma\in\Gamma} B_\gamma$. Thus, $\cap_{\gamma\in\Gamma} B_\gamma = \sigma$-$\zeta_\mu(\cap_{\gamma\in\Gamma} B_\gamma)$ and $\cap_{\gamma\in\Gamma} B_\gamma \in \sigma$-$\zeta_\mu(X,\mu)$.

(4) Let $B_\gamma \in \sigma$-$\zeta_\mu(X,\mu)$ for each $\gamma \in \Gamma$, then by Proposition 4.2(5) we have $\cup_{\gamma\in\Gamma} B_\gamma = \cup_{\gamma\in\Gamma} \sigma$-$\zeta_\mu(B_\gamma) = \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} B_\gamma) \supseteqq \cup_{\gamma\in\Gamma} B_\gamma$. Thus, $\cup_{\gamma\in\Gamma} B_\gamma = \sigma$-$\zeta_\mu(\cup_{\gamma\in\Gamma} B_\gamma)$ and $\cup_{\gamma\in\Gamma} B_\gamma \in \sigma$-$\zeta_\mu(X,\mu)$. □

References

[1] A. Al-Omari and T. Noiri, *A unified theory of contra-(μ,λ)-continuous functions in generalized topological spaces*, Acta Math. Hungar. **135** (2012), 31–41.

[2] Á. Császár, *Generalised open sets*, Acta Math. Hungar. **75** (1997), 65–87.

[3] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar. **96** (2002), 351–357.

[4] Á. Császár, *γ-connected sets*, Acta Math. Hungar. **101** (2003), 273–279.

[5] Á. Császár, *Generalized open sets in generalized topologies*, Acta Math. Hungar. **106** (2005), 53–66.

[6] Á. Császár, *Products of generalized topologies*, Acta Math. Hungar. **123** (2009), 127–132.

[7] Z. Li and W. ZhU, *Contra continuity on generalized topological spaces*, Acta Math. Hungar. **138** (2013), 34–43.

[8] B. Roy, *On weakly (μ,λ)-open functions*, Ukr. Math. J. **66** (2015), 1595–1602.

Rim Messaoud, Departement of Mathematics, Faculty of Sciences of Gafsa, Zarroug 2112, Tunisia
e-mail: `rimessaoud@yahoo.fr`

Amira Missaoui, Departement of Mathematics, Faculty of Sciences of Gafsa, Zarroug 2112, Tunisia
e-mail: `missaouiamira@yahoo.fr`