Math. Appl. **4** (2015), 101–108
DOI: 10.13164/ma.2015.08

BOOKMAKER'S DILEMMA

KJETIL K. HAUGEN

Abstract. In this paper, a game between two bookmakers is analysed. The simple game involves high or low odds strategies for each bookmaker, and an inherent Prisoner's Dilemma structure is revealed. However, this structure may be partially resolved given parametric changes in profit structures that might lead to a "Stag Hunt Game".

An argument for possible bookmaker arbitrage is added. Although this argument is well known, it is sparsely treated in research literature. A proof of the necessity of differing beliefs between gamblers to obtain bookmaker arbitrage is added.

1. Introduction

Wagering markets are surprisingly well described in Economic literature, both theoretically and empirically (see, e.g. [2, 4–6, 8, 11]). According to Thaler and Ziemba [9], there are good reasons for this. They argue that this industry contains transactions where the termination point of uncertain outcomes (of bets) are well defined. This, (for instance) in contrast to financial markets, where such future uncertainty resolvability is far less clearly defined. Combined with the fact that data from the betting industry is available, this industry is interesting as an empirical "laboratory" to test economists' hypotheses and models.

Of special interest for this paper is the concept of arbitrage. The main topic here, as the title and abstract suggest, is related to a strategic analysis of possible cooperation and conflict between bookmakers. For such a gaming situation to make sense, some basic structural points related to profit potential in this industry seems relevant.

It is a well known fact that arbitrage (or secure profit options) exists in real wagering markets [7]. A very simple example may for instance be observed in [10]. The point is simple. If more than one bookmaker exist, and at least two bookmakers give different odds on the same uncertain outcome, a possibility for an arbitrage on the hand of a gambler exists. As odds and probabilities are interlinked[1], it should not come as a surprise that an arbitrage opportunity on the bookmakers' hand also exists. This option, seemingly less discussed in academic literature (see [1, 3] for 2 exceptions) will be discussed in greater detail in what follows.

MSC (2010): primary 91; secondary 91A60.
Keywords: arbitrage, two player game, Nash equilibrium, bookmaker, gambler.

[1] $\left(p = \frac{o}{1+o}\right)$, where p is probability and o is odds.

Consider (as shown in Figure 1) a bookmaker offering two different gambles (G_1, G_2) to (at least) two different gamblers who have different probabilities (p and q) for some uncertain event of win/loose ($\mathcal{A}, \mathcal{B}$) type.

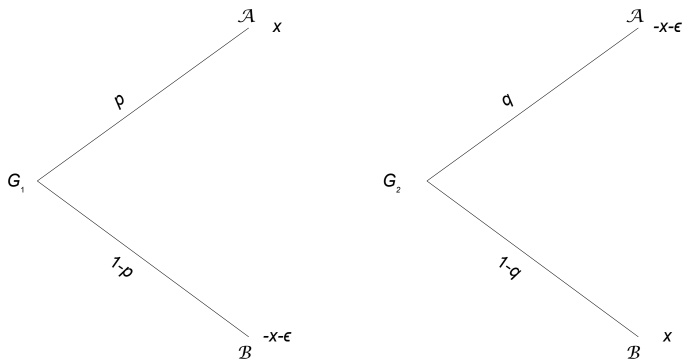

Figure 1. Two gambles offered to two gamblers.

The gambles G_1 and G_2 offered work as follows. If outcome $\mathcal{A}$ is realised, a gambler accepting G_1 will get paid $x, x > 0$ from the bookmaker, but will have to pay the bookmaker $x + \epsilon$ ($\epsilon > 0$) if the outcome is $\mathcal{B}$. Gamble G_2 is (pay-off-wise) inversely constructed so that the gambler receives x given $\mathcal{B}$, but must pay $x + \epsilon$ given $\mathcal{A}$.

