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DEFERRED STATISTICAL CLUSTER POINTS OF DOUBLE SEQUENCES

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Abstract. In this paper, deferred statistical cluster points for double sequences are defined and studied by using deferred double natural density of the subset of natural numbers. For $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ satisfying certain conditions, we obtain some results for the set of deferred statistical cluster points $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ in the case of $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$. Further, we provide some counterexamples regarding $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. Also, we give some inclusion results for the set of deferred statistical cluster points $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$.

1. Introduction

The concepts of statistical limit points and statistical cluster points of a number sequence was first introduced by Fridy [9]. Later on some generalizations and applications of these notions were investigated by many authors (see [5, 7, 8, 12, 14, 19, 23, 24]). Some equivalence results for Cesàro submethods were studied by Goffman and Petersen [10], Armitage and Maddox [2] and Osikiewicz [17]. In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequence $x = (x_k)$ by

$$(D_{p,q}x)_n := \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

In [4], the first study on double sequences was examined by Bromwich. And then it was investigated by many authors such as Hardy [11], Moricz [15], Tripathy [21], Başarır and Sonalcan [3]. The notion of regular convergence for double sequences was defined by Hardy [11]. After that both the theory of topological double sequence spaces and the theory of summability of double sequences were studied by Zeltser [25]. The statistical and Cauchy convergence for double sequences were examined recently by Mursaleen and Edely [16] and Tripathy [22] who also gave the relation between statistically convergent and strongly Cesàro

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summable double sequences. Many recent improvements containing the summability by four dimensional matrices might be found in [18].

As a continuation of these works, we investigate deferred statistical cluster points for double sequences by using deferred double natural density of the subset of natural numbers and give some results for deferred Cesàro mean of double sequences. We recall some notations and basic definitions used in this paper.

By the convergence of a double sequence we mean the convergence in Pringsheim's sense [20]. A double sequence $x = (x_{kl})$ is said to be convergent to L in the Pringsheim's sense if, for all $\varepsilon > 0$, there exists an $n_0 = n_0(\varepsilon)$ such that $|x_{nm} - L| < \varepsilon$ whenever $n, m \geq n_0$ [20]. In this case, we write $P-\lim_{k,l\to\infty} x_{kl} = L$. In [16], let $K \subset \mathbb{N}^2$ be a two-dimensional set of positive integers and let

$$K(n,m) := \{(k,l) \in K \ : \ (k,l) \leqslant (n,m)\}.$$

Then, the upper asymptoic density of the set $K \subset \mathbb{N}^2$ is defined as

$$\delta^{*(2)}(K(n,m)) := \lim \sup_{n,m\to\infty} \frac{|K(n,m)|}{nm},$$

if the limit exists and is finite. The vertical bars above indicate the cardinality of the set $K(n,m)$. If the sequence $\left(\frac{|K(n,m)|}{nm}\right)$ has a limit in Pringsheim's sense, we define the double natural density of K as

$$\delta_2(K(n,m)) := \lim_{n,m\to\infty} \frac{|K(n,m)|}{nm}.$$

Following Mursaleen [16] we say that a double sequence $x = (x_{kl})$ is statistically convergent to the number L if for each $\epsilon > 0$

$$\lim_{n,m\to\infty} \frac{1}{nm} |\{(k,l) : k \leq n, \ l \leq m, | \ x_{kl} - L \ |\geq \epsilon\}| = 0.$$

In this case, we write $st_2-\ \lim_{k,l\to\infty} x_{kl} = L$ and we denote the set of all double statistically convergent sequences by st_2.

It is well-known that a double sequence $x = (x_{kl})$ is said to be monotone increasing (decreasing) if $x_{kl} \geq x_{nm}$ for all $(k,l) \leq (n,m)$ ($x_{kl} \leq x_{nm}$ for all $(k,l) \leq (n,m)$).

Recall that the number σ is called a statistical cluster point of the double sequence $x = (x_{kl})$ provided that for any $\varepsilon > 0$ the set $\{(k,l) : |x_{kl} - \sigma| < \varepsilon \}$ does not have double natural density zero, i.e.,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : k \leq n, \ l \leq m, \ \ |x_{kl} - \sigma| < \varepsilon\}|}{nm} \neq 0.$$

The set of statistical cluster points of double sequence $x = (x_{kl})$ is denoted by $\Gamma^{(2)}(x)$ [13].

Definition 1.1. ([6]) Let $x = (x_{kl})$ be a double sequence and $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$. Then the deferred Cesàro mean $D_{\beta,\gamma}$ of the double sequence x is defined by

$$
\begin{aligned}
(D_{\beta,\gamma}x)_{nm} &= \frac{1}{\beta(n)\ \gamma(m)} \sum_{k=p(n)+1}^{q(n)} \sum_{l=t(m)+1}^{r(m)} x_{kl} \\
&= \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} x_{kl}
\end{aligned}
$$

where $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$, $t(m) < r(m)$ and $\lim_{n\to\infty} q(n) = \infty$, $\lim_{m\to\infty} r(m) = \infty$. We note that $D_{\beta,\gamma}$ is clearly regular for any choice of $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$.

