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ON THE POLYHEDRAL CONES OF CONVEX AND CONCAVE VECTORS

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Abstract. Convex or concave sequences of n positive terms, viewed as vectors in n-space, constitute convex cones with $2n-2$ and n extreme rays, respectively. Explicit description is given of vectors spanning these extreme rays, as well as of non-singular linear transformations between the positive orthant and the simplicial cones formed by the positive concave vectors. The simplicial cones of monotone convex and concave vectors can be described similarly.

In this note a sequence (vector) $\mathbf{a} = (a_1, \ldots, a_n)$, $n \geq 1$, of real numbers is called *positive* if, for all $1 \leq i \leq n$, $0 \leq a_i$, *increasing* if for all $1 \leq i < n$, $0 \leq a_{i+1} - a_i$ and *convex* if, for all $1 < i < n$, $0 \leq a_{i+1} - 2a_i + a_{i-1}$. The sequence $\mathbf{a}$ is *negative*, *decreasing*, or *concave*, when $-\mathbf{a}$ is positive, increasing, or convex, respectively. The vector $\mathbf{a}$ is *unimodal* if for some (not necessarily unique) index i, $(a_1, \ldots, a_i)$ is increasing and $(a_i, \ldots, a_n)$ is decreasing. All increasing, decreasing and concave vectors are unimodal. The question of unimodality of the members of certain sequence classes extensively studied in combinatorics can be difficult (e.g. Whitney numbers [2,7] and face vectors of certain classes of polytopes [5,8]). The proof of unimodality of $\mathbf{a} = (a_1, \ldots, a_n)$, when all $a_i > 0$, is sometimes based on proving the stronger property of concavity of $(\log a_1, \ldots, \log a_n)$, called log-concavity [1,2,5,9]. In turn, to prove that log-concavity is preserved in certain constructions of sequences, ordinary concavity of some coefficient sequences may be used [3]. These connections motivate our interest in the geometric description of the sets of vectors possessing one or another of the properties mentioned. Each of the sets of positive, negative, increasing, decreasing, convex, and concave vectors, and various intersections of these sets form closed *cones* (sets containing the null vector and closed under linear combinations with non-negative coefficients, called *conic combinations*). The cone of positive vectors is the positive orthant in $\mathbb{R}^n$. The cone of positive increasing vectors was described implicitly by Lovász ([4], p. 248, last equation) and by Marichal and Mathonet [6] in terms of its intersection with the unit hypercube (one of the $n!$ simplices of the standard triangulation of the hypercube). In this note we describe the cones of positive concave and positive convex vectors by determining their extreme rays. When this cone is simplicial, we describe a standardized matrix realizing the transformation of the orthant to the cone in question.

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Recall that the set of vectors in $\mathbb{R}^n$ is partially ordered by the *componentwise order* in which $(a_1,\dots,a_n) \le (b_1,\dots,b_n)$ if and only if $a_i \le b_i$ for all $1 \le i \le n$.

Proposition 1. (Lovász) *Every non-null positive vector* $\mathbf{c}$ *can be written uniquely as*

$$\mathbf{c} = \lambda_1 \mathbf{a}^{(1)} + \cdots + \lambda_k \mathbf{a}^{(k)},$$

where $\lambda_1 \dots, \lambda_k > 0$ *and* $\mathbf{a}^{(1)} \ge \mathbf{a}^{(2)} \ge \cdots \ge \mathbf{a}^{(k)}$ *are distinct non-null zero-one vectors.*

Essentially due to the above proposition and as apparent from [6], every positive increasing vector $\mathbf{c} \in \mathbb{R}^n$ can be written uniquely as

$$\mathbf{c} = \lambda_1 \mathbf{a}^{(1)} + \cdots + \lambda_n \mathbf{a}^{(n)},$$

where $\lambda_1 \dots, \lambda_n \ge 0$ and $\mathbf{a}^{(1)} \ge \mathbf{a}^{(2)} \ge \cdots \ge \mathbf{a}^{(n)}$ are distinct increasing zero-one vectors. The cone of positive increasing vectors is the image of the positive orthant under the linear transformation $\mathbf{v} \mapsto \mathbf{v}Z$ represented by the matrix Z whose i^{th} row is the vector $\mathbf{a}^{(i)}$ $(1 \le i \le n)$. The matrix Z^{-1}, whose non-zero entries are $Z^{-1}(i,i) = 1$ $(1 \le i \le n)$ and $Z^{-1}(i,i+1) = -1$ $(1 \le i < n)$, transforms the cone of positive increasing vectors to the positive orthant. For $n = 5$, for example, we have

$$Z = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Within $\mathbb{R}^n$, let C be the set of positive concave vectors having a maximal component value of at least 1, i.e.

