Math. Appl. **2** (2013), 35–42
DOI: 10.13164/ma.2013.04

SOME WOLSTENHOLME TYPE CONGRUENCES

ROMEO MEŠTROVIĆ

Abstract. In this paper we give an extension and another proof of the following Wolstenholme's type curious congruence established in 2008 by J. Zhao. Let s and l be two positive integers and let p be a prime such that $p \geq ls+3$. Then

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} & \pmod{p^3} \quad \text{if } 2 \nmid ls \\ (-1)^{l-1} \dfrac{sp}{ls+1} B_{p-ls-1} & \pmod{p^2} \quad \text{if } 2 \mid ls. \end{cases}$$

As an application, for given prime $p \geq 5$, we obtain explicit formulae for the sum $\sum_{1\leq k_1<\cdots<k_l\leq p-1} 1/(k_1\cdots k_l) \pmod{p^3}$ if $k \in \{1, 3, \dots, p-2\}$, and for the sum $\sum_{1\leq k_1<\cdots<k_l\leq p-1} 1/(k_1\cdots k_l) \pmod{p^2}$ if $k \in \{2, 4, \dots, p-3\}$.

1. Introduction and basic results

Our investigations are motivated by some recent results to multiple harmonic sums obtained by J. Zhao [12], Zhou and Cai [13]. These results are in fact, variations and generalizations of Wolstenholme's theorem. For more information on extensions and generalizations of Wolstenholme's theorem, see [5], [9], [10] and [11].

Throughout this paper we use the following definitions and notations.

For $n \in \mathbb{N}$, $l \in \mathbb{N}$ and $\mathbf{s} := (s_1, \dots, s_l) \in \mathbb{N}^l$, define the finite *harmonic sum*

$$H(\mathbf{s}; n) := H(s_1, \dots, s_l; n) = \sum_{1\leq k_1<\cdots<k_l\leq n} \frac{1}{k_1^{s_1}\cdots k_l^{s_l}}.$$

By convention we set $H(\mathbf{s}; r) = 0$ for $r = 0, \dots, l-1$. Further, we define the sum

$$S(\mathbf{s}; n) := S(s_1, \dots, s_l; n) = \sum_{1\leq k_1\leq\cdots\leq k_l\leq n} \frac{1}{k_1^{s_1}\cdots k_l^{s_l}}.$$

If $s_1 = \ldots = s_l = s$ then $H(\mathbf{s}; n)$ is a *homogeneous harmonic sum.* In this case, we shall denote such a sum by $H(\{s\}^l; n)$, and hence

$$H(\{s\}^l; n) = \sum_{1\leq k_1<\cdots<k_l\leq n} \frac{1}{(k_1\cdots k_l)^s}.$$

MSC (2010): primary 11B65; secondary 11A07, 05A10.
Keywords: multiple harmonic sum, Wolstenholme's theorem, Bernoulli numbers, Stirling numbers.

In particular, we write $H(s;n)$ instead of $H(\{s\}^1;n)$, that is,

$$H(s;n)=\sum_{1\le k\le n}\frac{1}{k^s}.$$

Analogously, we define $S(\{s\}^l;n)$ and $S(s;n)$ related to the sums $S(\mathbf{s};n)$.

Recall that *Stirling numbers $St(n,j)$ of the first kind* are defined by the expansion

$$x(x+1)(x+2)\cdots(x+n-1)=\sum_{j=1}^{n}St(n,j)x^j.$$

It is easy to see that for all $j=1,\dots,n-1$,

$$St(n,j)=\sum_{1\le k_1<\cdots<k_{n-j}\le n-1}k_1\cdots k_{n-j},$$

and that

$$St(n,j+1)=(n-1)!\cdot H(\{1\}^j;n-1).$$

For example, $St(n,n)=1$, $St(n,n-1)=n(n-1)/2$, and $St(n,1)=(n-1)!$.