Throughout this paper $\beta(n) = q(n)-p(n)$, $\gamma(m) = r(m)-t(m)$ are represented by β and γ, respectively.

Let $x = (x_{kl})$ be a double sequence and L a real number. Then, the double sequence x is said to be $D_{\beta,\gamma}$-summable to L if

$$
\lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),\ r(m)} (x_{kl} - L) = 0
$$

and we then write $(D_{\beta,\gamma}) - \lim_{n,m\to\infty} x_{nm} = L$ [6].
Let K be a subset of $\mathbb{N}^2$ and denote the set

$$
\{(k,l) : p(n) < k \le q(n),\ t(m) < l \le r(m),\ (k,\ l) \in K\}
$$

by $K_{\beta,\gamma}(n,m)$. The deferred double natural density of K is defined by

$$
\delta^{(2)}_{D_{\beta,\gamma}}(K) := \lim_{n,m\to\infty} \frac{1}{\beta(n)\ \gamma(m)} |K_{\beta,\gamma}(n,\ m)|
$$

whenever the limit exists. The vertical bars indicate the cardinality of the set $K_{\beta,\gamma}(n,m)$. Also, because of $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ does not exist for some $K \subset \mathbb{N}^2$, it is convenient to use upper deferred asymptotic density of K, defined by

$$
\begin{aligned}
&\delta^{*(2)}_{D_{\beta,\gamma}}(K) \\
= \ &\limsup_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ (k,l) \in K\}|}{\beta(n)\ \gamma(m)}.
\end{aligned}
$$

It is clear that, for the function $\delta^{*(2)}_{D_{\beta,\gamma}}(K)$, the following properties hold:

(i) if $\delta^{(2)}_{D_{\beta,\gamma}}(K)$ exists, then $\delta^{(2)}_{D_{\beta,\gamma}}(K) = \delta^{*(2)}_{D_{\beta,\gamma}}(K)$,

(ii) $\delta^{(2)}_{D_{\beta,\gamma}}(K) \neq 0$ if and only if $\delta^{*(2)}_{D_{\beta,\gamma}}(K) > 0$ and

(iii) the function $\delta^{*(2)}_{D_{\beta,\gamma}}(K)$ is monotone increasing.

A double sequence $x = (x_{kl})$ is said to be deferred statistically convergent to $L \in \mathbb{N}$ if for every $\varepsilon > 0$,

$$
\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1 \le k \le q(n),\ t(m)+1 \le l \le r(m),\ |x_{kl} - L| \ge \varepsilon\}|}{\beta(n)\ \gamma(m)} = 0.
$$

We then write $(D_{\beta,\gamma})\, st_2 - \lim_{n,m\to\infty} x_{nm} = L$ [6].

2. Deferred statistical cluster points for double sequences

Definition 2.1. The number Ω is called deferred statistical cluster point of the double sequence $x=(x_{kl})$ if for any $p(n)$, $q(n)$, $t(m)$ and $r(m)$ satisfying Definition 1.1 and any $\varepsilon>0$ the set

$$\{p(n)<k\le q(n),\ t(m)<l\le r(m) : |x_{kl}-\Omega|<\varepsilon\}$$

does not have deferred double natural density zero i.e.,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)<k\le q(n),\ t(m)<l\le r(m),\ |x_{kl}-\Omega|<\varepsilon\}|}{\beta(n)\ \gamma(m)} \ne 0. \tag{2.1}$$

The set of deferred statistical cluster points of the double sequence $x=(x_{kl})$ is denoted by $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$, i.e.,

$$\Gamma^{(2)}_{D_{\beta,\gamma}}(x) := \{\Omega : \Omega \text{ satisfies } (2.1)\}.$$

Now, we may obtain our main results.

Theorem 2.2. *If the double sequence $x=(x_{nm})$ is deferred statistically convergent to L, then $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ contains only the element L.*

Proof. Let the double sequence $x=(x_{nm})$ be deferred statistically convergent to L. Then, for every $\varepsilon>0$, the limit relation

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m),\ |x_{kl}-L|\ge\varepsilon\}|}{\beta(n)\ \gamma(m)} = 0$$

holds. It means that

$$\begin{aligned} &\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m),\ |x_{kl}-L|<\varepsilon\}|}{\beta(n)\ \gamma(m)} \\ = \quad & 1\ne 0. \end{aligned} \tag{2.2}$$

Therefore, $L\in\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. Now let us assume that the set $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ contains L' which is different from L, i.e., $L'\ne L$. Take into consideration $\varepsilon=\frac{1}{2}|L-L'|$. Since $x=(x_{nm})$ is deferred statistically convergent to L, (2.2) holds for this ε. It means that the deferred asymptotic density of the elements $x=(x_{nm})$ belonging to the ε-neighborhood of L is 1. Consequently, the deferred asymptotic density of the elements $x=(x_{nm})$ belonging to the ε-neighborhood of L' is zero. That is,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) : p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m),\ |x_{kl}-L'|<\varepsilon\}|}{\beta(n)\ \gamma(m)} = 0.$$

This is a contradiction to the assumption on L'. □

Remark 2.3. The inverse of Theorem 2.2 is not true.