$$C = \big\{(c_1,\dots,c_n) \in \mathbb{R}^n : (\forall i \;\; c_i \ge 0) \text{ and } (\exists i \;\; c_i \ge 1)\big\}.$$

The partially ordered set C (for the componentwise order) has exactly n minimal members, called *minimal standard concave vectors.* We shall denote these $\mathbf{c}^{(1)},\dots,\mathbf{c}^{(n)}$, where $\mathbf{c}^{(i)}$ is the vector whose j^{th} component $c_j^{(i)}$ $(1 \le j \le n)$ is given by

$$\begin{aligned} \min\left(\frac{j-1}{i-1}, \frac{n-j}{n-i}\right) &\quad \text{if } 1 < i < n, \\ \frac{j-1}{n-1} &\quad \text{if } i = n, \\ \frac{n-j}{n-1} &\quad \text{if } i = 1. \end{aligned}$$

For $n = 6$ and $i = 3$,for instance,

$$\mathbf{c}^{(3)} = \Big(0,\ 1/2,\ 1,\ 2/3,\ 1/3,\ 0\Big)$$

For any vector $\mathbf{c} = (c_1,\dots,c_n)$, an index i, $1 < i < n$, is called *singular* if $2c_i > c_{i-1} + c_{i+1}$.

Proposition 2. *The n minimal standard concave vectors $\mathbf{c}^{(1)}, \dots, \mathbf{c}^{(n)}$ form a basis of $\mathbb{R}^n$. Every positive concave vector $\mathbf{c}$ can be written uniquely as*

$$\mathbf{c} = \lambda_1 \mathbf{c}^{(1)} + \cdots + \lambda_n \mathbf{c}^{(n)} \tag{1}$$

with $\lambda_1, \dots, \lambda_n \geq 0$.

Proof. We prove only the second statement, which implies the first. For a given positive concave vector $\mathbf{c} = (c_1, \dots, c_n)$, $\mathbf{c} - c_1\mathbf{c}^{(1)} - c_n\mathbf{c}^{(n)}$ is still a positive concave vector and has 0 as its first and last components. Thus, it is sufficient to prove that any positive concave vector $\mathbf{c} = (c_1, \dots, c_n)$ with $c_1 = c_n = 0$ is a unique conic combination of $\mathbf{c}^{(2)}, \dots, \mathbf{c}^{(n-1)}$. We will proceed by induction on the number of singular indices. This number is 0 if and only if $\mathbf{c}$ is the null vector, in which case the assertion is obvious. Otherwise, let i be the first singular index in $\mathbf{c}$. There is a unique positive real number λ_i such that i is not a singular index in $\mathbf{c} - \lambda_i\mathbf{c}^{(i)}$. Every index $j \neq i$ is singular in $\mathbf{c}$ if and only if it is singular in $\mathbf{c} - \lambda_i\mathbf{c}^{(i)}$. Applying the induction hypothesis to $\mathbf{c} - \lambda_i\mathbf{c}^{(i)}$ completes the proof. $\square$

It follows from the above that the cone of positive concave vectors is the image of the positive orthant under the (non-singular) linear transformation represented by the matrix M whose rows are the minimal standard concave vectors $\mathbf{c}^{(1)}, \dots, \mathbf{c}^{(n)}$, which span the extreme rays of the cone. The matrix M is centrally symmetric, i.e., $\mathbf{c}^{(i)}$ is the reverse sequence of $\mathbf{c}^{(n-i+1)}$ (or, equivalently, $M(i,j) = M(n-i+1, n-j+1)$). In fact, M can be defined as the only centrally symmetric matrix with ones on the main diagonal, for which the entries $M(i,j)$ under the main diagonal ($i > j$) are given by $M(i,j) = \frac{j-1}{i-1}$. The linear transformation mapping the positive orthant to the positive concave vectors is then given, with the notation $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ as in (1), by $\boldsymbol{\lambda} \mapsto \boldsymbol{\lambda}M = \mathbf{c}$.
The inverse of M (also centrally symmetric) is the $n \times n$ matrix whose only non-zero entries are given by

(i) the main diagonal $1, \dots, \frac{2(i-1)(n-i)}{n-1}, \dots, 1$,

(ii) for $1 < j < n$, the entries $M^{-1}(j-1, j) = M^{-1}(j+1, j) = -\frac{1}{2}M^{-1}(j,j)$.

