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# LIFTING CONNECTIONS TO THE $r$-JET PROLONGATION OF THE COTANGENT BUNDLE

WŁODZIMIERZ M. MIKULSKI

*Abstract.* We show that the problem of finding all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ lifting classical linear connections $\nabla$ on $m$-manifolds $M$ into classical linear connections $C_M(\nabla)$ on the $r$-jet prolongation $J^rT^*M$ of the cotangent bundle $T^*M$ of $M$ can be reduced to that of finding all $\mathcal{M}f_m$-natural operators $D : Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ sending classical linear connections $\nabla$ on $M$ into tensor fields $D_M(\nabla)$ of type $(p, q)$ on $M$.

## 1. Introduction

All manifolds are assumed to be smooth, Hausdorff, finite dimensional, and without boundaries. Maps are assumed to be smooth (of class $C^\infty$). The category of $m$-dimensional manifolds and their embeddings is denoted by $\mathcal{M}f_m$.

In [5], M. Kureš described completely all $\mathcal{M}f_m$-natural operators $B : Q_\tau \rightsquigarrow QT^*$ lifting torsion free classical linear connections $\nabla$ on $m$-manifolds $M$ into classical linear connections $B_M(\nabla)$ on the cotangent bundle $T^*M$ of $M$.

In [4], the authors studied a similar problem of describing all $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow Q(\bigotimes^k T^*)$ transforming classical linear connections $\nabla$ on $m$-manifolds $M$ into classical linear connections $B_M(\nabla)$ on the $k$-th tensor power $\bigotimes^k T^*M$ of the cotangent bundle $T^*M$ of $M$. They proved that this problem can be reduced to the well known one of describing all $\mathcal{M}f_m$-natural operators $D : Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ sending classical linear connections $\nabla$ on $m$-manifolds $M$ into tensor fields $D_M(\nabla)$ of type $(p, q)$ on $M$.

In [7], we investigated a similar problem of describing all $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow QT^{r*}$ lifting classical linear connections $\nabla$ on $m$-manifolds $M$ into classical linear connections $B_M(\nabla)$ on the $r$-th order cotangent bundle $T^{r*}M = J^r(M, \mathbb{R})_0$ of $M$. We proved that this problem can also be reduced to the well known one of describing all $D : Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$.

In the present note, we consider a similar problem of describing all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ lifting classical linear connections $\nabla$ on $m$-manifolds $M$ into classical linear connections $C_M(\nabla)$ on the $r$-jet prolongation $J^rT^*M$ of the cotangent bundle $T^*M$ of $M$. Modifying paper [7], we show that this problem

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can be reduced to the well-known one of describing all
$D : Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$, too.

We recall that the $r$-jet prolongation of the cotangent bundle is a functor $J^rT^* : \mathcal{M}f_m \to \mathcal{VB}$ sending any $m$-manifold $M$ into $J^rT^*M$ (the vector bundle of $r$-jets of sections $M \to T^*M$ of the cotangent bundle $T^*M \to M$ of $M$) and any embedding $\varphi : M_1 \to M_2$ of two $m$-manifolds into $J^rT^*\varphi : J^rT^*M_1 \to J^rT^*M_2$ given by $J^rT^*\varphi(j^r_x\omega) = j^r_{\varphi(x)}(T^*\varphi \circ \omega \circ \varphi^{-1})$, $j^r_x\omega \in J^rT^*M$. If $r = 0$, $J^0T^*M \tilde{=} T^*M$ (the usual cotangent bundle).

Further, we inform that a linear connection on a vector bundle $E$ over a manifold $M$ is a bilinear map $D : \mathcal{X}(M) \times \Gamma E \to \Gamma E$ such that $D_{fX}\sigma = fD_X\sigma$ and $D_X f\sigma = Xf\sigma + fD_X\sigma$ for any smooth map $f : M \to \mathbb{R}$, any vector field $X \in \mathcal{X}(M)$ on $M$ and any smooth section $\sigma \in \Gamma E$ of $E \to M$. In particular, a linear connection $\nabla$ in the tangent space $TM$ of $M$ is called a classical linear connection on $M$.

We also inform that a general definition of natural operators can be found in [3]. In particular, an $\mathcal{M}f_m$-natural operator $C : Q \rightsquigarrow QJ^rT^*$ is an $\mathcal{M}f_m$-invariant system $C = \{C_M\}_{M \in obj(\mathcal{M}f_m)}$ of regular operators (functions)

$$C_M : \underline{Q}(M) \to \underline{Q}(J^rT^*M)$$

for any $m$-manifold $M$ where $\underline{Q}(M)$ is the set of all classical linear connections on $M$. More precisely, the $\mathcal{M}f_m$-invariance of $C$ means that if $\nabla_1 \in \underline{Q}(M_1)$ and $\nabla_2 \in \underline{Q}(M_2)$ are $\varphi$-related by an embedding $\varphi : M_1 \to M_2$ of $m$-manifolds (i.e. $\varphi$ is $(\nabla_1, \nabla_2)$-affine), then $C_{M_1}(\nabla_1)$ and $C_{M_2}(\nabla_2)$ are $J^rT^*\varphi$-related. The regularity means that $C_M$ transforms smoothly parametrized families of connections into smoothly parametrized ones.