Now, an interesting feature is the total pay-off received by the bookmaker. If outcome $\mathcal{A}$ is realised, he will have to pay x from G_1, but will receive $x + \epsilon$ from G_2. Obviously, the opposite happens if $\mathcal{B}$ is the outcome, and the bookmaker receives ϵ no matter what outcome. Hence, the bookmaker is guaranteed a profit of ϵ if he can find gamblers accepting both G_1 and G_2. We can check this option by checking the expected pay-offs and securing those to be positive for potential gamblers. Assuming risk neutral gamblers, we then get:

$$\begin{aligned} px + (1-p)(-x-\epsilon) &> 0, \\ q(-x-\epsilon) + (1-q)x &> 0, \end{aligned}$$

or, after some straightforward algebra,

$$p > \frac{x+\epsilon}{2x+\epsilon} \quad \text{and} \quad q < \frac{x}{2x+\epsilon}.$$

Given $\epsilon > 0$ (by assumption):

$$\frac{x+\epsilon}{2x+\epsilon} > \frac{x}{2x+\epsilon}$$

and p and q must be different. A small proof of the generality of this feature is offered in appendix A. Hence, we have shown and argued that the differences in the odds offered or in the gamblers' beliefs (probabilities) open up arbitraging opportunities. Furthermore, such opportunities only open up if such differing beliefs (in odds and probabilities) exist. The modern game theory has taken this discussion much further, it has even received a name – *The Harsanyi Doctrine* – refer, for instance, to [1]. The point here, is not to repeat this discussion, but rather to observe the following fact: One of these arbitraging opportunities acts in favour of the bookmakers (the last one), while the first one acts in favour of the gamblers. Gamblers and bookmakers play a zero-sum game, and what one group gains the other does not. The simple point here, is that given the information availability and openness of odds, it is hard to accept that arbitraging opportunities for gamblers would exist persistently. On the other hand, a bookmakers opportunity to construct gambles with arbitraging possibilities for himself may be much harder to reveal, even through the Internet. As such, it seems safe to conclude that these options to some extent explain how bookmakers can run their business profitably. The same conclusion seems obvious if we observe the reality. Bookmakers and casino owners exist and some of them are really big these days, with aggressive marketing in traditional marketing channels like TV-adverts. Thus, identifying real world profits is not hard.

2. The bookmaker's dilemma

2.1. A simplified bookmaker market

In Section 1, arguments for the existence of profits for bookmakers were given. Assume now, that the single bookmaker situation is changed into a two-bookmaker one. Hence, two bookmakers (with potential to earn profits) exist in a simplified bookmaker market. In this situation, competition between bookmakers arises, and, which is also obvious, the competitive means of the bookmakers are odds. As greedy gamblers incentives to choose one bookmaker over the other are solely related to the size of the odds, the main strategic choices for bookmakers are their odds setting. High odds (given low odds for the other) would draw gamblers to a bookmaker.

In a real world situation, other means than odds such as the quality of the website, brand, rumour etc. may influence the gamblers' decisions, but these effects are (for simplistic reasons) ruled out here.

As the main cost for a bookmaker is also related to odds, it seems evident that the profits earned by an individual bookmaker depend on his odds-strategy. That is, the higher the odds a bookmaker sets, the more he will have to pay to the winners, and the less he will earn. On the other hand, high odds increases the likelihood of attracting gamblers.

2.2. A two-player bookmaker game

Based on the arguments in Subsection 2.1, the following imperfect but complete information game is defined:

i) The game contains two players - bookmakers, named B_1 and B_2.
ii) Each player chooses between high and low odds strategies, o_H, o_L, respectively. ($o_L < o_H$.)
iii) The market situation is assumed such, that two possible profit options may arise. If both players choose the same odds-strategy, profits are divided equally among them. However, if they choose different strategies, the one choosing the low-odds strategy (o_L) takes the whole market. A choice of high odds reduces the profits compared to a low odds situation. Hence,

π_L, π_H[2], defines profits corresponding to odds choices o_H, o_L respectively. As a consequence, if one or both players choose the o_H-strategy, profits earned are π_L. If both choose the o_L strategy, the high profit option of π_H is obtained

iv) The bookmakers cannot observe each other's odds strategy before they have to decide their own odds. Hence, the game may be considered being simultaneous; that is, a game of imperfect information.

v) All information in points i) - iv) above is known by both players, and no additional information is available. Consequentially, the game has complete information.

This leads to a normal form model as shown in Figure 2.