Further, for any nonnegative integers j and $n\ge 1$, jth *power-sum symmetric function* is defined as

$$P(n,j)=\sum_{k=1}^{n}k^j.$$

By convention we set $P(0,j)=0$ for all $j\ge 0$.

By *Wolstenholme's theorem* (see, e.g., [6, p. 89]), if p is a prime greater than 3, then the numerator of the fraction

$$H(1,p-1)=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}$$

written in reduced form, is divisible by p^2.

Denote by $\mathfrak{p}$ the parity of m which is 1 if m is odd and 2 if m is even. Bayat proved

Theorem A ([1, Theorem 3]; also see Remark 2.3 in [12])**.** *For any positive integer s and a prime $p\ge s+3$ we have*

$$H(s;p-1)\equiv 0 \pmod{p^{\mathfrak{p}(s+1)}}.$$

Zhao in [12, p. 74] reported that one can find on the Internet the following generalization of Wolstenholme's theorem by Bruck [2], although no proof is given here.

Theorem B ([12, Theorem 1.2])**.** *For any prime number $p\ge 5$ and positive integers $l=1,\dots,p-3$, we have*

$$St(p,l+1)\equiv 0 \pmod{p^{\mathfrak{p}(l+1)}},$$

and

$$H(\{1\}^l;p-1)\equiv 0 \pmod{p^{\mathfrak{p}(l+1)}}.$$

The *Bernoulli numbers* B_k are defined by the generating function

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

It is easy to find the values $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, and $B_n = 0$ for odd $n \geq 3$. Furthermore, $(-1)^{n-1}B_{2n} > 0$ for all $n \geq 1$. These and many other properties can be found, for instance, in [7] and [3].

Recently, Zhao in [12] proved the following generalization of Theorem B to homogeneous multiple harmonic sums.

Theorem C ([12, Theorem 2.14]; cf. [12, Theorem 1.6])**.** *Let s and l be two positive integers. Let p be a prime such that $p \geq ls + 3$. Then*

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} & \pmod{p^3} \quad \text{if } 2 \nmid ls \\ (-1)^{l-1}\dfrac{sp}{ls+1} B_{p-ls-1} & \pmod{p^2} \quad \text{if } 2 \mid ls. \end{cases}$$

As an application, Zhao obtained the following result.

Theorem D ([12, Proposition 2.15]; also cf. [12, Theorem 1.5])**.** *Let s and l be two positive integers. Let p be a prime such that $p \geq l + 2$ and $p - 1$ divides none of ks and $ks + 1$ for $k = 1, \ldots, l$. Then*

$$H(\{s\}^l; p-1) \equiv S(\{s\}^l; p-1) \equiv 0 \pmod{p^{\mathfrak{p}(ls-1)}}.$$

In particular, if $p \geq ls+3$, then the above is always true, and so $p \mid H(\{s\}^l; p-1)$.

As noticed in [12, p. 85, Proof of Theorem 2.14], the above congruence for $H(\{s\}^l; p-1)$ follows immediately from [13, Lemma 2], while the above congruence for $S(\{s\}^l; p-1)$ then follows from the equality (2.11) in [12] and induction on l.

In Section 2 we give another proof of the above congruence for $H(\{s\}^l; p-1)$. Our result is as follows.

Theorem 1.1. *Let s and l be two positive integers. Let p be a prime such that $p \geq ls + 3$. Then*

$$H(\{s\}^l; p-1) \equiv (-1)^{l-1}\frac{H(ls; p-1)}{l} \equiv \begin{cases} -\dfrac{s(ls+1)p^2}{2(ls+2)} B_{p-ls-2} \pmod{p^3} \text{ if } 2 \nmid ls \\ (-1)^{l-1}\dfrac{sp}{ls+1} B_{p-ls-1} \pmod{p^2} \text{ if } 2 \mid ls. \end{cases}$$

Taking $s = 1$ in Theorem 1.1, we obtain the following result.