Similarly, an $\mathcal{M}f_m$-natural operator (natural tensor) $D : Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ is an $\mathcal{M}f_m$-invariant system $D = \{D_M\}_{M \in obj(\mathcal{M}f_m)}$ of regular operators

$$D_M : \underline{Q}(M) \to \mathcal{T}^{p,q}(M)$$

for any $M \in \mathcal{M}f_m$, where $\mathcal{T}^{p,q}(M)$ is the set of tensor fields of type $(p, q)$ on $M$.

By the general result in [6], since $J^rT^* : \mathcal{M}f \to \mathcal{VB}$ is a vector natural bundle, there exists an $\mathcal{M}f_m$-natural operator $C : Q \rightsquigarrow QJ^rT^*$. An explicit example of a natural operator $C : Q \rightsquigarrow QJ^rT^*$ (similar to Example 1 in [7]) will be presented in item 2, too.

A full description of all $\mathcal{M}f_m$-natural operators $Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ transforming torsion free classical connections on $m$-manifolds into tensor fields of types $(p, q)$ can be found in Lemma in Section 33.4 in [3]. For the reader's convenience, we present this description. Each covariant derivative of the curvature $\mathcal{R}(\nabla) \in C^\infty_M(\wedge^2 T^*M \otimes T^*M \otimes TM)$ of a classical linear connection $\nabla$ is an ($\mathcal{M}f_m$-)natural tensor. Further, every tensor multiplication of two natural tensors and every contraction on one covariant and one contravariant entry of a natural tensor gives a new natural tensor. Finally, we can multiply any natural tensor with a connection independent natural tensor, we can permute any number of entries in the tensor product and we can repeat these steps and take linear combinations. In this way, we can obtain any natural tensor of types $(p, q)$ depending on a torsion free classical linear connection. All natural tensors of a (not necessarily torsion free) classical linear connection $\nabla$ can be obtained provided we also include the torsion tensor $\mathcal{T}(\nabla)$ and their covariant derivatives in the above procedure.

## 2. Preparations

We are going to present an example of an $\mathcal{M}f_m$-natural operator $C^{(r)} : Q \rightsquigarrow QJ^rT^*$. We start with the following important proposition (similar to Proposition 1 in [7]).

**Proposition 2.1.** *Let $\nabla$ be a classical linear connection on $M$. Then, there is a (canonical in $\nabla$) vector bundle isomorphism*

$$I_\nabla : J^rT^*M \to \bigoplus_{k=0}^{r} \bigodot^{k} T^*M \otimes T^*M$$

*covering the identity map of $M$.*

*Proof.* We proceed as in the proof of Proposition 1 in [7]. Let $v \in T^{r*}_xM$, $x \in M$. Let $\varphi : (M,x) \to (\mathbb{R}^m, 0)$ be a $\nabla$-normal coordinate system with center $x$. We put

$$I_\nabla(v) = I^\varphi_\nabla(v) := \bigoplus_{k=0}^{r} \bigodot^{k} T^*\varphi^{-1} \otimes T^*\varphi^{-1} \circ I \circ J^rT^*\varphi(v),$$

where $I : J^rT^*_0\mathbb{R}^m \to \bigoplus_{k=1}^{r} \bigodot^{k} T^*_0\mathbb{R}^m \otimes T^*_0\mathbb{R}^m = \bigoplus_{k=0}^{r} \bigodot^{k} \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$ is the obvious $GL(m)$-invariant vector space isomorphism. If $\psi : (M,x) \to (\mathbb{R}^m, 0)$ is another $\nabla$-normal coordinate system with center $x$, then $\psi = A \circ \varphi$ (near $x$) for some $A \in GL(m)$. Using the $GL(m)$-invariance of $I$, we deduce that $I^\psi_\nabla(v) = I^\varphi_\nabla(v)$. So, the definition of $I_\nabla(v)$ is independent of the choice of $\varphi$. $\square$

In [1], J. Gancarzewicz presented a canonical construction of a classical linear connection on the total space of a vector bundle $E$ over $M$ from a linear connection $D$ in $E$ by means of a classical linear connection $\nabla$ on $M$. For the reader's convenience, we present the construction. If $X$ is a vector field on $M$ and $\sigma$ is a section of $E$, then $D_X\sigma$ is a section of $E$. Further, let $X^D$ denote the horizontal lift of a vector field $X$ with respect to $D$. Moreover, using the translations in the individual fibres of $E$, we derive from every section $\sigma : M \to E$ a vertical vector field $\sigma^V$ on $E$ called the vertical lift of $\sigma$. In [1], J. Gancarzewicz proved the following fact.