There exists a double sequence such that the set of deferred statistical cluster points has a unique element but it is not deferred statistical convergent to this point. Let us consider the double sequence $x=(x_{nm})$ where

$$x_{nm} := \begin{cases} \frac{1}{nm}, & n \text{ or } m \text{ even} \\ nm, & n \text{ and } m \text{ odd.} \end{cases}$$

It is clear that $0 \in \Gamma^{(2)}_{D_{n,m}}(x)$ but x is not deferred statistically convergent to zero.

Theorem 2.4. *Let $x = (x_{nm})$ be a monotone increasing (decreasing) double sequence. If* $\sup x_{nm} < \infty$ *(* $\inf x_{nm} < \infty$ *), then* $\sup x_{nm} \in \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ $(\inf x_{nm} \in \Gamma^{(2)}_{D_{\beta,\gamma}}(x))$.

Proof. We will only prove the theorem for a monotone increasing double sequence. By the definition of supremum for any $\varepsilon > 0$ there exist $n_0, m_0 \in \mathbb{N}$ such that the following inequality

$$\sup x_{nm} - \varepsilon < x_{n_0 m_0} \leq \sup x_{nm}$$

follows. Since the double sequence is monotone increasing, we have

$$\sup x_{nm} - \varepsilon < x_{n_0 m_0} < x_{nm} \leq \sup x_{nm} < \sup x_{nm} + \varepsilon$$

for all $n > n_0$ and $m > m_0$. It means that for any $\varepsilon > 0$ there exist $n_0(\varepsilon), m_0(\varepsilon) \in \mathbb{N}$ such that the inequality

$$|x_{nm} - \sup x_{nm}| < \varepsilon$$

holds for all $n > n_0$ and $m > m_0$. From these inequalities the following inclusion

$$\mathbb{N}^2 - \{(k,l) : 1 \leq k \leq n_0, 1 \leq l \leq m_0\} \subset \{(n,m) : |x_{nm} - \sup x_{nm}| < \varepsilon\}$$

results. Since

$$\delta^{(2)}_{D_{\beta,\gamma}}(\mathbb{N}^2 - \{(k,l) : 1 \leq k \leq n_0, 1 \leq l \leq m_0\}) = 1,$$

we have

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{(n,m) : |x_{nm} - \sup x_{nm}| < \varepsilon\}) \neq 0.$$

This completes the proof. □

Recall that the distance between $X \subset \mathbb{R}$ and $Y \subset \mathbb{R}$ is

$$d(X,Y) = \inf\{|a-b| : a \in X, b \in Y\}.$$

Theorem 2.5. *Let $x = (x_{nm})$ be a real valued double sequence. If $\Gamma^{(2)}_{D_{\beta,\gamma}}(x) \neq \emptyset$, then $d(\Gamma^{(2)}_{D_{\beta,\gamma}}(x), x) = 0$.*

Proof. Assume that $\Gamma^{(2)}_{D_{\beta,\gamma}}(x) \neq \emptyset$. Let us consider an arbitrary element $y \in \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. Then, we have for an arbitrary positive ε,

$$\lim_{n,m\to\infty} \frac{|\{(k,l) :\ p(n)+1 \leq k \leq q(n),\ t(m)+1 \leq l \leq r(m),\ |x_{kl} - y| < \varepsilon\}|}{\beta(n)\ \gamma(m)} \neq 0.$$

So, the set $X_\varepsilon := \{x_{kl} : |x_{kl} - y| < \varepsilon\}$ has at least countably many elements of the double sequence $x = (x_{nm})$ for an arbitrary $\varepsilon > 0$. Therefore,

$$0 \leq d(\Gamma^{(2)}_{D_{\beta,\gamma}}(x),\ x) = \inf\{|y - x_{kl}| : k,\ l \in \mathbb{N}\} \leq \varepsilon$$

holds. This gives the desired proof. □

Theorem 2.6. *If $\Gamma^{(2)}_{D_{\beta,\gamma}}(x) \neq \emptyset$ for any $p(n)$, $q(n)$, $t(m)$ and $r(m)$, then $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ is closed.*

