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ON A TWO-POINT BOUNDARY VALUE PROBLEM FOR THIRD-ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS. PART II.

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Abstract. Efficient conditions sufficient for the solvability of the problem

$$u'''(t) = g(t)u(\mu(t)) - p(t)u(\tau(t)) + q(t); \quad u(a) = c_1, \quad u'(a) = c_2, \quad u(b) = c_3$$

are derived using the general results obtained in our recent paper [1]. Here, $p, g \in L([a,b];\mathbb{R}^+)$, $q \in L([a,b];\mathbb{R})$, $\tau, \mu : [a,b] \to [a,b]$ are measurable functions, and $c_i \in \mathbb{R}$ $(i = 1,2,3)$. Sign-constant solutions are discussed as well.

1. Introduction

In [1], we have obtained general results on the existence, uniqueness and positivity of a solution to the two-point boundary value problem

$$u'''(t) = \ell(u)(t) + q(t) \qquad \text{for a.e. } t \in [a,b], \tag{1.1}$$

$$u(a) = c_1, \qquad u'(a) = c_2, \qquad u(b) = c_3, \tag{1.2}$$

where $q \in L([a,b];\mathbb{R})$, $c_i \in \mathbb{R}$ $(i = 1,2,3)$, and $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is a linear bounded operator. The present paper, which is the second part of [1], contains some nontrivial consequences of the general results of [1] for the equations with deviating arguments. The proofs essentially use the statements obtained in [1]. We refer to [1] for an overview of the topic and the related literature.

Here, we consider the problem (1.1), (1.2) with the operator ℓ having one of the following forms:

$$\ell(v)(t) = -p(t)v(\tau(t)) \qquad \text{for a.e. } t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{1.3}$$

$$\ell(v)(t) = g(t)v(\mu(t)) \qquad \text{for a.e. } t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{1.4}$$

and

$$\ell(v)(t) = g(t)v(\mu(t)) - p(t)v(\tau(t)) \qquad \text{for a.e. } t \in [a,b], \quad v \in C([a,b];\mathbb{R}), \tag{1.5}$$

where $p, g \in L([a,b];\mathbb{R}^+)$ and $\tau, \mu : [a,b] \to [a,b]$ are measurable functions. By a solution to the problem (1.1), (1.2), we understand a function $u : [a,b] \to \mathbb{R}$ which is absolutely continuous together with its first and second derivatives, satisfies the equality (1.1) almost everywhere in $[a,b]$, and (1.2) holds.

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The following notation is used throughout the paper:

$\mathbb{R}$ is a set of all real numbers, $\mathbb{R}^+ = [0, +\infty[$.

$C([a,b];\mathbb{R})$ is a Banach space of all continuous functions $u : [a,b] \to \mathbb{R}$ endowed with the norm

$$\|u\|_C = \max\big\{|u(t)| : t \in [a,b]\big\}.$$

$\widetilde{C}^2([a,b];\mathbb{R})$ is the set of all functions $u : [a,b] \to \mathbb{R}$ that are absolutely continuous together with their first and second derivatives.

$\widetilde{C}^2_{loc}(\,]a,b[\,;\mathbb{R})$ is the set of all functions $u :]a,b[\to \mathbb{R}$ such that $u \in \widetilde{C}^2([\alpha,\beta];\mathbb{R})$ for every $\alpha, \beta \in]a,b[$, $\alpha < \beta$.

Let $u :]a,b[\to \mathbb{R}$ be a continuous function and let there exist a finite or an infinite right, left, limit of u at the point a, b, respectively. Then we will write $u(a+)$, $u(b-)$, instead of $\lim\limits_{t\to a+} u(t)$, $\lim\limits_{t\to b-} u(t)$, respectively.

$\widetilde{C}_0(\,]a,b[\,;\mathbb{R})$ is a set of all functions $u \in \widetilde{C}^2_{loc}(\,]a,b[\,;\mathbb{R}) \cap C([a,b];\mathbb{R})$ such that there exist finite or infinite limits $u'(a+)$ and $u'(b-)$.

$L([a,b];\mathbb{R})$ is a Banach space of all Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ endowed with the norm

$$\|p\|_L = \int_a^b |p(s)|\,\mathrm{d}s.$$

$L([a,b];\mathbb{R}^+) = \big\{p \in L([a,b];\mathbb{R}) : p(t) \in \mathbb{R}^+ \text{ for a.e. } t \in [a,b]\big\}$.

$\mathcal{L}_{ab}$ is a set of all linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

For convenience we recall the definitions introduced in [1].

