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A NOTE ON FINITE GENERATED SUBSEMIGROUPS OF $T(X, Y)$

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Abstract. It is well known that a countable set of transformations on an infinite set X is contained in a two-generated subsemigroup of the full transformation semigroup on X. If $Y \subset X$, then $T(X, Y)$, the set of all transformations on X with an image in Y, forms a semigroup of transformations with restricted range, as shown in 1975 by Symons [10]. In this paper, we give a sufficient and necessary condition for a countable subset of $T(X, Y)$ to be contained in a three-generated subsemigroup of $T(X, Y)$.

1. Introduction and basic concepts

A subset A of the semigroup S is called a *generating system* or a *generating set* for S provided that A generates S, it is denoted by $\langle A \rangle = S$. The *rank* of a finite semigroup S is the smallest number of elements that generate S denoted by

$$\operatorname{rank} S := \min\{|A| : A \subseteq S, \langle A \rangle = S\}$$

and, for an uncountable semigroup S, the rank of S is $|S|$. For a semigroup S and a set $A \subseteq S$, the *relative rank* of S modulo A is the minimum cardinality of a set B such that $A \cup B$ generates S; we denote this by $\operatorname{rank}(S : A)$ [3]. We will call this briefly the relative rank of A in S. It can be seen immediately that $\operatorname{rank}(S : \emptyset) = |S|$, $\operatorname{rank}(S : S) = 0$, $\operatorname{rank}(S : A) = \operatorname{rank}(S : \langle A \rangle)$ and $\operatorname{rank}(S : A) = 0$ if and only if A is a generating set for S.

The full transformation semigroup on a set X is defined as consisting of all mappings of X into itself, the semigroup operation being the composition of mappings (we perform composition from left to right). We denote the full transformation semigroup on the set X as $T(X)$. Such semigroup is extremely important, any semigroup S embeds in $T(S^1)$ where S^1 is a semigroup S adjoining identity element $1 \notin S$ [2].

In 1935, W. Sierpiński [8] showed that any countable subset of the infinite full transformation semigroup $T(X)$ is contained in a two-generated subsemigroup of $T(X)$. A simple proof was given by S. Banach [1]. An immediate consequence of this result is that the relative rank of a subset of $T(X)$ is either uncountable or at most two [2]. In 2003, the same authors considered the relative rank of the

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full transformation semigroup $T(X)$ modulo the semigroup of all order preserving mappings $\mathcal{O}_X$ on X [4].

The range (or image) and the kernel of $\alpha \in T(X)$ are defined by

$$\operatorname{ran}\alpha := \{x\alpha : x \in X\}$$

and

$$\ker\alpha := \{(x, y) \in X \times X : x\alpha = y\alpha\}.$$

For $\alpha \in T(X)$ and $A \subseteq X$, $\alpha|_A$ denotes a mapping α with restricted domain to A. Recall that a set $\mathcal{P}$ of subsets of X is called a *partition* of X if

(1) $\emptyset \notin \mathcal{P}$,
(2) $\bigcup_{A\in\mathcal{P}} A = X$ and
(3) if $A, B \in \mathcal{P}$ and $A \neq B$, then $A \cap B = \emptyset$.

The sets in $\mathcal{P}$ are called the *blocks* of the partition. It is well known that a partition $\mathcal{P}$ of X defines an equivalence relation $\sim$ on X with $x \sim y$ if and only if x and y are together in a block in $\mathcal{P}$ and, conversely, each equivalence relation $\sim$ on X defines a partition $\mathcal{P}$ of X with two elements x and y belonging to the same block if and only if $x \sim y$. This gives a one-to-one correspondence between equivalence relation and partition on X. For this reason, $\ker\alpha$ can be regarded as relation as well as partition on X. In the present paper, we will switch between both concepts. Which concept is used for the kernel in each case will become clear by the context. Let $\mathcal{P}$ be a partition of X. A set $T \subseteq X$ is called a transversal of $\mathcal{P}$ if $|A \cap T| = 1$ for all blocks A in P.