Proof. Let us assume that $\Gamma^{(2)}_{D_{\beta,\gamma}}(x) \neq \emptyset$ for any $p(n),\ q(n),\ t(m)$ and $r(m)$. It is enough to show that $\mathbb{R} - \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ is an open set. Let $y \in \mathbb{R} - \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ be an arbitrary point. Since $y \notin \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$, there exists an $\varepsilon > 0$ such that

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : |x_{kl} - y| < \varepsilon\}) = 0.$$

If we denote the open interval $(y - \varepsilon,\ y + \varepsilon)$ by X, we have

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : x_{kl} \in X\}) = 0.$$

If we choose $\varepsilon_y := \frac{1}{2}\inf\{|x_{kl} - y| : x_{kl} \in X\}$, then it is clear that $\varepsilon_y < \varepsilon$ and $(y-\varepsilon_y, y+\varepsilon_y) \subset \mathbb{R} - \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. It means that y is an interior point of $\mathbb{R} - \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. Therefore, $\mathbb{R} - \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ is an open set. □

Theorem 2.7. *Let $x = (x_{nm})$ be a real valued double sequence and $\Omega \in \mathbb{R}$ be an arbitrary fixed point. If $d(\Omega,\ x) \neq 0$, then $\Omega \notin \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$ for any $p(n),\ q(n),\ t(m)$ and $r(m)$.*

Proof. By the hypothesis we have

$$d(\Omega,\ x) := \inf\{|x_{kl} - \Omega| : k, l \in \mathbb{N}\} = i > 0.$$

From the assumption, for all $k, l \in \mathbb{N}$, the following inequality results

$$|x_{kl} - \Omega| \geq i\ .$$

It means that the open interval $(\Omega - i,\ \Omega + i)$ has no elements of the double sequence $x = (x_{nm})$. So, we have

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : x_{kl} \in (\Omega - i,\ \Omega + i)\}) = 0 \tag{2.3}$$

therefore, if we choose an arbitrary $\varepsilon < i$, the relation

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : |x_{kl} - \Omega| < \varepsilon\}) = 0$$

holds. Otherwise it contradicts with (2.3) because of the inclusion

$$\begin{aligned} & \{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : |x_{kl} - \Omega| < \varepsilon\} \\ \subset\ & \{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : |x_{kl} - \Omega| < i\}. \end{aligned}$$

□

Remark 2.8. If $d(\Omega, x) = 0$, this does not necessarily imply $\Omega \in \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$.

Let us consider the double sequence $x = (x_{nm}) = \frac{1}{nm}$ for all $n, m \in \mathbb{N}$. If we take $\Omega = \frac{1}{2}$, then $d(\frac{1}{2},\ \frac{1}{nm}) = 0$ but $\frac{1}{2} \notin \Gamma^{(2)}_{D_{\beta,\gamma}}(x) = \{0\}$ when $q(n) = n,\ p(n) = 0 = t(m),\ r(m) = m$.

Theorem 2.9. *Let $x = (x_{nm})$ be a real valued double sequence and $X \subset \mathbb{R}$ be an arbitrary set. If $d(X, x) \neq 0$, then*

$$X \cap \Gamma^{(2)}_{D_{\beta,\gamma}}(x) = \emptyset.$$

Proof. If the subset $X \subset \mathbb{R}$ is a singleton, then the proof is obtained from Theorem 2.7. Let $\rho \in X$ be an arbitrary element. There is $m > 0$ such that

$$|\rho - x_{kl}| > m$$

since $d(X,\ x) > 0$. So, the intervals $(\rho - m, \rho + m)$ have no elements of the double sequence $x = (x_{nm})$. Therefore, if we choose $\varepsilon < m$, the set $(\rho - m, \rho + m)$ contains no element of the sequence. Consequently, we have

$$\delta^{(2)}_{D_{\beta,\gamma}}(\{p(n) < k \leq q(n),\ t(m) < l \leq r(m) : |\rho - x_{kl}| < \varepsilon\}) = 0$$

and $\rho \notin \Gamma^{(2)}_{D_{\beta,\gamma}}(x)$. $\square$

Remark 2.10. If $d(X, x) = 0$, this does not necessarily imply $X \cap \Gamma^{(2)}_{D_{\beta,\gamma}}(x) \neq \emptyset$.

Let us consider the sequence $x = (x_{nm}) = (\frac{1}{nm})$ for all $n, m \in \mathbb{N}$ and $X = (0,\ \infty)$. It is clear that $X \cap \Gamma^{(2)}_{D_{\beta,\gamma}}(x) = \emptyset$ but $d(X, x) = 0$.

3. Some inclusion results for $\Gamma^{(2)}_{D_{\beta,\gamma}}(x)$

Throughout this section, we consider the sequences of positive natural numbers $\{p(n)\}$, $\{p'(n)\}$, $\{q(n)\}$, $\{q'(n)\}$, $\{t(m)\}$, $\{t'(m)\}$, $\{r(m)\}$ and $\{r'(m)\}$.