Definition 1.1. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{V}([a,b])$ if every function $u \in \widetilde{C}^2([a,b];\mathbb{R})$ satisfying

$$\begin{gathered} u'''(t) \le \ell(u)(t) \qquad \text{for a.e. } t \in [a,b], \\ u(a) \ge 0, \qquad u'(a) \ge 0, \qquad u(b) \ge 0 \end{gathered} \tag{1.6}$$

admits the inequality

$$u(t) \ge 0 \qquad \text{for } t \in [a,b]. \tag{1.7}$$

Definition 1.2. An operator $\ell \in \mathcal{L}_{ab}$ is said to belong to the set $\mathcal{V}_0([a,b])$ if every function $u \in \widetilde{C}^2([a,b];\mathbb{R})$ satisfying (1.6) and

$$u(a) = 0, \qquad u'(a) \ge 0, \qquad u(b) = 0$$

admits the inequality (1.7).

2. Main results

Theorem 2.1. *Let*

$$(b-\tau(t))(\tau(t)-a)\int_a^{\tau(t)}(b-s)(s-a)p(s)\,\mathrm{d}s-\frac{(b-\tau(t))^2}{2}\int_a^{\tau(t)}(s-a)^2p(s)\,\mathrm{d}s$$

$$+\frac{(\tau(t)-a)^2}{2}\int_{\tau(t)}^{b}(b-s)^2p(s)\,\mathrm{d}s<(b-a)^2 \qquad \text{for a.e. } t\in[a,b]. \tag{2.1}$$

Then the operator ℓ defined by (1.3) *belongs to the set $\mathcal{V}([a,b])$.*

Corollary 2.2. *Let*

$$\int_a^b p(s)\,\mathrm{d}s\le\frac{16}{(b-a)^2}. \tag{2.2}$$

Then the operator ℓ defined by (1.3) *belongs to the set $\mathcal{V}([a,b])$.*

Theorem 2.3. *Let*

$$\left(\frac{b-\tau(t)}{b-t}\right)^{1-\frac{\sqrt{3}}{3}}\left(\frac{\tau(t)-a}{t-a}\right)^{1+\frac{\sqrt{3}}{3}}p(t)\le\frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for a.e. } t\in[a,b]. \tag{2.3}$$

Then the operator ℓ defined by (1.3) *belongs to the set $\mathcal{V}([a,b])$.*

Theorem 2.4. *Let there exist $c\in[a,b]$ and $\lambda_{ij}\in\mathbb{R}^+$, $\nu_i\in[0,1[$, $(i,j=1,2)$ such that*

$$\int_0^{+\infty}\frac{\mathrm{d}s}{s^2+\lambda_{11}s+\lambda_{12}}\ge\frac{(c-a)^{1-\nu_1}}{1-\nu_1}, \tag{2.4}$$

$$\int_0^{+\infty}\frac{\mathrm{d}s}{s^2+\lambda_{21}s+\lambda_{22}}\ge\frac{(b-c)^{1-\nu_2}}{1-\nu_2}, \tag{2.5}$$

and

$$-p(t)\frac{(t-a)^2}{2}+p(t)\sigma(t)\frac{(\tau(t)-t)^2}{2}\le\frac{\nu_1}{t-a}+\frac{\lambda_{11}}{(t-a)^{\nu_1}} \quad \text{for a.e. } t\in[a,c], \tag{2.6}$$

$$p(t)(t-a)+p(t)\sigma(t)(\tau(t)-t)\le\frac{\lambda_{12}}{(t-a)^{2\nu_1}} \qquad \text{for a.e. } t\in[a,c], \tag{2.7}$$

$$p(t)\frac{(t-a)^2}{2}-p(t)\sigma(t)\frac{(\tau(t)-t)^2}{2}\le\frac{\nu_2}{b-t}+\frac{\lambda_{21}}{(b-t)^{\nu_2}} \quad \text{for a.e. } t\in[c,b], \tag{2.8}$$

$$p(t)(t-a)+p(t)\sigma(t)(\tau(t)-t)\le\frac{\lambda_{22}}{(b-t)^{2\nu_2}} \qquad \text{for a.e. } t\in[c,b], \tag{2.9}$$

where

$$\sigma(t)=\frac{1}{2}\big(1+\operatorname{sgn}(\tau(t)-t)\big).$$

Then the operator ℓ defined by (1.3) *belongs to the set* $\mathcal{V}\big([a,b]\big)$.

Theorem 2.5. *Let $\mu(t)\le t$ for a.e. $t\in[a,b]$ and*

$$\int\limits_a^b (b-s)(s-a)g(s)\,\mathrm{d}s\le 2. \tag{2.10}$$

Then the operator ℓ defined by (1.4) *belongs to the set* $\mathcal{V}\big([a,b]\big)$.

Corollary 2.6. *Let $\mu(t)\le t$ for a.e. $t\in[a,b]$ and*

$$\int\limits_a^b g(s)\,\mathrm{d}s\le \frac{8}{(b-a)^2}.$$

Then the operator ℓ defined by (1.4) *belongs to the set* $\mathcal{V}\big([a,b]\big)$.