**Proposition 2.2.** *For every linear connection $D$ in a vector bundle $E$ over $M$ and every classical linear connection $\nabla$ on $M$, there exists a unique classical linear connection $\Theta = \Theta(D,\nabla)$ on the total space $E$ with the following properties*

$$\Theta_{X^D}Y^D = (\nabla_XY)^D,\ \Theta_{X^D}\sigma^V = (D_X\sigma)^V,$$

$$\Theta_{\sigma^V}X^D = 0,\ \Theta_{\sigma^V}\sigma_1^V = 0$$

*for all vector fields $X, Y$ on $M$ and all sections $\sigma, \sigma_1$ of $E$.*

It is well-known (see [2]) that every classical linear connection $\nabla$ on an $m$-manifold $M$ can be extended to a linear connection $D^{(r)}_{\nabla} = \nabla$ in $\bigoplus_{k=0}^r \overset{k}{\bigodot} T^*M \otimes T^*M$ by

$$(\nabla_X A)(X_0, \dots, X_k) = XA(X_0, \dots, X_k) - \sum_{i=0}^{k} A(X_0, \dots, \nabla_X X_i, \dots, X_k),$$
$$A \in \Gamma(\bigodot^k T^*M \otimes T^*M), X_0, \dots, X_k \in \mathcal{X}(M), k = 0, \dots, r.$$

Now, we are in a position to present a natural operator $C^{(r)} : Q \rightsquigarrow QJ^rT^*$.

**Example 2.3.** As in Example 1 in [7], given a classical linear connection $\nabla$ on $M$, by Propositions 2.1 and 2.2, we have the classical linear connection $\nabla^{(r)}$ on $J^rT^*M$ given by

$$\nabla^{(r)} := (I_\nabla)_*^{-1}\Theta(D^{(r)}_\nabla, \nabla).$$

Clearly, the family $C^{(r)} : Q \rightsquigarrow QJ^rT^*$ of operators

$$C^{(r)}_M : \underline{Q}(M) \to \underline{Q}(J^rT^*M),\ C^{(r)}_M(\nabla) := \nabla^{(r)},$$

where $M \in obj(\mathcal{M}f_m)$ and $\nabla \in \underline{Q}(M)$, is an $\mathcal{M}f_m$-natural operator.

## 3. A simple reduction

The set of all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ is an affine space with the corresponding vector space of all $\mathcal{M}f_m$-natural operators $\Delta : Q \rightsquigarrow (\bigotimes^2 T^* \otimes T)J^rT^*$ lifting classical linear connections $\nabla$ on $m$-manifolds $M$ into tensor fields $\Delta_M(\nabla)$ of type $(1, 2)$ on $J^rT^*M$ (the definition is quite similar to that of natural operators $Q \rightsquigarrow QJ^rT^*$). Actually, given $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ and $\Delta : Q \rightsquigarrow (\bigotimes^2 T^* \otimes T)J^rT^*$ we have $\mathcal{M}f_m$-natural operator $C + \Delta : Q \rightsquigarrow QJ^rT^*$ given by

$$(C + \Delta)_M(\nabla) := C_M(\nabla) + \Delta_M(\nabla),\ \nabla \in \underline{Q}(M),\ M \in obj(\mathcal{M}f_m).$$

So, as in [7], to describe all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$, it is sufficient to describe all $\mathcal{M}f_m$-natural operators $\Delta : Q \rightsquigarrow (\bigotimes^2 T^* \otimes T)J^rT^*$. Further, because of Proposition 2.1, we can put $\bigoplus_{k=0}^r \overset{k}{\bigodot} T^* \otimes T^*$ instead of $J^rT^*$, and our problem of describing all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ is reduced to that of finding all $\mathcal{M}f_m$-natural operators

$$\Delta : Q \rightsquigarrow (\bigotimes^2 T^* \otimes T) \bigoplus_{k=0}^{r} \bigodot^k T^* \otimes T^*$$

lifting classical linear connections $\nabla$ on $m$-manifolds into tensor fields $\Delta_M(\nabla)$ of type $(1, 2)$ on $\bigoplus_{k=0}^r \overset{k}{\bigodot} T^*M \otimes T^*M$.

As in [7], given a classical linear connection $\nabla$ on $M$ we have

$$T_v(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*M\otimes T^*M) = V_v(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*M\otimes T^*M)\oplus H_v^\nabla$$
$$\tilde{=}\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T_x^*M\otimes T_x^*M\oplus T_xM$$

for any $v\in\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T_x^*M\otimes T_x^*M$, $x\in M$ where $H_v^\nabla$ is the $\nabla$-horizontal subspace and the identification $\tilde{=}$ is the standard one. Then, by linear algebra,