For simplicity, we denote

$$\begin{aligned}\beta_1(n) &= q(n) - p(n),\ \beta_2(n) = q(n) - p'(n),\\ \beta_3(n) &= q'(n) - p(n),\ \beta_4(n) = q'(n) - p'(n),\end{aligned}$$

and

$$\begin{aligned}\gamma_1(m) &= r(m) - t(m),\ \gamma_2(m) = r(m) - t'(m),\\ \gamma_3(m) &= r'(m) - t(m),\ \gamma_4(m) = r'(m) - t'(m).\end{aligned}$$

Theorem 3.1. *Let the sets $G(q') \setminus G(q)$, $G(r') \setminus G(r)$ be finite and*

$$\lim_{n,m\to\infty} \frac{\beta_1(n)\ \gamma_1(m)}{\beta_3(n)\ \gamma_3(m)} = d \neq 0$$

hold. Then, $\delta^{(2)}_{D_{\beta_3,\gamma_3}}(K) \neq 0$ implies $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K) \neq 0$ for every $K \subseteq \mathbb{N}^2$ where

$$G(q) = \{q(n) : n \in \mathbb{N}\},\ G(q') = \{q'(n) : n \in \mathbb{N}\},$$

and

$$G(r) = \{r(m) : m \in \mathbb{N}\},\ G(r') = \{r'(m) : m \in \mathbb{N}\}.$$

Proof. Since the sets $G(q') \setminus G(q)$, $G(r') \setminus G(r)$ are finite, there exist positive natural numbers N, M such that

$$\{q'(n) : n \geq N\} \subset \{q(n) : n \in \mathbb{N}\} \text{ and } \{r'(m) : m \geq M\} \subset \{r(m) : m \in \mathbb{N}\}.$$

For $n \geq N$, $m \geq M$, let $i(n)$ and $j(m)$ be strictly increasing sequences such that $q(n) = q'(i(n))$ and $r(m) = r'(j(m))$. If $\delta^{(2)}_{D_{\beta_3,\gamma_3}}(K) \neq 0$, the equation

$$\begin{aligned}&\delta^{*(2)}_{D_{\beta_3,\gamma_3}}(K)\\ = \ &\limsup_{n,m\to\infty} \frac{|\{p(n)+1 \leq k \leq q'(n),\ t(m)+1 \leq l \leq r'(m) :\ (k,l) \in K\}|}{\beta_3(n)\ \gamma_3(m)} > 0\end{aligned}$$

holds. So, we have

$$\begin{aligned}
&\frac{|\{p(n)+1\le k\le q(i(n)),\ t(m)+1\le l\le r(j(m)):\ (k,l)\in K\}|}{(q(i(n))-p(n))(r(j(m))-t(m))}\\
\le\ &\frac{\beta_1(n)\ \gamma_1(m)}{(q(i(n))-p(n))(r(j(m))-t(m))}\\
&\times\ \frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}
\end{aligned}$$

and

$$\begin{aligned}
&\delta^{*(2)}_{D_{\beta_1,\gamma_1}}(K)\\
=\ &\limsup_{n,m\to\infty}\frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}>0.
\end{aligned}$$

Hence, $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$ and the proof is finished. □

Corollary 3.2. *Let us assume that*

$$\lim_{n,m\to\infty}\frac{\beta_1(n)\ \gamma_1(m)}{\beta_3\,(n)\ \ \gamma_3\,(m)}=d\neq 0.$$

Then, the following are true:

i) *If $G\left(q'\right)\setminus G\left(q\right)$, $G\left(r'\right)\setminus G\left(r\right)$ are finite, then*

$$\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x)\supset\Gamma^{(2)}_{D_{\beta_3,\gamma_3}}(x)\,,$$

ii) *If $G\left(q'\right)\vartriangle G\left(q\right)$, $G\left(r'\right)\vartriangle G\left(r\right)$ are finite, then*

$$\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x)=\Gamma^{(2)}_{D_{\beta_3,\gamma_3}}(x)\,.$$

Theorem 3.3. *Let $G\left(p'\right)\setminus G\left(p\right)$, $G\left(t'\right)\setminus G\left(t\right)$ be finite and*

$$\lim_{n,m\to\infty}\frac{\beta_2\,(n)\ \ \gamma_2\,(m)}{\beta_1(n)\ \gamma_1(m)}=d\neq 0$$

hold. Then, $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$ implies $\delta^{(2)}_{D_{\beta_2,\gamma_2}}(K)\neq 0$ for every $K\subseteq\mathbb{N}^2$ where

$$G\left(p\right)=\left\{p\left(n\right):n\in\mathbb{N}\right\},\ G\left(p'\right)=\left\{p'\left(n\right):n\in\mathbb{N}\right\}$$

and

$$G\left(t\right)=\left\{t\left(m\right):m\in\mathbb{N}\right\},\ G\left(t'\right)=\left\{t'\left(m\right):m\in\mathbb{N}\right\}.$$