The present paper is devoted to a particular subsemigroup of $T(X)$. In 1975, Symons [10] introduced and studied the subsemigroup $T(X, Y)$ consisting of $\alpha \in T(X)$ with the range of α being a subset of Y, where $Y \subseteq X$. That is

$$T(X, Y) := \{\alpha \in T(X) : \operatorname{ran}\alpha \subseteq Y\}.$$

We know that $T(X, Y)$ is a subsemigroup of $T(X)$. If $Y = X$, then $T(X, Y)$ coincides with the full transformation semigroup $T(X)$. The semigroup $T(X, Y)$ has been studied extensively, for example, in [7] where the authors gave a necessary and sufficient condition for $T(X, Y)$ to be regular. In [5], the authors described the ideal structure of $T(X, Y)$. In [6], the author introduced the set $F(X, Y) = \{\alpha \in T(X, Y) : \operatorname{ran}(\alpha) \subseteq \operatorname{ran}(\alpha|_Y)\}$ and proved that every regular semigroup S can be embedded in $F(S^1, S)$, described Green's relations and ideals of $F(X, Y)$ and applied these results to get all of its maximal regular subsemigroups when Y is a nonempty finite subset of X. In [9], the authors compute the rank of $F(X, Y)$.

In this paper, we study the rank of $T(X, Y)$ where X is an infinite set and present a sufficient and necessary condition for a countable subset of $T(X, Y)$ to be contained in a three-generated subsemigroup of $T(X, Y)$. We also show that a countable subset S of $T(X, Y)$ is not contained in a finite-generated subsemigroup of $T(X, Y)$ if Y is finite.

2. Main results

We start with a sufficient condition for a countable subset of $T(X, Y)$ to be contained in a three-generated subsemigroup of $T(X, Y)$.

Lemma 2.1. *Let X and Y be infinite sets, $Y \subseteq X$, and $S \subseteq T(X,Y)$ any countable set, say $S = \{\theta_1, \theta_2, \ldots\}$. If there exists a partition $\{A_y : y \in Y\}$ of X such that each of its blocks is a subset of a block of $\ker\theta_i$ for all $i \in \mathbb{N}$, then S is contained in a three-generated subsemigroup of $T(X,Y)$.*

Proof. Let T be a transversal of $\{A_y,\ y \in Y\}$. Partition Y into a countable disjoint union of infinitely many sets $Y_0, Y_1, \ldots, Y_n, \ldots$ such that all are of the same cardinality as Y. Similarly, we partition Y_0 into $Y_{0,1}, Y_{0,2}, \ldots, Y_{0,n}, \ldots$ all of the same cardinality as Y.

We define $\alpha : X \to Y$ by

$$x\alpha = y$$

if $x \in A_y$, $y \in Y$ for all $x \in X$. Since $\{A_y : y \in Y\}$ is a partition of X, the mapping α is well-defined. Let $\tilde{\beta}$ and $\tilde{\gamma}$ be mappings that map Y_n bijectively onto Y_{n+1} for all $n \in \mathbb{N} \cup \{0\}$ and Y_n bijectively onto $Y_{0,n}$ for all $n \in \mathbb{N}$, respectively.

We let $x_0 \in Y \setminus Y_0$, $y_0 \in Y_0$ and define $\beta : X \to Y \setminus Y_0$ by

$$x\beta = \begin{cases} x\tilde{\beta} & \text{if } x \in Y, \\ x_0 & \text{if } x \in X \setminus Y \end{cases}$$

for all $x \in X$ and define $\gamma : X \setminus Y_0 \to Y_0$ by

$$y\gamma = \begin{cases} y\tilde{\gamma} & \text{if } y \in Y \setminus Y_0, \\ y_0 & \text{if } y \in X \setminus Y \end{cases}$$

for all $y \in X \setminus Y_0$. We see that γ is not defined on Y_0.

We let $n \in \mathbb{N}$ and $\delta_n := \alpha\beta\gamma\beta^n\gamma$ be a mapping of X into $Y_{0,n}$ and $\hat{\delta}_n := \alpha|_T\beta\gamma\beta^n\gamma$ be mapping of T into $Y_{0,n}$. Let $x, y \in T$ be such that $x \neq y$. Since x and y belong to different blocks of the partition $\{A_y : y \in Y\}$, we get $x\,\alpha|_T \neq y\,\alpha|_T$. By the injectivity of $\beta|_Y$ and $\gamma|_{Y\setminus Y_0}$, $x\,\hat{\delta}_n \neq y\,\hat{\delta}_n$. This shows $\hat{\delta}_n$ is an injective mapping of T into $Y_{0,n}$. Now, let $y \in Y_{0,n}$. Then there is $y_1 \in Y_n$ such that $y = y_1\tilde{\gamma}$, so $y = y_1\gamma$. Since $y_1 \in Y_n$, there is $y_2 \in Y_0$ such that $y_1 = y_2\tilde{\beta}^n$, so $y_1 = y_2\beta^n$, that is $y = y_2(\beta^n\gamma)$. Since $y_2 \in Y_0$, assume that $y_2 \in Y_{0,k}$ for some $k \in \mathbb{N}$. So there is $y_3 \in Y_k$ such that $y_2 = y_3\tilde{\gamma}$, that is, $y_2 = y_3\gamma$, so $y = y_3(\gamma\beta^n\gamma)$. Since $y_3 \in Y_k$, there is $y_4 \in Y_{k-1}$ such that $y_3 = y_4\tilde{\beta}$. That is $y_3 = y_4\beta$, so $y = y_4(\beta\gamma\beta^n\gamma)$. Since $y_4 \in Y_{k-1} \subseteq Y$, we choose $y_5 \in A_{y_4} \cap T \neq \emptyset$ such that $y_4 = y_5\alpha|_T$. So $y = y_5\,(\alpha|_T\beta\gamma\beta^n\gamma)$. This shows that $\hat{\delta}_n$ is a mapping of T onto $Y_{0,n}$. Altogether, we have $\hat{\delta}_n$ is a bijective mapping of T onto $Y_{0,n}$.