$$(T_v(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*M\otimes T^*M))^*\otimes(T_v(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*M\otimes T^*M))^*$$
$$\otimes T_v(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*M\otimes T^*M)$$
$$= (T_x^*M\otimes T_x^*M\otimes T_xM)\oplus\bigoplus_{l=0}^{r}(T_x^*M\otimes T_x^*M\otimes\overset{l}{\bigodot} T_x^*M\otimes T_x^*M)$$
$$\oplus\bigoplus_{l=0}^{r}(T_x^*M\otimes\overset{l}{\bigodot} T_xM\otimes T_xM\otimes T_xM)$$
$$\oplus\bigoplus_{l,l_1=0}^{r}(T_x^*M\otimes\overset{l}{\bigodot} T_xM\otimes T_xM\otimes\overset{l_1}{\bigodot} T_x^*M\otimes T_x^*M)$$
$$\oplus\bigoplus_{l=0}^{r}(\overset{l}{\bigodot} T_xM\otimes T_xM\otimes T_x^*M\otimes T_xM)$$
$$\oplus\bigoplus_{l,l_1=0}^{r}(\overset{l}{\bigodot} T_xM\otimes T_xM\otimes T_x^*M\otimes\overset{l_1}{\bigodot} T_x^*M\otimes T_x^*M)$$
$$\oplus\bigoplus_{l,l_1=0}^{r}(\overset{l}{\bigodot} T_xM\otimes T_xM\otimes\overset{l_1}{\bigodot} T_xM\otimes T_xM\otimes T_xM)$$
$$\oplus\bigoplus_{l,l_1,l_2=0}^{r}(\overset{l}{\bigodot} T_xM\otimes T_xM\otimes\overset{l_1}{\bigodot} T_xM\otimes T_xM\otimes\overset{l_2}{\bigodot} T_x^*M\otimes T_x^*M).$$

Consequently, our problem of finding of all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^rT^*$ is reduced to that of finding systems $\Delta^C = ((\Delta^1),\dots,(\Delta^8_{l,l_1,l_2}))$ of systems $(\Delta^1),\dots,(\Delta^8_{l,l_1,l_2})$ of $\mathcal{M}f_m$-natural operators

$$\Delta^1 : Q\rightsquigarrow(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*\otimes T^*, T^*\otimes T^*\otimes T),$$
$$\Delta^2_l : Q\rightsquigarrow(\bigoplus_{k=0}^{r}\overset{k}{\bigodot} T^*\otimes T^*, T^*\otimes T^*\otimes\overset{l}{\bigodot} T^*\otimes T^*),$$

$$\Delta^3_l : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, T^* \otimes \bigodot^{l} T \otimes T \otimes T),$$

$$\Delta^4_{l,l_1} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, T^* \otimes \bigodot^{l} T \otimes T \otimes \bigodot^{l_1} T^* \otimes T^*),$$

$$\Delta^5_l : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, \bigodot^{l} T \otimes T \otimes T^* \otimes T),$$

$$\Delta^6_{l,l_1} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, \bigodot^{l} T \otimes T \otimes T^* \otimes \bigodot^{l_1} T^* \otimes T^*),$$

$$\Delta^7_{l,l_1} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, \bigodot^{l} T \otimes T \otimes \bigodot^{l_1} T \otimes T \otimes T),$$

$$\Delta^8_{l,l_1,l_2} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \bigodot^{k} T^* \otimes T^*, \bigodot^{l} T \otimes T \otimes \bigodot^{l_1} T \otimes T \otimes \bigodot^{l_2} T^* \otimes T^*)$$

transforming classical linear connections $\nabla$ on $m$-manifolds $M$ into fibred maps $\Delta^1_M(\nabla) : \bigoplus_{k=0}^r \overset{k}{\bigodot} T^*M \otimes T^*M \to T^*M \otimes T^*M \otimes TM, \dots, \Delta^8_{l,l_1,l_2 M}(\nabla) :$ $\bigoplus_{k=0}^r \overset{k}{\bigodot} T^*M \otimes T^*M \to \overset{l}{\bigodot} TM \otimes TM \otimes \overset{l_1}{\bigodot} TM \otimes TM \otimes \overset{l_2}{\bigodot} T^*M \otimes T^*M$ covering the identity map of $M$, where $l, l_1, l_2 = 0, \dots, r$.

## 4. A more reduction

To obtain a more reduction than the above one, we need a preparation.

As in [7], a tensor natural sub-bundle (of type $(p, q)$) is a natural vector bundle $F : \mathcal{M}f_m \to \mathcal{VB}$ such that (modulo a natural vector bundle isomorphism) $FM \subset \bigotimes^p TM \otimes \bigotimes^q T^*M$ and $F\varphi = \bigotimes^p T\varphi \otimes \bigotimes^q T^*\varphi_{|FM}$ for any $m$-manifold $M$ and any $\mathcal{M}f_m$-map $\varphi : M \to M^1$.

**Proposition 4.1.** *Let $F : \mathcal{M}f_m \to \mathcal{VB}$ be a tensor natural sub-bundle of type $(p, q)$. The $\mathcal{M}f_m$-natural operators $B : Q \rightsquigarrow (\bigoplus_{k=0}^r \overset{k}{\bigodot} T^* \otimes T^*, F)$ transforming classical linear connections $\nabla$ on $m$-manifolds $M$ into fibred maps $B_M(\nabla) :$ $\bigoplus_{k=0}^r \overset{k}{\bigodot} T^*M \otimes T^*M \to FM$ covering $id_M$ are in bijection with the systems $E = (E^{(k_1,\dots,k_j)})$ of $\mathcal{M}f_m$-natural operators*
*$E^{(k_1,\dots,k_j)} : Q \rightsquigarrow ((\overset{k_1}{\bigodot} T \otimes T) \odot \dots \odot (\overset{k_j}{\bigodot} T \otimes T)) \otimes F$ for systems $(k_1, \dots, k_j)$ of integers $k_1, \dots, k_j$ with $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le q - p - j$, $j = 0, 1, 2, \dots$ If $j = 0$, $(k_1, \dots, k_j) = \emptyset$, and $E^\emptyset : Q \rightsquigarrow F$. If $q - p - j < 0$, any $B$ is the zero operator. (For $\odot$, see below.)*