Theorem 2.7. *Let $\mu(t)\le t$ for a.e. $t\in[a,b]$ and*

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}}\left(\frac{\mu(t)-a+\omega}{t-a+\omega}\right)^{1-\frac{\sqrt{3}}{3}} g(t)\le \frac{2\sqrt{3}(b-a+\omega)^3}{9(b-t)^3(t-a+\omega)^3} \quad \text{for a.e. } t\in[a,b] \tag{2.11}$$

with $\omega=\frac{3-\sqrt{3}}{3+\sqrt{3}}(b-a)$. Then the operator ℓ defined by (1.4) *belongs to the set* $\mathcal{V}\big([a,b]\big)$.

Theorem 2.8. *Let $\mu(t)\le t$ for a.e. $t\in[a,b]$. Let, moreover, there exist $c\in[a,b]$ and $\lambda_{ij}\in\mathbb{R}^+$, $\nu_i\in[0,1[$, $(i,j=1,2)$ such that* (2.4), (2.5) *hold and*

$$g(t)\frac{(b-t)^2}{2}-g(t)\frac{(t-\mu(t))^2}{2}\le\frac{\nu_1}{t-a}+\frac{\lambda_{11}}{(t-a)^{\nu_1}} \qquad \text{for a.e. } t\in[a,c], \tag{2.12}$$

$$g(t)(b-t)+g(t)(t-\mu(t))\le\frac{\lambda_{12}}{(t-a)^{2\nu_1}} \qquad \text{for a.e. } t\in[a,c], \tag{2.13}$$

$$-g(t)\frac{(b-t)^2}{2}+g(t)\frac{(t-\mu(t))^2}{2}\le\frac{\nu_2}{b-t}+\frac{\lambda_{21}}{(b-t)^{\nu_2}} \qquad \text{for a.e. } t\in[c,b], \tag{2.14}$$

$$g(t)(b-t)+g(t)(t-\mu(t))\le\frac{\lambda_{22}}{(b-t)^{2\nu_2}} \qquad \text{for a.e. } t\in[c,b]. \tag{2.15}$$

Then the operator ℓ defined by (1.4) *belongs to the set* $\mathcal{V}\big([a,b]\big)$.

Theorem 2.9. *Let $\mu(t)\le t$ for a.e. $t\in[a,b]$ and*

$$(b-\mu(t))(\mu(t)-a)\int\limits_{\mu(t)}^b (b-s)(s-a)g(s)\,\mathrm{d}s-\frac{(\mu(t)-a)^2}{2}\int\limits_{\mu(t)}^b (b-s)^2 g(s)\,\mathrm{d}s$$

$$+\frac{(b-\mu(t))^2}{2}\int\limits_a^{\mu(t)} (s-a)^2 g(s)\,\mathrm{d}s\le (b-a)^2 \qquad \text{for a.e. } t\in[a,b]. \tag{2.16}$$

Then the operator ℓ defined by (1.4) *belongs to the set* $\mathcal{V}_0\big([a,b]\big)$.

Corollary 2.10. *Let $\mu(t) \le t$ for a.e. $t \in [a,b]$ and*

$$\int_a^b g(s)\,\mathrm{d}s \le \frac{16}{(b-a)^2}. \tag{2.17}$$

Then the operator ℓ defined by (1.4) *belongs to the set $\mathcal{V}_0\big([a,b]\big)$.*

Theorem 2.11. *Let $\mu(t) \le t$ for a.e. $t \in [a,b]$ and*

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}} \left(\frac{\mu(t)-a}{t-a}\right)^{1-\frac{\sqrt{3}}{3}} g(t) \le \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \quad \text{for a.e. } t \in [a,b]. \tag{2.18}$$

Then the operator ℓ defined by (1.4) *belongs to the set $\mathcal{V}_0\big([a,b]\big)$.*

The results listed below immediately follow from [1, Theorems 2.10–2.13], Theorems 2.1–2.11, and Corollaries 2.2–2.10.

Theorem 2.12. *Let functions p, τ satisfy the assumptions of at least one of Theorems 2.1–2.4 or Corollary 2.2 and let functions g, μ satisfy the assumptions of at least one of Theorems 2.5–2.8 or Corollary 2.6. Then the problem* (1.1), (1.2) *with ℓ defined by* (1.5) *has a unique solution u. If, in addition,*

$$q(t) \le 0 \qquad \text{for a.e. } t \in [a,b], \tag{2.19}$$
$$c_i \ge 0 \qquad (i = 1,2,3),$$
$$\|q\|_L + \sum_{i=1}^{3} c_i > 0,$$

then

$$u(t) > 0 \qquad \text{for } t \in \,]a,b[\,. \tag{2.20}$$

Theorem 2.13. *Let functions p, τ satisfy the assumptions of at least one of Theorems 2.1–2.4 or Corollary 2.2 and let functions g, μ satisfy the assumptions of either Theorem 2.9 or Theorem 2.11 or Corollary 2.10. Then the problem* (1.1), (1.2) *with ℓ defined by* (1.5) *has a unique solution u. If, in addition,* (2.19) *holds and*

$$c_1 = 0, \qquad c_2 \ge 0, \qquad c_3 = 0,$$
$$\|q\|_L + c_2 > 0,$$

then (2.20) *holds.*

Theorem 2.14. *Let functions p, τ satisfy the assumptions of at least one of Theorems 2.1–2.4 or Corollary 2.2 and let functions g, μ satisfy at least one of the following items:*