Let $x \in Y_0$. Then there is $n \in \mathbb{N}$ such that $x \in Y_{0,n}$. We now define a transformation $\eta : X \to Y$ by

$$x\eta = \begin{cases} x\gamma & \text{if } x \in X \setminus Y_0, \\ x\hat{\delta}_n^{-1}\theta_n & \text{if } x \in Y_0. \end{cases}$$

Next, we show that $\theta_n = \delta_n \eta$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x \in X$. If $x \in T$, then

$$\begin{aligned} x\delta_n\eta &= x\hat{\hat{\delta}}_n(\hat{\delta}_n^{-1}\theta_n) \\ &= x(\hat{\delta}_n\hat{\delta}_n^{-1})\theta_n \\ &= x\theta_n. \end{aligned}$$

If $x \notin T$, then $x \in A_y$ for some $y \in Y$, so $x\alpha = y$. Thus, we choose $x_0 \in A_y \cap T \neq \emptyset$ such that

$$x\alpha = y = x_0\alpha.$$

Since $x, x_0 \in A_y$ and A_y is a subset of some block of $\ker \theta_n$, we have

$$x\theta_n = x_0\theta_n. \tag{2.1}$$

Next, $x\alpha = x_0\alpha$ implies $x_0\alpha\beta\eta\beta^n\eta^2 = x\alpha\beta\eta\beta^n\eta^2$. That is,

$$x_0\delta_n\eta = x\delta_n\eta. \tag{2.2}$$

Since $x_0\delta_n\eta = x_0\theta_n$ is already shown and, by (2.1) and (2.2), we have

$$x\delta_n\eta = x_0\delta_n\eta = x_0\theta_n = x\theta_n.$$

Hence, $\theta_n = \alpha\beta\eta\beta^n\eta^2 \in \langle \alpha, \beta, \eta \rangle$. Therefore, S is contained in the three-generated subsemigroup $\langle \alpha, \beta, \eta \rangle$ of $T(X, Y)$. □

In the previous lemma, we considered a countable subset S of $T(X, Y)$ where X, Y are infinite sets. Now, we consider a countable subset S of $T(X, Y)$ where X is an infinite set and Y is finite. Then, we obtain the following lemma.

Lemma 2.2. *Let X be an infinite set and S be a countable subset of $T(X, Y)$ where $Y \subseteq X$. If Y is finite, then there is no finite subset $B \subseteq T(X, Y)$ such that $S \subseteq \langle B \rangle$.*

Proof. We first let $|Y| = k$ and assume that there is a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of $T(X, Y)$ such that $S \subseteq \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$. We let $A_i := \{\ker \alpha_i\beta : \beta \in T(X, Y)\}$ where $1 \leq i \leq n$ and $|A_i| = p_i \in \mathbb{N}$ since $|Y| \in \mathbb{N}$. Then $|\bigcup_{i=1}^n A_i| \leq \sum_{i=1}^n p_i$. Now, we consider any partition $\rho \in \bigcup_{i=1}^n A_i$ of X with $k_\rho := |\rho|$. Then, the number of all k_ρ-element subsets of Y is $\binom{k}{k_\rho}$. Thus $k_\rho!\binom{k}{k_\rho}$ is the maximal number of transformations in $T(X, Y)$ with kernel ρ. Any element in $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ has the form $\alpha_i\beta$ for some $i \in \{1, 2, \ldots, n\}$ and $\beta \in T(X, Y)$. Hence, each element in $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ has a kernel in $\bigcup_{i=1}^n A_i$. Thus, $\sum_{\rho \in \bigcup_{i=1}^n A_i} k_\rho!\binom{k}{k_\rho} \in \mathbb{N}$ is the maximal number of elements in $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ and $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ is finite. This contradicts the fact that $S \subseteq \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ is countable. □

The following proposition gives a necessary and sufficient condition for a countable set $S \subseteq T(X, Y)$ to be contained in a finite-generated subsemigroup of $T(X, Y)$.