*More precisely, the natural operator $B^E : Q \rightsquigarrow (\bigoplus_{k=0}^r \overset{k}{\bigodot} T^* \otimes T^*, F)$ corresponding to a system $E = (E^{(k_1,\dots,k_j)})$ (as above) is defined by*

$$B^E_M(\nabla)_x(v) = \sum < E^{(k_1,\dots,k_j)}_M(\nabla)_x, v_{k_1} \otimes \dots \otimes v_{k_j} >,$$

$\nabla \in \underline{Q}(M)$, $M \in obj(\mathcal{M}f_m)$, $x \in M$, $v = (v_0, \dots, v_r) \in \bigoplus_{k=0}^{r} \overset{k}{\bigodot} T_x^*M \otimes T_x^*M$, *where the (finite) sum* $\sum$ *is over all systems* $(k_1, \dots, k_j)$ *of integers with* $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le q - p - j$, $j = 0, 1, 2, \dots$

*Conversely, the system* $E^B = (E^{B;(k_1,\dots,k_j)})$ *corresponding to a natural operator* $B \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, F)$ *is well-defined by*

$$\begin{aligned}&< E_M^{B;(k_1,\dots,k_j)}(\nabla)_x, v_{k_1} \otimes \dots \otimes v_{k_j} > \\ &= \frac{1}{\alpha!}\frac{\partial}{\partial t^{k_1}} \dots \frac{\partial}{\partial t^{k_j}} B_M(\nabla)_x(t^0 v_0, \dots, t^r v_r)_{|t^0,\dots,t^r=0}\end{aligned}$$

*where* $v = (v_0, \dots, v_r) = \bigoplus_{k=0}^{r} \overset{k}{\bigodot} T_x^*M \otimes T_x^*M$, $x \in M$, $\alpha = 1_{k_1} + \dots + 1_{k_j} \in \mathbb{N}^r$.

As in [7], in Proposition 4.1, we used the following notation. Given a sequence $V_0, \dots, V_r$ of different vector spaces and a system $(k_1, \dots, k_j)$ of integers with $0 \le k_1 \le \dots \le k_j \le r$, $V_{k_1} \odot \dots \odot V_{k_j}$ denotes the factor space $V_{k_1} \otimes \dots \otimes V_{k_j} / \sim$ where, for any $u, w \in V_{k_1} \otimes \dots \otimes V_{k_j}$, $u \sim w$ iff $< u, \varphi_{k_1} \otimes \dots \otimes \varphi_{k_j} >=< w, \varphi_{k_1} \otimes \dots \varphi_{k_j} >$ (the usual pairing (contraction)) for any $(\varphi_0, \dots, \varphi_r) \in \bigoplus_{k=0}^{r} V_k^*$.

*Proof.* The proof of Proposition 4.1 is almost the same as that of Proposition 3 in [7]. By the non-linear Petree theorem (see [3]) $B$ is of finite order. Further, by the invariance with respect to manifold charts, $B$ is determined by the values

$$(B_{\mathbb{R}^m}(\nabla))_0(v) \in F_0\mathbb{R}^m$$

for all classical linear connections $\nabla$ on $\mathbb{R}^m$ and all points $v = (v_0, \dots, v_r) \in \bigoplus_{k=0}^{r} \overset{k}{\bigodot} T_0^*\mathbb{R}^m \otimes T_0^*\mathbb{R}^m$. We can assume that the coordinates (symbols) of $\nabla$ are polynomials of a degree that is the finite order of $B$. Next, by the invariance of $B$ with respect to the homotheties, we have

$$B_{\mathbb{R}^m}((tid_{\mathbb{R}^m})_*\nabla)_0(\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^*(tid_{\mathbb{R}^m}) \otimes T^*(tid_{\mathbb{R}^m})(v)) = t^{p-q} B_{\mathbb{R}^m}(\nabla)_0(v)$$

for $t > 0$. So, the homogeneous function theorem and the Taylor theorem end the proof. □

## 5. Corollaries

Applying Proposition 4.1 to natural operators $\Delta^1, \dots, \Delta^8_{l,l_1,l_2}$ in item 3, we obtain.