(i) *$\mu(t) \le t$ for a.e. $t \in [a,b]$ and*

$$\int_a^b g(s)\,\mathrm{d}s \le \frac{32}{(b-a)^2};$$

(ii) $\mu(t) \le t$ *for a.e.* $t \in [a,b]$ *and*

$$(b-\mu(t))(\mu(t)-a)\int_{\mu(t)}^{b}(b-s)(s-a)g(s)\,\mathrm{d}s - \frac{(\mu(t)-a)^2}{2}\int_{\mu(t)}^{b}(b-s)^2 g(s)\,\mathrm{d}s$$
$$+\frac{(b-\mu(t))^2}{2}\int_{a}^{\mu(t)}(s-a)^2 g(s)\,\mathrm{d}s \le 2(b-a)^2 \qquad \text{for a.e. } t \in [a,b];$$

(iii) $\mu(t) \le t$ *for a.e.* $t \in [a,b]$ *and*

$$\left(\frac{b-\mu(t)}{b-t}\right)^{1+\frac{\sqrt{3}}{3}}\left(\frac{\mu(t)-a}{t-a}\right)^{1-\frac{\sqrt{3}}{3}} g(t) \le \frac{4\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3} \qquad \text{for a.e. } t \in [a,b];$$

(iv) $\mu(t) \le t$ *for a.e.* $t \in [a,b]$ *and there exist* $c \in [a,b]$ *and* $\lambda_{ij} \in \mathbb{R}^+$, $\nu_i \in [0,1[$, $(i,j=1,2)$ *such that* (2.4), (2.5) *hold and*

$$g(t)\frac{(b-t)^2}{2} - g(t)\frac{(t-\mu(t))^2}{2} \le \frac{2\nu_1}{t-a} + \frac{2\lambda_{11}}{(t-a)^{\nu_1}} \qquad \text{for a.e. } t \in [a,c],$$
$$g(t)(b-t) + g(t)(t-\mu(t)) \le \frac{2\lambda_{12}}{(t-a)^{2\nu_1}} \qquad \text{for a.e. } t \in [a,c],$$
$$-g(t)\frac{(b-t)^2}{2} + g(t)\frac{(t-\mu(t))^2}{2} \le \frac{2\nu_2}{b-t} + \frac{2\lambda_{21}}{(b-t)^{\nu_2}} \qquad \text{for a.e. } t \in [c,b],$$
$$g(t)(b-t) + g(t)(t-\mu(t)) \le \frac{2\lambda_{22}}{(b-t)^{2\nu_2}} \qquad \text{for a.e. } t \in [c,b].$$

Then the problem (1.1), (1.2) *with* ℓ *defined by* (1.5) *is uniquely solvable.*

Theorem 2.15. *Let functions* p, τ *satisfy the assumptions of at least one of Theorems 2.1–2.4 or Corollary 2.2. Let, moreover,* $\tau(t) \le t$ *and* $\mu(t) \le t$ *for a.e.* $t \in [a,b]$. *Then the problem* (1.1), (1.2) *with* ℓ *defined by* (1.5) *is uniquely solvable.*

3. Proofs

Proof of Theorem 2.1. If $p \equiv 0$, then the conclusion of theorem follows from [1, Remark 2.3]. Therefore, we can assume that

$$\int_a^b p(s)\,\mathrm{d}s > 0. \tag{3.1}$$

Put

$$\begin{aligned}\gamma(t) = \frac{1}{(b-a)^2}\Bigg(&(b-t)(t-a)\int_a^t (b-s)(s-a)p(s)\,\mathrm{d}s \\ &-\frac{(b-t)^2}{2}\int_a^t (s-a)^2 p(s)\,\mathrm{d}s + \frac{(t-a)^2}{2}\int_t^b (b-s)^2 p(s)\,\mathrm{d}s\Bigg) \quad \text{for } t \in [a,b].\end{aligned} \tag{3.2}$$