Proposition 2.3. *Let X and Y be infinite sets, $Y \subseteq X$ and $S = \{\theta_1, \theta_2, \ldots\} \subseteq T(X, Y)$ a countable subset of $T(X, Y)$. Then, the following conditions are equivalent:*

(i) *There exists a partition $\{A_y : y \in Y\}$ of X such that each of its blocks is a subset of a block of $\ker \theta_i$ for all $i \in \mathbb{N}$.*

(ii) *S is contained in a finite-generated subsemigroup of $T(X,Y)$.*

Proof. (i) $\Rightarrow$ (ii) : This is clear by Lemma 2.1.
(ii) $\Rightarrow$ (i) : Assume that S is contained in a finite-generated subsemigroup of $T(X,Y)$. Then, there are $\alpha_1, \alpha_2, \ldots, \alpha_m \in T(X,Y)$ for some $m \in \mathbb{N}$ such that $S \subseteq \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$. Let $\mathcal{P}$ be the partition of X associated to the equivalence relation $\bigcap_{i=1}^{m} \ker \alpha_i$. Let $B \in \mathcal{P}$ and $i \in \mathbb{N}$. Then, there is $j \in \{1, 2, \ldots, m\}$ such that $\theta_i = \alpha_j \beta$ for some $\beta \in T(X,Y)$ so we have $\ker \alpha_j \subseteq \ker \theta_i$ regarded as relations. Since $B \in \mathcal{P}$, there is a block $\bar{x} \in \ker \alpha_j$ with $B \subseteq \bar{x}$. Since $\ker \alpha_j \subseteq \ker \theta_i$, there is a block $\bar{y} \in \ker \theta_i$ with $\bar{x} \subseteq \bar{y}$. Hence $B \subseteq \bar{y}$. □

Summarizing all observations, we obtain the following proposition.

Proposition 2.4. *Let X be an infinite set, $Y \subseteq X$ and $S \subseteq T(X,Y)$ any countable set, say $S = \{\theta_1, \theta_2, \ldots\}$. Then, the following statements are equivalent:*

(i) *S is contained in a three-generated subsemigroup of $T(X,Y)$.*
(ii) *S is contained in a finite-generated subsemigroup of $T(X,Y)$.*
(iii) *Y is infinite and there exists a partition $\{A_y : y \in Y\}$ of X such that each block in this partition is a subset of a block of $\ker \theta_i$ for all $i \in \mathbb{N}$.*

Proof. (i)$\Rightarrow$(ii) : Straightforward.
(ii)$\Rightarrow$(iii) : This is clear by Lemma 2.2 and Proposition 2.3.
(iii)$\Rightarrow$(i) : This is clear by Lemma 2.1. □

In our main theorem, we characterize the sets $Y \subseteq X$ such that any countable set $S \subseteq T(X,Y)$ is contained in a three-generated subsemigroup of $T(X,Y)$.

Theorem 2.5. *Let X be an infinite set and let $Y \subseteq X$. Then, the following statements are equivalent:*

(i) *Each countable set $S \subseteq T(X,Y)$ is contained in a three-generated subsemigroup of $T(X,Y)$.*
(ii) *$|X| = |Y|$ or $|X| > |Y| > \aleph_0$.*

Proof. (i)$\Rightarrow$(ii): Suppose that $|X| \neq |Y|$. Then $|X| > |Y|$. We show that $|Y| > \aleph_0$. The case $|Y| < \aleph_0$ is not possible by Lemma 2.2. Assume that $|Y| = \aleph_0$, that is, $Y = \{y_i : i \in \mathbb{N}\}$. Now, we construct transformations $\theta_1, \theta_2, \theta_3, \ldots$ in $T(X,Y)$ as follows.