Corollary 1.2. *Let p be a prime greater than 3, and let l be a positive integer such that $l \leq p - 3$. Then*

$$H(\{1\}^l; p-1) := \sum_{1 \leq k_1 < \cdots < k_l \leq p-1} \frac{1}{k_1 \cdots k_l} \equiv \begin{cases} -\dfrac{(l+1)p^2}{2(l+2)} B_{p-l-2} & \pmod{p^3} \text{ if } 2 \nmid l \\ -\dfrac{p}{l+1} B_{p-l-1} & \pmod{p^2} \text{ if } 2 \mid l. \end{cases}$$

Corollary 1.3. *Let p be a prime greater than 3, and let l be a positive integer such that $2 \le l \le p-2$. Then*

$$St(p,l) \equiv \begin{cases} \dfrac{lp^2}{2(l+1)} B_{p-1-l} & \pmod{p^3} \ if\ 2 \mid l \\ \dfrac{p}{l} B_{p-l} & \pmod{p^2} \ if\ 2 \nmid l. \end{cases}$$

Proof. Since by Wilson theorem, $(p-1)! \equiv -1 \pmod{p}$, we get for all $l = 2, \dots, p-2$

$$St(p,l) = (p-1)! \cdot H(\{1\}^{l-1}; p-1) \equiv -H(\{1\}^{l-1}; p-1) \pmod{p}.$$

Therefore, both congruences follow immediately from the congruences given in Corollary 1.2. □

Remark 1.4. Observe that Corollary 1.2 does not contain a related congruence for the Stirling number $St(p,1) = (p-1)!$. In 1900 Glaisher [4] showed that $(p-1)! \equiv pB_{p-1} - p \pmod{p^2}$.

In order to prove Theorem 1.1, we use Theorem A in the proof of Lemma 2.2 given in Section 2. Note that Theorem A is an immediate consequence of a classical result of E. Lehmer [8] given by Lemma 2.7. On the other hand, Theorem B is an immediate consequence of Corollaries 1.2 and 1.3 and the fact that by Lemma 2.5, the denominator of the Bernoulli number B_l, written in reduced form, is not divisible by p for each integer l such that $0 \le l \le p-3$. Further, the congruence for $H(\{s\}^l; p-1)$ in Theorem C is given by Theorem 1.1. Finally, note that (see Remark 2.3) the proof of Theorem D is the same as that of Lemma 2.2.

2. Proof of Theorem 1.1

For the proof of Theorem 1.1, we will need some auxiliary results.

Lemma 2.1. *Let n, s and l be positive integers such that $l \ge 2$ and $n \ge ls$. Then*

$$\sum_{j=1}^{l-1} (-1)^{j-1} H(js; n) \cdot H(\{s\}^{l-j}; n) = lH(\{s\}^l; n) + (-1)^l H(ls; n). \tag{2.1}$$

Proof. First note that (2.1) is trivially satisfied for $l = 2$. For simplicity, here we write $H(\{\mathbf{s}\}^l)$ instead of $H(\{\mathbf{s}\}^l; n)$, and denote

$$\sigma(j) = \sum_{i=0}^{l-j} H(\underbrace{s, \dots, s}_{i}, js, \underbrace{s, \dots, s}_{l-j}; n), \quad j = 1, \dots l,$$

whence we see that $\sigma(l) = H(ls;n)$. Now if $l \geq 3$, then for all j with $2 \leq j \leq l-1$, we have

$$\begin{aligned}
H(js)\cdot H(\{s\}^{l-j}) &= \left(\sum_{1\le k\le n}\frac{1}{k^{js}}\right)\left(\sum_{1\le k_1<\cdots<k_{l-j}\le n}\frac{1}{(k_1\cdots k_{l-j})^s}\right)\\
&= \sum_{k\notin\{k_1,\ldots,k_{l-j}\}} + \sum_{k\in\{k_1,\ldots,k_{l-j}\}}\\
&= \sum_{i=0}^{l-j} H(\underbrace{s,\ldots,s}_{i}, js, \underbrace{s,\ldots,s}_{l-j-i})\\
&+ \sum_{i=0}^{l-j-1} H(\underbrace{s,\ldots,s}_{i}, (j+1)s, \underbrace{s,\ldots,s}_{l-j-1-i})\\
&= \sigma(j) + \sigma(j+1).
\end{aligned}$$