We will show that γ satisfies the assumptions of [1, Theorem 2.1] with ℓ defined by (1.3). It can be easily verified that

$$\gamma'''(t) = -p(t) \qquad \text{for a.e. } t \in [a,b], \tag{3.3}$$

$$\gamma(a) = 0, \qquad \gamma'(a) = 0, \qquad \gamma(b) = 0. \tag{3.4}$$

Therefore, according to [1, Remark 2.3, Theorem 2.10] and the inequality (3.1), we have

$$\gamma(t) > 0 \qquad \text{for } t \in]a,b[\,. \tag{3.5}$$

Furthermore, (2.1) and (3.2) imply

$$\gamma(\tau(t)) < 1 \qquad \text{for a.e. } t \in [a,b], \tag{3.6}$$

which, when used in (3.3), yields

$$\gamma'''(t) \le -p(t)\gamma(\tau(t)) \qquad \text{for a.e. } t \in [a,b], \tag{3.7}$$

$$\operatorname{meas}\big\{t \in [a,b] : \gamma'''(t) < -p(t)\gamma(\tau(t))\big\} > 0. \tag{3.8}$$

Finally, $\gamma \in \widetilde{C}_0\big(]a,b[\,;\mathbb{R}\big)$ and (3.4), (3.5), (3.7), and (3.8) imply that all the assumptions of [1, Theorem 2.1] are fulfilled. □

Proof of Corollary 2.2. If $p \equiv 0$, then the conclusion of the corollary follows from [1, Remark 2.3]. Therefore, assume that (3.1) holds. It is sufficient to show that (2.1) is fulfilled. For this purpose, we will estimate the maximum value of the function γ defined by (3.2). Obviously, (3.3)–(3.5) hold. In view of (3.4) and (3.5), there exists $t_0 \in]a,b[$ such that

$$\gamma(t_0) = \max\big\{\gamma(t) : t \in [a,b]\big\}. \tag{3.9}$$

Consequently, $\gamma'(t_0) = 0$, i.e.,

$$\begin{aligned} &(a+b-2t_0)\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s + (b-t_0)\int_a^{t_0}(s-a)^2p(s)\,\mathrm{d}s \\ &+(t_0-a)\int_{t_0}^{b}(b-s)^2p(s)\,\mathrm{d}s = 0. \end{aligned} \tag{3.10}$$

From (3.10) we obtain

$$\begin{aligned} &(t_0-a)\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s - (b-t_0)\int_a^{t_0}(s-a)^2p(s)\,\mathrm{d}s \\ &= (t_0-a)\int_{t_0}^{b}(b-s)^2p(s)\,\mathrm{d}s + (b-t_0)\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s. \end{aligned} \tag{3.11}$$

From (3.2) we have

$$\gamma(t_0) = \frac{t_0-a}{2(b-a)^2}\left((t_0-a)\int_{t_0}^{b}(b-s)^2p(s)\,\mathrm{d}s + (b-t_0)\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s\right)$$
$$+\frac{b-t_0}{2(b-a)^2}\left((t_0-a)\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s - (b-t_0)\int_a^{t_0}(s-a)^2p(s)\,\mathrm{d}s\right).$$

Now using (3.11) in the latter equality, we obtain

$$\gamma(t_0) = \frac{t_0-a}{2(b-a)}\int_{t_0}^{b}(b-s)^2p(s)\,\mathrm{d}s + \frac{b-t_0}{2(b-a)}\int_a^{t_0}(b-s)(s-a)p(s)\,\mathrm{d}s,$$

whence, on account of the relation $4AB \leq (A+B)^2$, we get

$$\begin{aligned}\gamma(t_0) &\leq \frac{(b-t_0)^2(t_0-a)}{2(b-a)}\int_{t_0}^{b}p(s)\,\mathrm{d}s + \frac{(b-a)(b-t_0)}{8}\int_a^{t_0}p(s)\,\mathrm{d}s\\ &\leq \frac{(b-a)(b-t_0)}{8}\int_a^b p(s)\,\mathrm{d}s.\end{aligned} \tag{3.12}$$

On the other hand, the equality (3.10) yields

$$a+b-2t_0<0, \qquad \text{i.e.} \qquad t_0 > \frac{a+b}{2}. \tag{3.13}$$

Therefore, the inequality (3.12) with respect to (2.2), (3.1), and (3.13) results in

$$\gamma(t_0) < \frac{(b-a)^2}{16}\int_a^b p(s)\,\mathrm{d}s \leq 1. \tag{3.14}$$

Now in view of (3.9), we have (3.6), whence, on account of (3.2), we get (2.1). □

Proof of Theorem 2.3. Put

$$\gamma(t) = (b-t)^{1-\frac{\sqrt{3}}{3}}(t-a)^{1+\frac{\sqrt{3}}{3}} \qquad \text{for } t\in[a,b]. \tag{3.15}$$