**Corollary 5.1.** *For* $l = 0, \dots, r$ *any* $\mathcal{M}f_m$*-natural operator* $\Delta_l^3 : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^*M \otimes T^*, T^* \oplus \overset{l}{\bigodot} T \otimes T \otimes T)$ *is the zero one.*

**Corollary 5.2.** *For* $l = 0, \dots, r$ *any* $\mathcal{M}f_m$*-natural operator* $\Delta_l^5 : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, \overset{l}{\bigodot} T \otimes T \otimes T^* \otimes T)$ *is the zero one.*

**Corollary 5.3.** *For* $l, l_1 = 0, \dots, r$ *any* $\mathcal{M}f_m$*-natural operator* $\Delta_{l,l_1}^7 : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, \overset{l}{\bigodot} T \otimes T \otimes \overset{l_1}{\bigodot} T \otimes T \otimes T)$ *is the zero one.*

**Corollary 5.4.** *The $\mathcal{M}f_m$-natural operators* $\Delta^1 : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, T^* \otimes T^* \otimes T)$ *are in (the) bijection with the systems* $E^{\Delta^1} = (E^{\Delta^1;\emptyset}, E^{\Delta^1;(0)})$ *of $\mathcal{M}f_m$-natural operators* $E^{\Delta^1;\emptyset} : Q \rightsquigarrow T^* \otimes T^* \otimes T$ *and* $E^{\Delta^1;(0)} : Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T$.

**Corollary 5.5.** *Given natural numbers $l = 0, \dots, r$ the $\mathcal{M}f_m$-natural operators* $\Delta^2_l : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, T^* \otimes T^* \otimes \overset{l}{\bigodot} T^* \otimes T^*)$ *are in (the) bijection with the systems* $E^{\Delta^2_l} = (E^{\Delta^2_l;(k_1,\dots,k_j)})$ *of $\mathcal{M}f_m$-natural operators* $E^{\Delta^2_l;(k_1,\dots,k_j)} : Q \rightsquigarrow ((\overset{k_1}{\bigodot} T \otimes T) \odot \dots \odot (\overset{k_j}{\bigodot} T \otimes T)) \otimes T^* \otimes T^* \otimes \overset{l}{\bigodot} T^* \otimes T^*$ *for systems* $(k_1, \dots, k_j)$ *of integers with* $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le l + 3 - j$, $j = 0, 1, 2, \dots$.

**Corollary 5.6.** *Given natural numbers $l, l_1 = 0, \dots, r$ the $\mathcal{M}f_m$-natural operators* $\Delta^4_{l,l_1} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, T^* \otimes \overset{l}{\bigodot} T \otimes T \otimes \overset{l_1}{\bigodot} T^* \otimes T^*)$ *are in (the) bijection with the systems* $E^{\Delta^4_{l,l_1}} = (E^{\Delta^4_{l,l_1};(k_1,\dots,k_j)})$ *of $\mathcal{M}f_m$-natural operators* $E^{\Delta^4_{l,l_1};(k_1,\dots,k_j)} : Q \rightsquigarrow ((\overset{k_1}{\bigodot} T \otimes T) \odot \dots \odot (\overset{k_j}{\bigodot} T \otimes T)) \otimes T^* \otimes \overset{l}{\bigodot} T \otimes T \otimes \overset{l_1}{\bigodot} T^* \otimes T^*$ *for systems* $(k_1, \dots, k_j)$ *of integers with* $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le l_1 + 1 - l - j$, $j = 0, 1, \dots$

**Corollary 5.7.** *Given natural numbers $l, l_1 = 0, \dots, r$ the $\mathcal{M}f_m$-natural operators* $\Delta^6_{l,l_1} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, \overset{l}{\bigodot} T \otimes T \otimes T^* \otimes \overset{l_1}{\bigodot} T^* \otimes T^*)$ *are in (the) bijection with the systems* $E^{\Delta^6_{l,l_1}} = (E^{\Delta^6_{l,l_1};(k_1,\dots,k_j)})$ *of $\mathcal{M}f_m$-natural operators* $E^{\Delta^6_{l,l_1};(k_1,\dots,k_j)} : Q \rightsquigarrow ((\overset{k_1}{\bigodot} T \otimes T) \odot \dots \odot (\overset{k_j}{\bigodot} T \otimes T)) \otimes \overset{l}{\bigodot} T \otimes T \otimes T^* \otimes \overset{l_1}{\bigodot} T^* \otimes T^*$ *for systems* $(k_1, \dots, k_j)$ *of integers with* $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le l_1 + 1 - l - j$, $j = 0, 1, \dots$

**Corollary 5.8.** *Given natural numbers $l, l_1, l_2 = 0, \dots, r$ the $\mathcal{M}f_m$-natural operators* $\Delta^8_{l,l_1,l_2} : Q \rightsquigarrow (\bigoplus_{k=0}^{r} \overset{k}{\bigodot} T^* \otimes T^*, \overset{l}{\bigodot} T \otimes T \otimes \overset{l_1}{\bigodot} T \otimes T \otimes \overset{l_2}{\bigodot} T^* \otimes T^*)$ *are in (the) bijection with the systems* $E^{\Delta^8_{l,l_1,l_2}} = (E^{\Delta^8_{l,l_1,l_2};(k_1,\dots,k_j)})$ *of $\mathcal{M}f_m$-natural operators*
$E^{\Delta^8_{l,l_1,l_2};(k_1,\dots,k_j)} : Q \rightsquigarrow ((\overset{k_1}{\bigodot} T \otimes T) \odot \dots \odot (\overset{k_j}{\bigodot} T \otimes T)) \otimes \overset{l}{\bigodot} T \otimes T \otimes \overset{l_1}{\bigodot} T \otimes T \otimes \overset{l_2}{\bigodot} T^* \otimes T^*$ *for systems* $(k_1, \dots, k_j)$ *of integers with* $0 \le k_1 \le \dots \le k_j \le r$, $k_1 + \dots + k_j \le l_2 - l_1 - l - 1 - j$, $j = 0, 1, \dots$

## 6. The main result

Summing up, we have proved the following result.