Furthermore, for $j = 1$, we have

$$\begin{aligned}
H(s)\cdot H(\{s\}^{l-1}) &= \left(\sum_{1\le k\le n}\frac{1}{k^{s}}\right)\left(\sum_{1\le k_1<\cdots<k_{l-1}\le n}\frac{1}{(k_1\cdots k_{l-1})^s}\right)\\
&= \sum_{k\notin\{k_1,\ldots,k_{l-1}\}} + \sum_{k\in\{k_1,\ldots,k_{l-1}\}}\\
&= \sum_{i=0}^{l-1} H(\underbrace{s,\ldots,s}_{i}, s, \underbrace{s,\ldots,s}_{l-1-i}) + \sum_{i=0}^{l-1} H(\underbrace{s,\ldots,s}_{i}, 2s, \underbrace{s,\ldots,s}_{l-2-i})\\
&= lH(\{s\}^l) + \sigma(2).
\end{aligned}$$

The above two equalities imply

$$\begin{aligned}
\sum_{j=1}^{l-1}(-1)^{j-1}H(js)\cdot H(\{s\}^{l-j}) &= lH(\{s\}^l) + \sigma(2) + \sum_{j=2}^{l-1}(-1)^{j-1}(\sigma(j)+\sigma(j+1))\\
&= lH(\{s\}^l) + \sigma(2) + \sum_{j=2}^{l-1}(-1)^{j-1}(\sigma(j)+\sigma(j+1))\\
&= lH(\{s\}^l) + \sigma(2) - \sigma(2) + (-1)^{l-1}\sigma(l)\\
&= lH(\{s\}^l) + (-1)^l H(ls;n),
\end{aligned}$$

as desired. □

The following lemma is an extension of the congruence for harmonic sums $H(\{1\}^l;p-1)$ given by Theorem B. This is in fact the congruence for $H(\{s\}^l;p-1)$ from Theorem D when $p \geq ls+3$.

Lemma 2.2 (cf. Theorem C)**.** *Let s and l be two positive integers, and let p be a prime such that $p \geq ls+3$. Then $p^2 \mid H(\{s\}^l;p-1)$ if ls is odd, and $p \mid H(\{s\}^l;p-1)$ if ls is even.*

Proof. Putting $n = p-1$ in (2.1) of Lemma 2.1, we obtain

$$\sum_{j=1}^{l-1}(-1)^{j-1}H(js;p-1)\cdot H(\{s\}^{l-j};p-1) + (-1)^{l-1}H(ls;p-1) = lH(\{s\}^l;p-1). \quad (2.2)$$

We proceed by induction on the sum $\sigma := l+s \geq 2$. If $\sigma = 2$ then $l = s = 1$, and $p^2 \mid H(1;p-1)$ by Wolstenholme's theorem. Now suppose that the assertion is true for some σ with $\sigma \geq 2$. This means that $p^2 \mid H(\{s'\}^{l'};p-1)$ whenever l' and s' are both odd such that $l'+s' \leq \sigma$ and $p \geq l's'+3$, and that $p \mid H(\{s'\}^{l'};p-1)$ whenever $l's'$ is even such that $l'+s' \leq \sigma$ and $p \geq l's'+3$ In order to prove the assertion for all pairs l and s with $l+s = \sigma$ and $p \geq ls+3$, we consider the following two cases.

`Case 1.` ls is odd; that is both integers l and s are odd. Then, for odd j with $1 \leq j \leq l-1$, we have $s+(l-j) = \sigma - j < \sigma$ and $s(l-j)$ is even. Therefore, by the inductive hypothesis, $p \mid H(\{s\}^{l-j}$. Furthermore, for such a j, by Theorem A, $p^2 \mid H(js;p-1)$.