Obviously, $\gamma \in \widetilde{C}_0\big(\,]a,b[\,;\mathbb{R}\big)$, $\gamma(t)>0$ for $t\in\,]a,b[\,$,

$$\gamma(a)=0, \qquad \gamma'(a)=0, \qquad \gamma(b)=0,$$

and

$$\gamma'''(t) = -\frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3}(b-t)^{1-\frac{\sqrt{3}}{3}}(t-a)^{1+\frac{\sqrt{3}}{3}} \qquad \text{for a.e. } t\in[a,b].$$

Using (2.3) in the latter equality, in view of (3.15), we get

$$\gamma'''(t) \leq -p(t)\gamma(\tau(t)) \qquad \text{for a.e. } t\in[a,b].$$

Moreover, (3.8) holds because $p(\cdot)\gamma(\tau(\cdot)) \in L\big([a,b];\mathbb{R}\big)$ and $\gamma''' \notin L\big([a,b];\mathbb{R}\big)$. Thus, all the assumptions of [1, Theorem 2.1] are fulfilled. □

Proof of Theorem 2.4. Assume $c \in\,]a, b[$; the cases $c = a$ and $c = b$ can be proved analogously. Without loss of generality we can assume that (2.4) and (2.5) are fulfilled as equalities. Define functions ρ_i $(i = 1, 2)$ as follows:

$$\int\limits_{\rho_1(t)}^{+\infty} \frac{\mathrm{d}s}{s^2 + \lambda_{11}s + \lambda_{12}} = \frac{(t-a)^{1-\nu_1}}{1-\nu_1} \qquad \text{for } t \in\,]a, c], \tag{3.16}$$

$$\int\limits_{\rho_2(t)}^{+\infty} \frac{\mathrm{d}s}{s^2 + \lambda_{21}s + \lambda_{22}} = \frac{(b-t)^{1-\nu_2}}{1-\nu_2} \qquad \text{for } t \in [c, b[\,. \tag{3.17}$$

Then

$$\rho_1(t) > 0 \quad \text{for } t \in\,]a, c[\,, \qquad \rho_2(t) > 0 \quad \text{for } t \in\,]c, b[\,, \tag{3.18}$$

$$\rho_i(c) = 0 \quad (i = 1, 2), \qquad \lim_{t \to a+} \rho_1(t) = +\infty, \qquad \lim_{t \to b-} \rho_2(t) = +\infty, \tag{3.19}$$

and

$$\rho_1'(t) = -(t-a)^{-\nu_1}\big(\rho_1^2(t) + \lambda_{11}\rho_1(t) + \lambda_{12}\big) \qquad \text{for } t \in\,]a, c], \tag{3.20}$$

$$\rho_2'(t) = (b-t)^{-\nu_2}\big(\rho_2^2(t) + \lambda_{21}\rho_2(t) + \lambda_{22}\big) \qquad \text{for } t \in [c, b[\,. \tag{3.21}$$

Put

$$z(t) = \begin{cases} \exp\left(-\int\limits_t^c (s-a)^{-\nu_1}\rho_1(s)\,\mathrm{d}s\right) & \text{for } t \in\,]a, c] \\ \exp\left(-\int\limits_c^t (b-s)^{-\nu_2}\rho_2(s)\,\mathrm{d}s\right) & \text{for } t \in\,]c, b[\end{cases} \tag{3.22}$$

and

$$\gamma(t) = \int\limits_a^t z(s)\,\mathrm{d}s \qquad \text{for } t \in [a, b]. \tag{3.23}$$

We will show that γ satisfies the assumptions of [1, Theorem 2.1]. Obviously, $\gamma \in \widetilde{C}_0\big(\,]a, b[\,; \mathbb{R}\big)$ and

$$\gamma(a) = 0, \qquad \gamma(t) > 0 \qquad \text{for } t \in\,]a, b]. \tag{3.24}$$

Moreover, in view of (3.19) and (3.22), we have

$$\gamma'(a+) = 0. \tag{3.25}$$

Furthermore, (3.22) and (3.23) yield

$$\gamma''(t) = \begin{cases} (t-a)^{-\nu_1}\rho_1(t)\gamma'(t) & \text{for } t \in\,]a, c], \\ -(b-t)^{-\nu_2}\rho_2(t)\gamma'(t) & \text{for } t \in\,]c, b[\,. \end{cases} \tag{3.26}$$

Obviously,

$$\gamma'(t) > 0 \qquad \text{for } t \in\,]a, b[\tag{3.27}$$

and, in view of (3.18), we have

$$\gamma''(t) > 0 \quad \text{for } t \in\,]a, c[\,, \qquad \gamma''(t) < 0 \quad \text{for } t \in\,]c, b[\,. \tag{3.28}$$