For $n = 5$, for example, we have

$$M = \frac{1}{12} \cdot \begin{bmatrix} 12 & 9 & 6 & 3 & 0 \\ 0 & 12 & 8 & 4 & 0 \\ 0 & 6 & 12 & 6 & 0 \\ 0 & 4 & 8 & 12 & 0 \\ 0 & 3 & 6 & 9 & 12 \end{bmatrix}, \quad M^{-1} = \frac{1}{12} \cdot \begin{bmatrix} 12 & -9 & 0 & 0 & 0 \\ 0 & 18 & -12 & 0 & 0 \\ 0 & -9 & 24 & -9 & 0 \\ 0 & 0 & -12 & 18 & 0 \\ 0 & 0 & 0 & -9 & 12 \end{bmatrix}.$$

It is easy to verify that the product of M and M^{-1} is indeed always the identity matrix. For this, we compute the product of the i^{th} row of M and the j^{th} column of M^{-1} for each i and j. For $1 \leq i < j \leq n$ (or $1 \leq j < i \leq n$), we have $M(i, j-1) + M(i, j+1) = 2M(i,j)$, therefore, the product in question equals 0. For $1 < i = j < n$, the three non-zero terms of this product are $-\frac{j-2}{j-1} \cdot \frac{(j-1)(n-j)}{n-1}$, $\frac{2(j-1)(n-j)}{n-1}$, and $-\frac{n-j-1}{n-j} \cdot \frac{(j-1)(n-j)}{n-1}$ and the sum of these is 1. The remaining cases $i = j = 1$ and $i = j = n$ are obvious.

The matrix M^{-1} transforms the cone of positive concave vectors to the positive orthant. All non-zero entries of M^{-1} are on three diagonals. For comparison, the non-zero entries of the matrix Z^{-1} are on two diagonals. Both Z^{-1} and M^{-1} have column sums equal to 0, except for the first column of Z^{-1} and the first and last columns of M^{-1}.

For $1 \le i < n$ let C_i (D_i) be the set of those positive increasing (decreasing) convex vectors with maximal component value 1 that have exactly i components equal to 0. Then C_i (D_i) has a unique maximal vector $\mathbf{a}_i$ ($\mathbf{b}_i$) with respect to the componentwise ordering, called the i^{th} *standard increasing (resp. decreasing) convex vector*. Note that $\mathbf{1} > \mathbf{a}^{(1)} > \cdots > \mathbf{a}^{(n-1)}$ and $\mathbf{1} > \mathbf{b}^{(1)} > \cdots > \mathbf{b}^{(n-1)}$ where $\mathbf{1} = (1, \ldots, 1)$. For example, for $n = 6$ and $i = 2$,

$$\mathbf{a}^{(2)} = \Big(0,\ 0,\ 0.25,\ 0.5,\ 0.75,\ 1\Big),$$

$$\mathbf{b}^{(2)} = \Big(1,\ 0.75,\ 0.5,\ 0.25,\ 0,\ 0\Big).$$

Proposition 3. *The $n-1$ standard increasing convex vectors are linearly independent and, together with $\mathbf{1}$, form a basis of $\mathbb{R}^n$. Every positive increasing convex vector $\mathbf{c}$ can be written uniquely as*

$$\mathbf{c} = \lambda_1 \mathbf{a}^{(1)} + \cdots + \lambda_{n-1}\mathbf{a}^{(n-1)} + \lambda_n \mathbf{1}$$

with $\lambda_1, \ldots, \lambda_n \ge 0$.