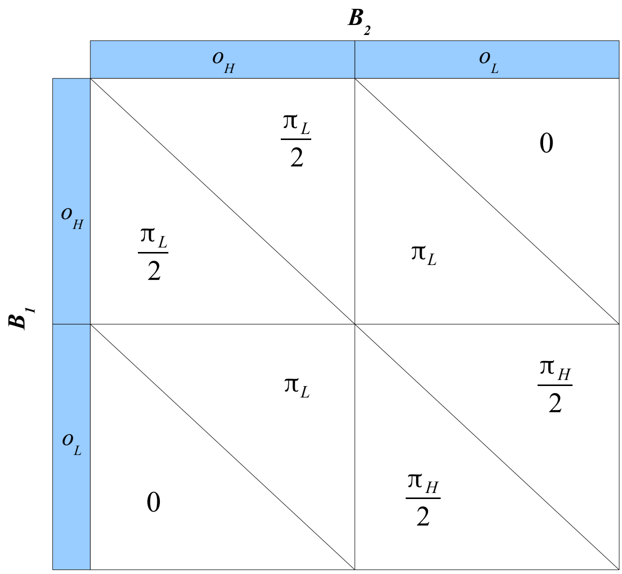

Figure 2. A simultaneous game of complete information between two bookmakers.

2.3. Game analysis – Nash equilibria

Figure 3 contains best replies for the game. The solid (red) circles (best reply function for B_1) and squares (best reply function for B_2) will be unchangeable as $\frac{\pi_L}{2}$ is always positive. However, the dotted (red) circles and squares are parameter dependent.

If

$$\pi_L > \frac{\pi_H}{2},$$

the game contains a single unique Nash equilibrium (NE) where both players chose the high-odds strategy. We name this NE: $\{o_H, o_H\}$. Refer to the left part of Figure 4. If we rule out the highly unlikely $\pi_L = \frac{\pi_H}{2}$-alternative, the only

[2] $\pi_L, \pi_H > 0$ by assumption and by the discussion in Section 1.

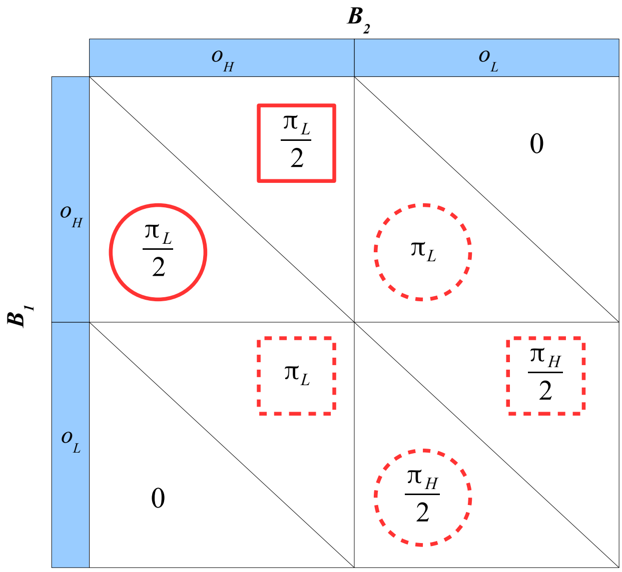

Figure 3. Game with best replies.

remaining option leads to a "Stag-Hunt"– situation – as illustrated on the right in Figure 4.

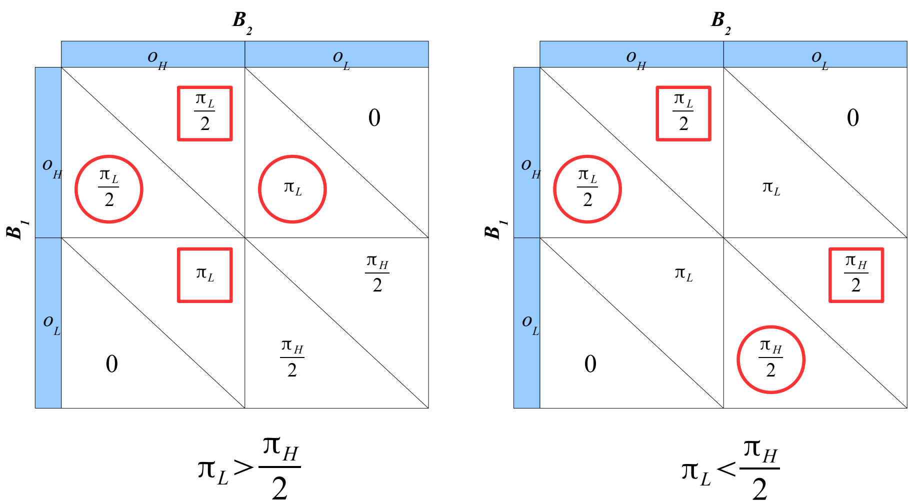

Figure 4. The two only possible NE-structures in the game.