Finally, with respect to (3.20) or (3.21), from (3.26), we obtain

$$\begin{aligned}\gamma'''(t) &= -\nu_1(t-a)^{-\nu_1-1}\rho_1(t)\gamma'(t) - (t-a)^{-2\nu_1}\lambda_{11}\rho_1(t)\gamma'(t)\\ &-(t-a)^{-2\nu_1}\lambda_{12}\gamma'(t) \qquad \text{for } t \in\,]a,c],\end{aligned} \tag{3.29}$$

or

$$\begin{aligned}\gamma'''(t) &= -\nu_2(b-t)^{-\nu_2-1}\rho_2(t)\gamma'(t) - (b-t)^{-2\nu_2}\lambda_{21}\rho_2(t)\gamma'(t)\\ &-(b-t)^{-2\nu_2}\lambda_{22}\gamma'(t) \qquad \text{for } t \in\,]c,b[\,,\end{aligned} \tag{3.30}$$

respectively. Now using (3.26) in (3.29) and (3.30), we get

$$\gamma'''(t) = -\left(\frac{\nu_1}{t-a} + \frac{\lambda_{11}}{(t-a)^{\nu_1}}\right)\gamma''(t) - \frac{\lambda_{12}}{(t-a)^{2\nu_1}}\gamma'(t) \qquad \text{for } t \in\,]a,c], \tag{3.31}$$

$$\gamma'''(t) = \left(\frac{\nu_2}{b-t} + \frac{\lambda_{21}}{(b-t)^{\nu_2}}\right)\gamma''(t) - \frac{\lambda_{22}}{(b-t)^{2\nu_2}}\gamma'(t) \qquad \text{for } t \in\,]c,b[\,. \tag{3.32}$$

Note that, on account of (3.27) and (3.28), we have $\gamma'''(t) \le 0$ for $t \in\,]a,b[$ and, consequently, γ'' is a nonincreasing function. Therefore,

$$\begin{aligned}\gamma(t) = \int_a^t \gamma'(s)\,\mathrm{d}s &= (t-a)\gamma'(t) - \int_a^t (s-a)\gamma''(s)\,\mathrm{d}s\\ &\le (t-a)\gamma'(t) - \frac{(t-a)^2}{2}\gamma''(t) \qquad \text{for } t \in\,]a,b[\end{aligned}$$

and thus (3.31), or (3.32), results in

$$\begin{aligned}\gamma'''(t) &\le -\left(\frac{\nu_1}{t-a} + \frac{\lambda_{11}}{(t-a)^{\nu_1}} + p(t)\frac{(t-a)^2}{2}\right)\gamma''(t)\\ &-\left(\frac{\lambda_{12}}{(t-a)^{2\nu_1}} - p(t)(t-a)\right)\gamma'(t) - p(t)\gamma(t) \qquad \text{for a.e. } t \in\,]a,c],\end{aligned}$$

or

$$\begin{aligned}\gamma'''(t) &\le \left(\frac{\nu_2}{b-t} + \frac{\lambda_{21}}{(b-t)^{\nu_2}} - p(t)\frac{(t-a)^2}{2}\right)\gamma''(t)\\ &-\left(\frac{\lambda_{22}}{(b-t)^{2\nu_2}} - p(t)(t-a)\right)\gamma'(t) - p(t)\gamma(t) \qquad \text{for a.e. } t \in\,]c,b[\,,\end{aligned}$$

respectively. In view of (2.6)–(2.9), (3.27), and (3.28), the latter two inequalities yield

$$\begin{aligned}\gamma'''(t) \le -p(t)\sigma(t)\frac{(\tau(t)-t)^2}{2}\gamma''(t) - p(t)\sigma(t)(\tau(t)-t)\gamma'(t) - p(t)\gamma(t)&\\ \text{for a.e. } t \in\,]a,b[\,.&\end{aligned} \tag{3.33}$$

On the other hand, in view of (3.27),

$$\int_t^{\tau(t)} \gamma'(s)\,\mathrm{d}s \le 0 \qquad \text{if } \tau(t) \le t \tag{3.34}$$

and

$$\int_t^{\tau(t)} \gamma'(s)\,\mathrm{d}s = (\tau(t)-t)\gamma'(t) + \int_t^{\tau(t)} (\tau(t)-s)\gamma''(s)\,\mathrm{d}s$$
$$\leq (\tau(t)-t)\gamma'(t) + \frac{(\tau(t)-t)^2}{2}\gamma''(t) \qquad \text{if } \tau(t) > t. \tag{3.35}$$

Thus, from (3.34) and (3.35), we have

$$\int_t^{\tau(t)} \gamma'(s)\,\mathrm{d}s \leq \sigma(t)(\tau(t)-t)\gamma'(t) + \sigma(t)\frac{(\tau(t)-t)^2}{2}\gamma''(t) \quad \text{for a.e. } t \in [a,b]. \tag{3.36}$$