Similarly, if j is even with $1 \leq j \leq l-1$, we also have $s+(l-j) = \sigma - j < \sigma$, and $s(l-j)$ is odd. Thus, by the inductive hypothesis, $p^2 \mid H(\{s\}^{l-j}$. Furthermore, for such a j, by Theorem A, $p \mid H(js;p-1)$.

Hence, in both cases it follows that $p^3 \mid H(js;p-1)\cdot H(\{s\}^{l-j};p-1)$. This together with the fact that, by Theorem A, $p^3 \mid H(ls;p-1)$, implies that the sum on the right hand side of (2.2) is divisible by p^3. Therefore, $p^3 \mid lH(\{s\}^l;p-1)$, whence, because $l \leq ls \leq p-3$, it follows that $p^3 \mid H(\{s\}^l;p-1)$. This concludes the inductive proof when ls is odd.

`Case 2.` ls is even. Then in the same way as in the first case, we obtain by induction on the sum $l+s$ that $p^2 \mid H(\{s\}^l;p-1)$.

This completes the inductive proof. □

Remark 2.3. Observe that the above proof holds if we replace the condition $p \geq ls+3$ of the Lemma by the following conditions of Theorem D: $p \geq l+2$ and $p-1$ divides none of ks and $ks+1$ for $k = 1,\dots,l$. In other words, Theorem D can be proved in the same manner as Lemma 2.2.

We are now ready to state the following

Proposition 2.4. *Let s and l be two positive integers, and let $t = \max\{1, l-1\}$. Let p be a prime such that $p \geq ts+3$. Then*

$$lH(\{s\}^l;p-1) \equiv (-1)^{l-1}H(ls;p-1) \begin{cases} \pmod{p^3} & if\ 2 \nmid ls \\ \pmod{p^2} & if\ 2 \mid ls. \end{cases}$$

Proof. If $l = 1$ and $p \geq s+3$, then the above congruences reduce to the congruence for $H(s;p-1)$ given in Theorem A. If $l \geq 2$ and $p \geq (l-1)s+3$, by (2.2) of the proof of Lemma 2.2, we have

$$\sum_{j=1}^{l-1}(-1)^{j-1}H(js;p-1)\cdot H(\{s\}^{l-j};p-1) = lH(\{s\}^l;p-1) - (-1)^l H(ls;p-1).$$

From the proof of Lemma 2.2, we see that each term of the sum on the left hand side of the above identity is divisible by p^3 if ls is odd, and by by p^2 if ls is even. Clearly, this fact implies both congruences from our Proposition. □

Lemma 2.5. *Let $p \geq 3$ be a prime, and let s be any even integer such that $0 \leq s \leq p-3$. Then the denominator of the Bernoulli number B_s, written in reduced form, is not divisible by p.*

Proof. If $p = 3$, then $s = 0$, that is, $B_0 = 1$. Suppose now that $p \geq 5$. It is well known (see [7]) that Bernoulli numbers can be defined recursively as

$$B_s = -\frac{1}{s+1}\sum_{i=0}^{s-1}\binom{s+1}{i}B_i.$$

Now, by induction on even s with $0 \leq s \leq p-3$, the above equality immediately implies that the denominator of B_s, written in reduced form, is not divisible by p. □

Remark 2.6. The above lemma is an immediate consequence of the von Staudt-Clausen theorem, which asserts that $B_{2m} + \sum_{p-1|2m} 1/p$ is an integer for all $m \in \mathbb{N}$, where the summation is over all primes p such that $p-1 \mid 2m$ (see, for example, [7, p. 233, Theorem 3]). If $B_{2m} = N_{2m}/D_{2m}$ with $\gcd(N_{2m}, D_{2m}) = 1$, then, by this result, it follows that the denominator D_{2m} of B_{2m} is given by

$$D_{2m} = \prod_{\substack{p\ prime\\ p-1|2m}} p,$$

whence Lemma 2.5 follows.

The following result is closely related to congruences of Glaisher [4], as quoted and proved by E. Lehmer (two congruences after (16) in [8]).