Partition X into $X_1, X_2, X_3, \ldots$ such that $|X_i| = |X|$ for each $i \in \mathbb{N}$. Let $i_1 i_2 \cdots i_n, n \in \mathbb{N}$, be a sequence of natural numbers. Then, we partition $X_{i_1 i_2 \cdots i_n}$ into $X_{i_1 i_2 \cdots i_n 1}, X_{i_1 i_2, \ldots, i_n 2}, \ldots$ such that $|X_{i_1 i_2 \cdots i_n j}| = |X|$ for each $j \in \mathbb{N}$. We let θ_1 be a transformation with $\operatorname{ran} \theta_1 = Y$ and $y_i \theta_1^{-1} = X_i$ for all $i \in \mathbb{N}$. For $n \in \mathbb{N}$, let θ_{n+1} be the transformation with $\operatorname{ran} \theta_{n+1} = Y$ and $y_k \theta_{n+1}^{-1} = \bigcup_{i_1, i_2, \ldots, i_n \in \mathbb{N}} X_{i_1 i_2 \cdots i_n k}$ for all $k \in \mathbb{N}$. Thus, we have a countable set $S := \{\theta_1, \theta_2, \theta_3, \ldots\} \subseteq T(X,Y)$. By our assumption and by Proposition 2.4, there is a partition $\{A_y : y \in Y\}$ of X such that, for each $y \in Y$ and each $i \in \mathbb{N}$, there is $B \in \ker \theta_i$ such that $A_y \subseteq B$. This means that there is a countable sequence $i_1 i_2 \cdots$ of natural numbers such that $A_y \subseteq X_{i_1 i_2 \cdots i_k}$ for all $k \in \mathbb{N}$. Since $\{A_y : y \in Y\}$ is a partition of X, the cardinality of $\{A_y : y \in Y\}$ is $|\{i_1 i_2 \cdots : i_1, i_2, \ldots \in \mathbb{N}\}| = \aleph_0^{\aleph_0}$. That is, $|Y| = |\{A_y : y \in Y\}| = \aleph_0^{\aleph_0} > \aleph_0$ which is a contradiction. Hence $|Y| > \aleph_0$.

(ii)$\Rightarrow$(i): Let $\{\theta_1, \theta_2, \ldots\} = S \subseteq T(X, Y)$ be a countable set. If $|X| = |Y|$, then $\mathcal{A} = \{\{x\} : x \in X\}$ is a partition of X with $|\mathcal{A}| = |X| = |Y|$. Let $x \in X$ and $i \in \mathbb{N}$. Then, we have the block which contains x, namely $\bar{x} \in \ker\theta_i$ such that $\{x\} \subseteq \bar{x}$, i.e., this partition satisfies (iii) in Proposition 2.4 and S is contained in a three-generated subsemigroup of $T(X, Y)$. In the case of $|X| > |Y| > |\aleph_0|$, we know that $\bigcap_{i\in\mathbb{N}} \ker\theta_i$ is a partition of X and $\bigcap_{i\in\mathbb{N}} \ker\theta_i \subseteq \ker\theta_j$ considered as relation for each $j \in \mathbb{N}$. But $|\ker\theta_j| \leq |Y|$ for each $j \in \mathbb{N}$, so $|\bigcap_{i\in\mathbb{N}} \ker\theta_i| \leq |Y|^{\aleph_0}$. Since $|Y| > \aleph_0$, we obtain $|Y| = |Y|^{\aleph_0}$. That means that $\bigcap_{i\in\mathbb{N}} \ker\theta_i$ is a partition of X of cardinality less than or equal to $|Y|$. If $|\bigcap_{i\in\mathbb{N}} \ker\theta_i| < |Y|$, then there is a block $B \in \bigcap_{i\in\mathbb{N}} \ker\theta_i$ with $|B| \geq |Y|$. We partition B into $|Y|$ many blocks $B_y, y \in Y$, and consider the partition $\left[\left(\bigcap_{i\in\mathbb{N}} \ker\theta_i\right) \setminus B\right] \cup \{B_y : y \in Y\}$ of X with cardinality $|Y|$. So without loss of generality, we can assume that $|\bigcap_{i\in\mathbb{N}} \ker\theta_i| \leq |Y|$. Hence, $\bigcap_{i\in\mathbb{N}} \ker\theta_i$ is a partition of cardinality $|Y|$. Let $B \in \bigcap_{i\in\mathbb{N}} \ker\theta_i$ and $j \in \mathbb{N}$. Since $\bigcap_{i\in\mathbb{N}} \ker\theta_i \subseteq \ker\theta_j$, there is a block $\bar{x} \in \ker\theta_j$ with $B \subseteq \bar{x}$. Now, $\bigcap_{i\in\mathbb{N}} \ker\theta_i$ satisfies the condition (iii) in Proposition 2.4. Thus, we have completed the proof. $\square$

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