Proof. If $G\left(p'\right)\setminus G\left(p\right)$, $G\left(t'\right)\setminus G\left(t\right)$ are finite, then there exist positive numbers N,M such that

$$\{p'(n):n\ge N\}\subset\{p(n):n\in\mathbb{N}\}\text{ and }\{t'(m):m\ge M\}\subset\{t(m):m\in\mathbb{N}\}.$$

For $n\ge N$, $m\ge M$ let $i(n)$ and $j(m)$ be monotone increasing sequences such that $p'(n)=p(i(n))$ and $t'(m)=t(j(m))$. If $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$, then

$$\begin{aligned}
&\delta^{*(2)}_{D_{\beta_1,\gamma_1}}(K)\\
=\ &\limsup_{n,m\to\infty}\frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}>0.
\end{aligned}$$

From this we have

$$\begin{aligned}
&\frac{|\{p(n)+1\leq k\leq q(n),\ t(m)+1\leq l\leq r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}\\
\leq\ &\frac{(q(n)-p(i(n)))(r(m)-t(j(m)))}{\beta_1(n)\ \gamma_1(m)}\\
&\times\frac{|\{p(i(n))+1\leq k\leq q(n),\ t(j(m))+1\leq l\leq r(m):\ (k,l)\in K\}|}{(q(n)-p(i(n)))(r(m)-t(j(m)))}
\end{aligned}$$

and

$$\begin{aligned}
&\delta^{*(2)}_{D_{\beta_2,\gamma_2}}(K)\\
=\ &\limsup_{n,m\to\infty}\frac{|\{p'(n)+1\leq k\leq q(n),\ t'(m)+1\leq l\leq r(m):\ (k,l)\in K\}|}{\beta_2\,(n)\ \ \gamma_2\,(m)}>0.
\end{aligned}$$

This yields $\delta^{(2)}_{D_{\beta_2,\gamma_2}}(K)\neq 0$ and the desired result is obtained. ∎

Corollary 3.4. *Under the assumption of Theorem 3.3 the following statements are true:*

i) *If $G\,(p')\setminus G\,(p)$, $G\,(t')\ \setminus G\,(t)$ are finite, then*

$$\Gamma^{(2)}_{D_{\beta_2,\gamma_2}}(x)\supset\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x)\,,$$

ii) *If $G\,(p')\vartriangle G\,(p)$, $G\,(t')\vartriangle G\,(t)$ are finite, then*

$$\Gamma^{(2)}_{D_{\beta_2,\gamma_2}}(x)=\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x)\,.$$

Theorem 3.5. *Let us assume that*

$$p(n)\leq p'(n)<q'(n)\leq q(n),\ \ t(m)\leq t'(m)<r'(m)\leq r(m)$$

and

$$\lim_{n,m\to\infty}\frac{\beta_1(n)\ \gamma_1(m)}{\beta_4\,(n)\ \ \gamma_4\,(m)}=d\neq 0.$$

Then, $\delta^{(2)}_{D_{\beta_4,\gamma_4}}(K)\neq 0$ implies $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$ for every $K\subseteq\mathbb{N}^2$.

Proof. If $\delta^{(2)}_{D_{\beta_4,\gamma_4}}(K)\neq 0$, then we have

$$\begin{aligned}
&\delta^{*(2)}_{D_{\beta_4,\gamma_4}}(K)\\
=\ &\limsup_{n,m\to\infty}\frac{|\{p'(n)+1\leq k\leq q'(n),\ t'(m)+1\leq l\leq r'(m):\ (k,l)\in K\}|}{\beta_4\,(n)\ \ \gamma_4\,(m)}>0
\end{aligned}$$

and the relation

$$\begin{aligned}
&\frac{|\{p'(n)+1\leq k\leq q'(n),\ t'(m)+1\leq l\leq r'(m):\ (k,l)\in K\}|}{\beta_4\,(n)\ \ \gamma_4\,(m)}\\
\leq\ &\frac{\beta_1(n)\ \gamma_1(m)}{\beta_4\,(n)\ \ \gamma_4\,(m)}\\
&\times\frac{|\{p(n)+1\leq k\leq q(n),\ t(m)+1\leq l\leq r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}
\end{aligned}$$

holds. Therefore,

$$\begin{aligned}&\delta^{*(2)}_{D_{\beta_1,\gamma_1}}(K)\\ =\ &\limsup_{n,m\to\infty}\frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}>0.\end{aligned}$$

Hence, $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$ and the proof is completed. □

Corollary 3.6. *Under the assumptions of Theorem 3.5, we have*

$$\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x)\supset\Gamma^{(2)}_{D_{\beta_4,\gamma_4}}(x).$$

Theorem 3.7. *Let* $\{p(n)\}$ *and* $\{t(m)\}$ *be arbitrary sequences,* $q(n)\le n$, $r(m)\le m$ *for all* $n,m\in\mathbb{N}$ *and*

$$\lim_{n,m\to\infty}\frac{nm}{\beta_1(n)\ \gamma_1(m)}=d\neq 0.$$

Then, $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$ *implies* $\delta_2(K)\neq 0$ *for every* $K\subseteq\mathbb{N}^2$.