The game on the right in Figure 4 contains 3 NE's, two in pure strategies and one in mixed strategies. The two pure strategy NE's involve a coordinated solution, both players do the same, but they either choose to set high odds - $\{o_H, o_H\}$ or low odds - $\{o_L, o_L\}$. The "invisible" mixed strategy NE involves both players choosing either o_H or o_L with (easily computable) probabilities. The actual probability-values of the mixed strategy NE are of no further importance here and, thus, are not added.

3. Conclusions

The main interesting observation in the analysis in Subsection 2.2, is the structure of the leftmost NE of Figure 4. Here, a unique NE where each bookmaker chooses to set high odds ($\{o_H, o_H\}$) is the game prediction. This may seem as good news for gamblers, but the Prisoner's Dilemma structure is evident. Observe that the total profit generated by the two bookmakers in this market is $\frac{\pi_L}{2} + \frac{\pi_L}{2} = \pi_L$. At the same time, it is easily observed that the "opposite" coordination alternative – $\{o_H, o_H\}$ provides a larger total profit for each of the players – as $\frac{\pi_H}{2} + \frac{\pi_H}{2} = \pi_H$. ($\pi_H > \pi_L$ by assumption.) As a consequence, cooperative activity by bookmakers should be expected. A cartel (much like OPEC) may be expected.

If the bookmaker market is compared to the oil market, it is easy to see the significant differences. The oil producers must keep their cartel steady after observing price changes while the real relevant (more or less unobservable) variable is the production quantities. However, the bookmakers can easily (at least in principle through the Internet) control each other's odds and, thus, it should (logically) be expected that it is far easier to maintain a cartel for bookmakers than oil producers. As OPEC has managed to keep a high oil price (well above marginal costs) for more than 40 years, it is easy to suspect that bookmaker odds perhaps are not as high as they should be.

Furthermore, to some extent, this situation may be viewed as follows: If π_L is relatively close (in value) to π_H, we can interpret the market situation as reasonably competitive. Such a situation would make the $\{o_H, o_H\}$ NE more probable. And, (obviously) if a cartel formation tendency is evident, the probability of such an occurrence will increase. On the other hand, if $\pi_H >> \pi_L$, the $\{o_H, o_H\}$ becomes an NE. Not a unique one though, but still.

Thus, to some extent, our model indicates that, no matter how profits are distributed, the bookmakers can end up as winners. As such, this conclusion should not be very surprising, not many sound people recommend gambling activities. However, our simple game model indicates another type of argument against gambling activities.

Appendix A. A proof of generality for $p \neq q$

The gambles (or lotteries) of Figure 1 are generalised[3] as in Figure 5.

As before; $x, y, \epsilon_1, \epsilon_2 > 0$ To secure arbitrage for the bookmaker, the following two inequalities must be satisfied:

$$\begin{aligned} y + \epsilon_2 - x &> 0, \\ x + \epsilon_1 - y &> 0, \end{aligned}$$

[3]These gambles are of course not completely generalized by given sign restrictions. However, it seems unnecessary to argue for this assumption.

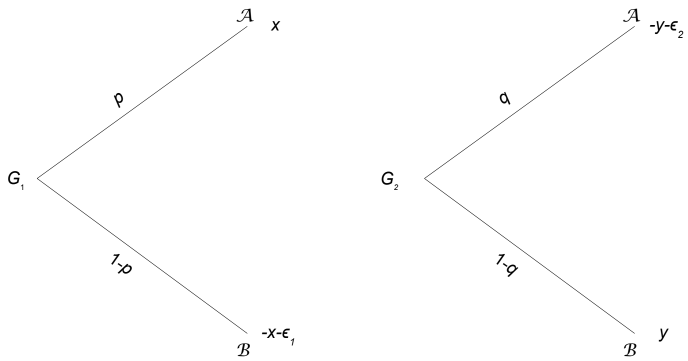

Figure 5. Generalized gambles - G_1 and G_2.

or

$$\epsilon_2 > x - y, \tag{A.1}$$
$$\epsilon_1 > y - x. \tag{A.2}$$

As inequalities (A.1) and (A.2) indicate, this poses no problems, as simply adding left and right sides of these inequalities produce $\epsilon_1 + \epsilon_2 > 0$ which, given the assumption of positive ϵ-s, poses no problems.