**Theorem 6.1.** *The $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QT^{r*}$ are in (the) bijection with the systems* $\Delta^C = ((\Delta^1), (\Delta^2_l), (\Delta^4_{l,l_1}), (\Delta^6_{l,l_1}), (\Delta^8_{l,l_1,l_2}))$ *of systems* $(\Delta^1), \dots, (\Delta^8_{i,l_1,l_2})$ *of $\mathcal{M}f_m$-natural operators corresponding to systems of $\mathcal{M}f_m$-natural operators(of the form $Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$) as in Corollaries 5.4—5.8.*

*If $r=0$, then $l, l_1, l_2 = 0$ only. Consequently $E^{\Delta^1} = (E^{\Delta^1;\emptyset}, E^{\Delta^1;(0)})$, $E^{\Delta^2_0} = (E^{\Delta^2_0;\emptyset}, E^{\Delta^2_0;(0)}, E^{\Delta^2_0;(0,0)}, E^{\Delta^2_0;(0,0,0)})$, $E^{\Delta^4_{0,0}} = (E^{\Delta^4_{0,0};\emptyset}, E^{\Delta^4_{0,0};(0)})$, $E^{\Delta^8_{0,0,0}} = (0)$, $E^{\Delta^6_{0,0}} = (E^{\Delta^6_{0,0};\emptyset}, E^{\Delta^6_{0,0};(0)})$. So, Theorem 6.1 for $r=0$ can be read as follows.*

*The $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QT^*$ lifting classical linear connections to the cotangent bundle are in the bijection with the 10-tuples $(E^{\Delta^1;\emptyset}, \dots, E^{\Delta^6_{0,0};(0)})$ containing of $\mathcal{M}f_m$-natural operators*

$$
\begin{aligned}
&E^{\Delta^1;\emptyset} : Q \rightsquigarrow T^* \otimes T^* \otimes T,\\
&E^{\Delta^1;(0)} : Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T,\\
&E^{\Delta^2_0;\emptyset} : Q \rightsquigarrow T^* \otimes T^* \otimes T^*,\\
&E^{\Delta^2_0;(0)} : Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T^*,\\
&E^{\Delta^2_0;(0,0)} : Q \rightsquigarrow (T \odot T) \otimes T^* \otimes T^* \otimes T^*,\\
&E^{\Delta^2_0;(0,0,0)} : Q \rightsquigarrow (T \odot T \odot T) \otimes T^* \otimes T^* \otimes T^*,\\
&E^{\Delta^4_{0,0};\emptyset} : Q \rightsquigarrow T^* \otimes T \otimes T^*,\\
&E^{\Delta^4_{0,0};(0)} : Q \rightsquigarrow T \otimes T^* \otimes T \otimes T^*,\\
&E^{\Delta^6_{0,0};\emptyset} : Q \rightsquigarrow T^* \otimes T \otimes T^*,\\
&E^{\Delta^6_{0,0};(0)} : Q \rightsquigarrow T \otimes T^* \otimes T \otimes T^*.
\end{aligned}
$$

Using the description of $\mathcal{M}f_m$-natural operators $Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ of [3] (see item 1), we can explicitly describe the above 10-tuples. For example, any $\mathcal{M}f_m$-natural operator $E^{\Delta^1;\emptyset} : Q \rightsquigarrow T^* \otimes T^* \otimes T$ is the linear combination (with real coefficients) of three $\mathcal{M}f_m$-natural operators (the connection torsion operator $T_\nabla$, the operator $\delta_M \otimes C^1_1 T_\nabla$ (the tensor multiplication of the identity tensor field $\delta_M : TM \to TM$ and the contraction of the connection torsion) and the operator $C^1_1 T_\nabla \otimes \delta^*_M$). In the case of torsion free connections, any such operator is the zero one. Similarly, any $\mathcal{M}f_m$-natural operator $E^{\Delta^1;(0)} : Q \rightsquigarrow T \otimes T^* \otimes T^* \otimes T$ is the linear combination of two connection independent natural tensors (from the identity tensor $TM \otimes TM \to TM \otimes TM$ by means of the permutations of indices), e.t.c. In this way, we may reobtain (in another form) the result of M. Kureš [5] (in the case of natural operators $Q_\tau \rightsquigarrow QT^*$) and get a more general similar result in the case of not necessarily torsion free connections.