Proof. If $\delta^{(2)}_{D_{\beta_1,\gamma_1}}(K)\neq 0$, then

$$\begin{aligned}&\delta^{*(2)}_{D_{\beta_1,\gamma_1}}(K)\\ =\ &\limsup_{n,m\to\infty}\frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}>0,\end{aligned}$$

and the relation

$$\begin{aligned}&\frac{|\{p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\ (k,l)\in K\}|}{\beta_1(n)\ \gamma_1(m)}\\ \le\ &\frac{nm}{\beta_1(n)\ \gamma_1(m)}\frac{|\{k\le n,\ l\le m:\ (k,l)\in K\}|}{nm}\end{aligned}$$

holds. Therefore,

$$\delta^{*(2)}(K)=\limsup_{n,m\to\infty}\frac{|\{k\le n,l\le m:\ (k,l)\in K\}|}{nm}>0.$$

Hence, $\delta_2(K)\neq 0$. The proof is completed. □

Let us note that, if $q(n)=n$ and $r(m)=m$ for all $n,m\in\mathbb{N}$, the condition

$$\lim_{n,m\to\infty}\frac{nm}{\beta_1(n)\ \gamma_1(m)}=d\neq 0$$

is omitted in Theorem 3.7.

Corollary 3.8. *Under the conditions of Theorem 3.7, the following inclusion holds*

$$\Gamma^{(2)}(x)\supset\Gamma^{(2)}_{D_{\beta_1,\gamma_1}}(x).$$

4. Some inclusion results for $\Gamma^{(2)}_{D_{\lambda,\mu}}(x)$

In this section, we consider the case $q(n) := \lambda(n)$, $p(n) := \lambda(n-1)$, $r(m) := \mu(m)$ and $t(m) := \mu(m-1)$ when the sequences $\{\lambda(n)\}_{n\in\mathbb{N}}$, $\{\mu(m)\}_{m\in\mathbb{N}}$ are strictly increasing sequences of positive natural numbers and $\lambda(0) = 0$, $\mu(0) = 0$.

Theorem 4.1. *If*

$$\lim_{n,m} \frac{\lambda(n)\ \mu(m)}{\Delta\lambda\,(n)\,\Delta\mu\,(m)} = d \neq 0,$$

then $\delta^{(2)}_{D_{\lambda,\mu}}(K) \neq 0$ implies $\delta^{(2)}_{C_{\lambda,\mu}}(K) \neq 0$ for every subset K of $\mathbb{N}$, where $\Delta\lambda\,(n) = \lambda(n) - \lambda(n-1)$, $\Delta\mu\,(m) = \mu(m) - \mu(m-1)$.

Proof. If $\delta^{(2)}_{D_{\lambda,\mu}}(K) \neq 0$, then we have

$$\begin{aligned}&\delta^{*(2)}_{D_{\lambda,\mu}}(K)\\ =&\limsup_{n,m\to\infty} \frac{|\{(k,l) : \lambda(n-1)+1 \le k \le \lambda(n), \mu(m-1)+1 \le l \le \mu(m), (k,l)\in K\}|}{\Delta\lambda\,(n)\,\Delta\mu\,(m)}\\ >&0,\end{aligned}$$

and the following inequality

$$\begin{aligned}&\frac{|\{(k,l) : \lambda(n-1)+1 \le k \le \lambda(n), \mu(m-1)+1 \le l \le \mu(m), (k,l)\in K\}}{\Delta\lambda\,(n)\,\Delta\mu(m)} \le\\ &\frac{\lambda(n)\mu(m)|\{(k,l) : \lambda(n-1)+1 \le k \le \lambda(n), \mu(m-1)+1 \le l \le \mu(m), (k,l)\in K\}|}{\Delta\lambda\,(n)\,\Delta\mu(m)\lambda(n)\mu(m)}\end{aligned}$$

holds. Therefore, we have

$$\begin{aligned}&\delta^{*(2)}_{C_{\lambda,\mu}}(K) :=\\ &\limsup_{n,m\to\infty} \frac{|\{(k,l) : \lambda(n-1)+1 \le k \le \lambda(n), \mu(m-1)+1 \le l \le \mu(m), (k,l)\in K\}|}{\lambda(n)\mu(m)}\\ &> 0.\end{aligned}$$