Lemma 2.7 ([8]; also cf. [12, Theorem 2.8])**.** *Let $p \geq 5$ be a prime, and let m be any integer such that $1 \leq m \leq (p-3)/2$. Then*

$$H(2m-1;p-1) := \sum_{k=1}^{p-1}\frac{1}{k^{2m-1}} \equiv \frac{m(1-2m)p^2}{2m+1}B_{p-1-2m} \pmod{p^3}, \tag{2.3}$$

and

$$H(2m;p-1) := \sum_{k=1}^{p-1}\frac{1}{k^{2m}} \equiv \frac{2mp}{2m+1}B_{p-1-2m} \pmod{p^2}. \tag{2.4}$$

Proof. Consider the case when ls is odd, that is, when both l and s are odd. By the first congruence of Proposition 2.4 and the congruence (2.3) of the above lemma with $ls = 2m-1$, we obtain

$$\begin{aligned} H(\{s\}^l;p-1) &\equiv \frac{H(ls;p-1)}{l} && \pmod{p^3}\\ &\equiv -\frac{1}{l}\frac{sl(ls+1)p^2}{2(ls+2)}B_{p-ls-2} && \pmod{p^3}\\ &= -\frac{s(ls+1)p^2}{2(ls+2)}B_{p-ls-2} && \pmod{p^3}. \end{aligned}$$

Similarly, using the second congruence of Proposition 2.4 and the congruence (2.4) of the above lemma with $ls = 2m$, we obtain the congruence of Theorem 1.1 for even ls. □

References

[1] M. Bayat, *A generalization of Wolstenholme's Theorem*, Amer. Math. Monthly **104** (1997), 557–560.

[2] R. Bruck, *Wolstenholme's theorem, Stirling numbers, and binomial coefficients*, available at `mathlab.usc.edu/∼bruck/research/stirling/`.

[3] K. Dilcher, L. Skula and I. Sh. Slavutsky: *Bernoulli Numbers Bibliography* (1713–1990), *Queen's papers in Pure and Appl. Math.*, vol. **87**, 175 pp.; Appendix, Queen's University, Kingston, Ontario, 1991, updated on-line version: `http://www.mathstat.dal.ca/∼dilcher/bernoulli.html`. Zbl 0741.11001.

[4] J. W. L. Glaisher, *On the residues of the sums of products of the first* $p-1$ *numbers and their powers, to modulus* p^2 *or* p^3, Quart. J. Math. **31** (1900), 321–353.

[5] A. Granville, *Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers*, in Organic mathematics (Burnaby, BC, 1995), CMS Conf. Proc., 20, Amer. Math. Soc., Providence, RI, 1997, 253–276.

[6] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Clarendon press, Oxford, 1980.

[7] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1982.

[8] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. Math. **39** (1938), 350–360.

[9] R. Meštrović, *A note on the congruence* $\binom{np^k}{mp^k} \equiv \binom{n}{m} (\mathrm{mod}\, p^r)$, Czechoslovak Math. J. **62** (2012), 59–65.

[10] R. Meštrović, *On the mod* p^7 *determination of* $\binom{2p-1}{p-1}$, accepted for publication in Rocky Mountain J. Math., preprint `arXiv:1108.1174 [mathNT]`, 2011.

[11] R. Meštrović, *Wolstenholme's theorem: Its Generalizations and Extensions in the last hundred and fifty years* (1862–2012) preprint `arXiv: 0911.4433v3 [math.NT]`, 2011.

[12] J. Zhao, *Wolstenholme type theorem for multiple harmonic sum*, Int. J. Number Theory **4** (2008), 73–106.

[13] X. Zhou and T. Cai, *A generalization of a curious congruence on harmonic sums*, Proc. Amer. Math. Soc. **135** (2007), 1329–1333.

Romeo Meštrović, Maritime Faculty, University of Montenegro, Dobrota 36, 85330 Kotor, Montenegro
e-mail: `romeo@ac.me`