However, arbitrage for the bookmaker is not enough. It is also necessary that positive expected values for the gamblers can be constructed. Then:

$$px + (1-p)(-x - \epsilon_1) > 0, \tag{A.3}$$
$$q(-y - \epsilon_2) + (1-q)y > 0. \tag{A.4}$$

Straightforward algebra applied to inequalities (A.3) and (A.4) yields:

$$p > \frac{x+\epsilon_1}{2x+\epsilon_1},$$
$$q < \frac{y}{2y+\epsilon_2}.$$

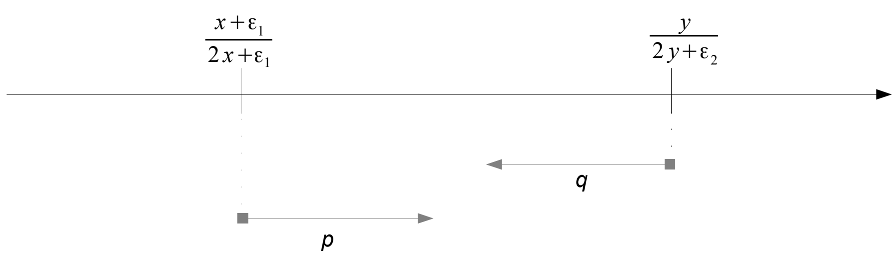

Figure 6. Position of p and q when $p = q$ is a possibility.

Now, let us show that p must be unequal to q by an "ad absurdum" argument, starting with $p = q$ and show that this is impossible. If p should be able to equal

q, the following must hold (refer to Figure 6 for a graphical explanation):

$$\frac{x+\epsilon_1}{2x+\epsilon_1} < \frac{y}{2y+\epsilon_2}. \tag{A.5}$$

As, $x, y, \epsilon_1, \epsilon_2 > 0 \Rightarrow (2x+\epsilon_1), (2y+\epsilon_2) > 0$, and (A.5) becomes:

$$\Rightarrow (x+\epsilon_1)(2y+\epsilon_2) < y(2x+\epsilon_1)$$

or

$$\Rightarrow 2xy + x\epsilon_2 + 2y\epsilon_1 + \epsilon_1\epsilon_2 < 2xy + y\epsilon_1$$

and finally

$$\Rightarrow \epsilon_2(x+\epsilon_1) < -y\epsilon_1. \tag{A.6}$$

Now, given non negativity assumptions on all variables, $-y\epsilon_1$ (on the right-hand side in (A.6)) must be negative, while the expression on the left-hand side is clearly positive. Consequently, $p = q$ is impossible, and in order to achieve arbitrage and gambles with positive expected values (which risk neutral gamblers would accept), p and q must be different. $\square$

References

[1] K. Binmore, *Fun and Games*, D. C. Heath, Lexington, MA 02173, USA, 1992.

[2] R. Bird and M. McCrae, *The test of the efficiency of racetrack betting using bookmaker odds*, Manage. Sci. **33** (1987), 1552–1562.

[3] F. E. Su et al, *Risk-free betting on different beliefs*, retrieved from the internet `http://www.math.hmc.edu/funfacts`, September 2015.

[4] S. Figlewski, *Subjective information and market efficiency in a betting market*, J. Polit. Econ. **87** (1979), 75–88.

[5] J. Fingleton and P. Waldron, *Optimal determination of bookmakers' betting odds: Theory and tests*, Trinity Economic Papers, Series Technical Paper No. 96/9, 1999.

[6] V. Makropoulou and R. N. Markellos, *Optimal price setting in fixed-odds betting markets under information uncertainty*, Scot. J. Polit. Econ. **58** (2011), 519–536.

[7] R. D. Sauer, *The economics of wagering markets*, J. Econ. Lit. **36** (1998), 2021–2064.

[8] H. S. Shin, *Optimal betting odds against insider traders*, Econ. J. **101** (1991), 1179–1185.

[9] R. H. Thaler and W. T. Ziemba, *Parimutuel betting markets: Racetracks and lotteries*, J. Econ. Perspect. **2** (1988), 161–174.

[10] The Free Encyclopedia, Wikimedia Foundation Wikipedia, *Arbitrage betting*, retrieved from the internet `https://en.wikipedia.org`, September 2015.

[11] L. V. Williams and D. S. Siegel (eds.), *The Oxford Handbook of the Economics of Gambling*, Oxford Handbooks in Economics, Oxford University Press, 2014.

Kjetil K. Haugen, Faculty of Logistics, Molde University College, Specialized University in Logistics, Britvegen 2, 6410 Molde, Norway
e-mail: `kjetil.haugen@himolde.no`