So, $\delta^{(2)}_{C_{\lambda,\mu}}(K) \neq 0$. □

Corollary 4.2. *Under the condition of Theorem 4.1, the following inclusion holds*

$$\Gamma^{(2)}_{C_{\lambda,\mu}}(x) \supset \Gamma^{(2)}_{D_{\lambda,\mu}}(x).$$

Theorem 4.3. *Let $G = \{\lambda(n)\}_{n\in\mathbb{N}}$, $G' = \{\lambda'(n)\}_{n\in\mathbb{N}}$, $H = \{\mu(m)\}_{m\in\mathbb{N}}$ and $H' = \{\mu'(m)\}_{m\in\mathbb{N}}$ be infinite subsets of positive natural numbers. If $G \setminus G'$ and $H \setminus H'$ are finite and*

$$\lim_{n,m\to\infty} \frac{\Delta\lambda'(n)\ \Delta\mu'(m)}{\Delta\lambda\,(n)\ \ \Delta\mu\,(m)} = d \neq 0,$$

then $\delta_{D_{\lambda,\mu}}(K) \neq 0$ implies $\delta_{D_{\lambda',\mu'}}(K) \neq 0$ for every $K \subseteq \mathbb{N}^2$.

Proof. If the sets $G \setminus G'$ and $H \setminus H'$ are finite, then there exist positive natural numbers N, M such that the inclusions

$$\{\lambda'(n) : n \geq N\} \subset E \qquad \text{and} \qquad \{\mu'(m) : m \geq M\} \subset F$$

hold. It means that there exist monotone increasing sequences $(i(n))$, $(j(m))$ tending to infinity such that $\lambda'(n) = \lambda(i(n))$ and $\mu'(m) = \mu(j(m))$.

If $\delta_{D_{\lambda,\mu}}(K) \neq 0$, then we have

$$\delta^{*(2)}_{D_{\lambda,\mu}}(K) := \limsup_{n,m\to\infty} \frac{|\{(k,l) : \lambda(n-1)+1 \leq k \leq \lambda(n), \mu(m-1)+1 \leq l \leq \mu(m), (k,l) \in K\}|}{\Delta\lambda(n)\ \Delta\mu(m)} > 0.$$

Also, the inequality

$$\begin{aligned} &\frac{|\{(k,l) : \lambda(n-1)+1 \leq k \leq \lambda(n), \mu(m-1)+1 \leq l \leq \mu(m), (k,l) \in K\}|}{\Delta\lambda(n)\ \Delta\mu(m)} \\ &\leq \frac{(\lambda(i(n)) - \lambda(i(n)-1))\,(\mu(j(m)) - \mu(j(m)-1))}{\Delta\lambda(n)\ \Delta\mu(m)} \\ &\times \frac{|A_{i(n),j(m)}|}{(\lambda(i(n)) - \lambda(i(n)-1))\,(\mu(j(m)) - \mu(j(m)-1))} \end{aligned}$$

holds where

$$A_{i(n),j(m)} = \{(k,l) : \lambda(i(n)-1)+1 \leq k \leq \lambda(i(n)), \mu(j(m)-1)+1 \leq l \leq \mu(j(m)), (k,l) \in K\}.$$

Taking the upper limit we have

$$\limsup_{n,m\to\infty} \frac{|A_{i(n),j(m)}|}{(\lambda(i(n)) - \lambda(i(n)-1))\,(\mu(j(m)) - \mu(j(m)-1))} > 0.$$

Therefore,

$$\limsup_{n,m\to\infty} \frac{|B_{\lambda',\mu'}|}{\Delta\lambda'(n)\Delta\mu'(m)} > 0$$

where

$$\begin{aligned} B_{\lambda',\mu'} = &\{(k,l) : \lambda(i(n)-1)+1 \leq k \leq \lambda(i(n)), \mu(j(m)-1)+1 \leq l \leq \mu(j(m)), \\ &(k,l) \in K\} \end{aligned}$$

and this yields $\delta_{D_{\lambda',\mu'}}(K) \neq 0$. This completes the proof. □

Corollary 4.4. *Let us assume that*

$$\lim_{n,m\to\infty} \frac{\Delta\lambda'(n)\ \Delta\mu'(m)}{\Delta\lambda(n)\ \Delta\mu(m)} = d \neq 0.$$

Then, the following statements are true:

(i) *If $G \setminus G'$ and $H \setminus H'$ are finite, then*

$$\Gamma^{(2)}_{D_{\lambda,\mu}}(x) \supset \Gamma^{(2)}_{D_{\lambda',\mu'}}(x),$$

(ii) *If $G \vartriangle G'$ and $H \vartriangle H'$ are finite, then*

$$\Gamma^{(2)}_{D_{\lambda,\mu}}(x) = \Gamma^{(2)}_{D_{\lambda',\mu'}}(x).$$

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