The explanation of our result in the case $r=1$ (i.e. in the case of natural operators $Q \rightsquigarrow QJ^1T^*$ lifting classical linear connections to the first jet prolongation of the cotangent bundle) is more sophisticated but feasible. Indeed, if $r=1$, then $l, l_1, l_2 = 0, 1$. Consequently $E^{\Delta^1} = (E^{\Delta^1;\emptyset}, E^{\Delta^1;(0)})$. Next, $E^{\Delta^2_0} = (E^{\Delta^2_0 \emptyset}, E^{\Delta^2_0;(0)}, E^{\Delta^2_0;(1)}, E^{\Delta^2_0;(0,0)}, E^{\Delta^2_0;(0,1)}, E^{\Delta^2_0;(1,1)}, E^{\Delta^2_0;(0,0,0)}, E^{\Delta^2_0;(0,0,1)}, E^{\Delta^2_0;(0,1,1)}, E^{\Delta^2_0;(1,1,1)})$ and $E^{\Delta^2_1} = (E^{\Delta^2_1;\emptyset}, E^{\Delta^2_1;(0)}, E^{\Delta^2_1;(1)}, E^{\Delta^2_1;(0,0)}, E^{\Delta^2_1;(0,1)}, E^{\Delta^2_1;(1,1)}, E^{\Delta^2_1;(0,0,0)}, E^{\Delta^2_1;(0,0,1)}, E^{\Delta^2_1;(0,1,1)}, E^{\Delta^2_1;(1,1,1)}, E^{\Delta^2_1;(0,0,0,0)}, E^{\Delta^2_1;(0,0,0,1)}, E^{\Delta^2_1;(0,0,1,1)}, E^{\Delta^2_1;(0,1,1,1)}, E^{\Delta^2_1;(1,1,1,1)})$. Next $E^{\Delta^4_{1,0}} = (E^{\Delta^4_{1,0};\emptyset}; E^{\Delta^4_{1,0};(0)}, E^{\Delta^4_{1,0};(1)}, E^{\Delta^4_{1,0};(0,0)}, E^{\Delta^4_{1,0};(0,1)}, E^{\Delta^4_{1,0};(1,1)})$, $E^{\Delta^4_{0,0}} = (E^{\Delta^4_{0,0};\emptyset}, E^{\Delta^4_{0,0};(0)}, E^{\Delta^4_{0,0};(1)})$, $E^{\Delta^4_{1,1}} = (E^{\Delta^4_{1,1};\emptyset}, E^{\Delta^4_{1,1};(0)}, E^{\Delta^4_{1,1};(1)})$ and $E^{\Delta^4_{0,1}} = (E^{\Delta^4_{0,1};\emptyset})$. Similarly $E^{\Delta^6_{0,1}} = (E^{\Delta^6_{0,1};\emptyset},$

$E^{\Delta^6_{0,1};(0)}, E^{\Delta^6_{0,1};(1)}, E^{\Delta^6_{0,1};(0,0)}, E^{\Delta^6_{0,1};(0,1)}, E^{\Delta^6_{0,1};(1,1)})$, $E^{\Delta^6_{0,0}} = (E^{\Delta^6_{0,0};\emptyset}, E^{\Delta^6_{0,0};(0)}, E^{\Delta^6_{0,0};(1)})$, $E^{\Delta^6_{1,1}} = (E^{\Delta^6_{1,1};\emptyset}, E^{\Delta^6_{1,1};(0)}, E^{\Delta^6_{1,1};(1)})$ and $E^{\Delta^6_{1,0}} = (E^{\Delta^6_{1,0};\emptyset})$. Next $E^{\Delta^8_{0,0,1}} = (E^{\Delta^8_{0,0,1};\emptyset})$, and $E^{\Delta^8_{l,l_1,l_2}} = (0)$ if $l = 1$ or $l_1 = 1$ or $l_2 = 0$.

So, Theorem 6.1 for $r = 1$ can be read as follows.

*The $\mathcal{M}f_m$-natural operators $Q \rightsquigarrow QJ^1T^*$ are in (the) bijection with the* 54-*tuples of $\mathcal{M}f_m$-natural operators*

$$E^{\Delta^1;\emptyset} : Q \rightsquigarrow T^* \otimes T^* \otimes T,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$E^{\Delta^8_{0,0,1};\emptyset} : Q \rightsquigarrow T \otimes T \otimes T^* \otimes T^*.$$

Clearly, the dots denote the respective types $\mathcal{M}f_m$-natural operators (we do not present the dots explicitly because this would require about two pages). Using the description of $\mathcal{M}f_m$-natural operators $Q \rightsquigarrow \bigotimes^p T \otimes \bigotimes^q T^*$ of [3] (see item 1) we may describe explicitly all above 54-tuples and describe explicitly all $\mathcal{M}f_m$-natural operators $C : Q \rightsquigarrow QJ^1T^*$.

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Włodzimierz M. Mikulski, Institute of Mathematics, Jagiellonian University, ul. S. Łojasiewicza 6 , Kraków, Poland
*e-mail*: `wlodzimierz.mikulski@im.uj.edu.pl`