Proof. The first statement is obvious. It is sufficient to prove the second statement for $\mathbf{c} = (c_1, \ldots, c_n)$ with $c_1 = 0$ (because $c_1\mathbf{1}$ can be subtracted). We proceed by induction on the number of singular indices. This number is 0 only in the obvious case of $\mathbf{c}$ being the null vector. Otherwise, let i be the first singular index in $\mathbf{c}$. There is a unique positive real number λ_i such that i is not a singular index in $\mathbf{c} - \lambda_i \mathbf{a}^{(i)}$ and, to this vector, we can apply the induction hypothesis. □

Corollary 4. *The $n-1$ standard decreasing convex vectors are linearly independent and, with $\mathbf{1}$, form a basis of $\mathbb{R}^n$. Every positive decreasing convex vector $\mathbf{c}$ can be written uniquely as*

$$\mathbf{c} = \lambda_1 \mathbf{b}^{(1)} + \cdots + \lambda_{n-1}\mathbf{b}^{(n-1)} + \lambda_n \mathbf{1}$$

with $\lambda_1, \ldots, \lambda_n \ge 0$.

Proposition 5. *In $\mathbb{R}^n$, the cone of positive convex vectors has $2n-2$ extreme rays spanned by the standard increasing and standard decreasing convex vectors.*

Proof. Every positive convex vector can be written (not uniquely) as the sum of an increasing and a decreasing positive convex vector. The vector $\mathbf{1}$ equals $\mathbf{a}^{(1)} + \mathbf{b}^{(n)}$. Further, since none of the $\mathbf{a}^{(i)}$ can be a conic combination of the other vectors $\mathbf{a}^{(j)}$ and the various $\mathbf{b}^{(k)}$ and, similarly, none of the $\mathbf{b}^{(i)}$ is a combination of the other vectors, all the rays generated by the standard convex vectors are extremal. □

Obviously, the representation of a positive convex vector as a conic combination of standard convex vectors is not unique. However, every positive convex vector

$\mathbf{c}$ in $\mathbb{R}^n$ can be written with coefficients $\lambda_1 \dots, \lambda_{n-1}$ and $\theta_1 \dots, \theta_{n-1}$, uniquely determined by $\mathbf{c}$, in the form

$$\mathbf{c} = (\min \mathbf{c})\mathbf{1} + \sum \lambda_i \mathbf{a}^{(i)} + \sum \theta_i \mathbf{b}^{(i)}$$

where the vectors $\mathbf{a}^{(i)}$ and $\mathbf{b}^{(i)}$ are the standard convex vectors. Not all possible combinations of coefficients λ_i, θ_i can appear in such a representation.

The cone of positive increasing convex vectors is the image of the positive orthant under the (non-singular) linear transformation represented by the matrix N whose rows are the vector $\mathbf{1}$ and the standard increasing convex vectors $\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(n-1)}$. For all $1 \leq i < n$, the first i components of the vector $\mathbf{a}^{(i)}$ equal 0, and $(\mathbf{a}_i^{(i)}, \dots, \mathbf{a}_n^{(i)})$ is an increasing arithmetic progression from 0 to 1. Thus, the matrix N is an upper triangular matrix, whose first row is the vector $\mathbf{1}$ and, for all $1 < i \leq j \leq n$, $N(i,j) = \frac{j-i+1}{n-i+1}$.

The inverse matrix N^{-1} is the $n \times n$ upper triangular matrix for which $N^{-1}(i,i) = (N(i,i))^{-1}$ $(1 \leq i \leq n)$, $N^{-1}(i,j) \neq 0$ for $0 \leq j-i \leq 2$ only, and all column sums and row sums are equal to 0, except for the first column and the last row. For example, for $n = 5$, we have

$$N = \frac{1}{12} \cdot \begin{bmatrix} 12 & 12 & 12 & 12 & 12 \\ 0 & 3 & 6 & 9 & 12 \\ 0 & 0 & 4 & 8 & 12 \\ 0 & 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & 0 & 12 \end{bmatrix}, \quad N^{-1} = \begin{bmatrix} 1 & -4 & 3 & 0 & 0 \\ 0 & 4 & -6 & 2 & 0 \\ 0 & 0 & 3 & -4 & 1 \\ 0 & 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly to the case of matrices M and M^{-1}, one can show that the product of N and N^{-1} is indeed always the identity matrix.

The cone of positive decreasing convex vectors, its n extreme rays, and the transformation matrix between that cone and the positive orthant have entirely analogous descriptions. The same can be done for the cones of positive increasing concave and positive decreasing concave vectors. These cones are all simplicial cones, images of the positive orthant under a linear transformation, whose inverse is represented by a special form matrix, as the matrices M^{-1} and N^{-1} above. This matrix is almost diagonal, in the sense that all non-zero entries are on two or three diagonals. Moreover, the row as well as the column sums of these inverse matrices are constant with the exception of a single special row and a single column.

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