Now using (3.36) in (3.33), we obtain

$$\gamma'''(t) \leq -p(t)\int_t^{\tau(t)} \gamma'(s)\,\mathrm{d}s - p(t)\gamma(t) = -p(t)\gamma(\tau(t)) \qquad \text{for a.e. } t \in [a,b]. \tag{3.37}$$

Consequently, (3.22)–(3.25), and (3.37) imply that all the assumptions of [1, Theorem 2.1] are fulfilled. $\square$

Proof of Theorem 2.5. Put

$$\beta(t) = 1 - \frac{1}{(b-a)^2}\Bigg((b-t)(t-a)\int_a^t (b-s)(s-a)g(s)\,\mathrm{d}s + (t-a)^2$$
$$-\frac{(b-t)^2}{2}\int_a^t (s-a)^2 g(s)\,\mathrm{d}s + \frac{(t-a)^2}{2}\int_t^b (b-s)^2 g(s)\,\mathrm{d}s\Bigg) \qquad \text{for } t \in [a,b].$$

We will show that the assumptions of [1, Theorem 2.4] are fulfilled. Obviously, $\beta \in \widetilde{C}_0(\,]a,b[\,;\mathbb{R})$,

$$\beta(a) = 1, \qquad \beta'(a) = 0, \qquad \beta(b) = 0, \tag{3.38}$$

and it can be easily verified that

$$\beta'(b) = \frac{1}{b-a}\left(\int_a^b (b-s)(s-a)g(s)\,\mathrm{d}s - 2\right), \tag{3.39}$$

$$\beta'''(t) = g(t) \qquad \text{for a.e. } t \in [a,b]. \tag{3.40}$$

From (3.38)–(3.40), in view of (2.10), it follows that

$$\beta'(t) \leq 0 \qquad \text{for } t \in [a,b]. \tag{3.41}$$

Further, put

$$\gamma(t) = \beta(a+b-t) \qquad \text{for } t \in [a,b]. \tag{3.42}$$

Then, on account of (3.38), (3.40), and (3.41), we have

$$\gamma'''(t) = -g(t) \qquad \text{for a.e. } t \in [a,b],$$
$$\gamma(a) = 0, \qquad \gamma'(a) \geq 0, \qquad \gamma(b) = 1,$$

whence, according to [1, Remark 2.3, Theorem 2.10], it follows that

$$\gamma(t) > 0 \qquad \text{for } t \in]a,b[\,.$$

However, the latter inequality together with (3.38) and (3.42) results in

$$\beta(t) > 0 \qquad \text{for } t \in [a,b[\,. \tag{3.43}$$

Finally, in view of (3.38) and (3.41), we have

$$\beta(\mu(t)) \le 1 \qquad \text{for a.e. } t \in [a,b], \tag{3.44}$$

which, together with (3.40), results in

$$\beta'''(t) \ge g(t)\beta(\mu(t)) \qquad \text{for a.e. } t \in [a,b]. \tag{3.45}$$

Consequently, (3.41), (3.43), and (3.45) imply that all the assumptions of [1, Theorem 2.4] are fulfilled. □

Proof of Corollary 2.6. It immediately follows from Theorem 2.5 because

$$\int_a^b (b-s)(s-a)g(s)\,\mathrm{d}s \le \frac{(b-a)^2}{4}\int_a^b g(s)\,\mathrm{d}s.$$

□

Proof of Theorem 2.7. Put

$$\beta(t) = (b-t)^{1+\frac{\sqrt{3}}{3}}(t-a+\omega)^{1-\frac{\sqrt{3}}{3}} \qquad \text{for } t \in [a,b]. \tag{3.46}$$

Then, obviously, $\beta \in \widetilde{C}_0\big(\,]a,b[\,;\mathbb{R}\big)$, (3.43) holds,

$$\beta'(a) = 0, \qquad \beta'(b) = 0, \tag{3.47}$$

and

$$\beta'''(t) = \frac{2\sqrt{3}(b-a+\omega)^3}{9(b-t)^3(t-a+\omega)^3}(b-t)^{1+\frac{\sqrt{3}}{3}}(t-a+\omega)^{1-\frac{\sqrt{3}}{3}} \quad \text{for } t \in]a,b[\,. \tag{3.48}$$

From (3.47) and (3.48), it follows that (3.41) holds. Moreover, using (2.11) in (3.48), on account of (3.46), we get (3.45). Thus, all the assumptions of [1, Theorem 2.4] are fulfilled. □

Proof of Theorem 2.8. Assume $c \in]a,b[\,$; the cases $c = a$ and $c = b$ can be proved analogously. Without loss of generality we can assume that (2.4) and (2.5) are fulfilled as equalities. Define functions ρ_i $(i = 1,2)$ by (3.16) and (3.17), respectively. Then (3.18)–(3.21) hold. Define z by (3.22) and put

$$\beta(t) = \int_t^b z(s)\,\mathrm{d}s \qquad \text{for } t \in [a,b]. \tag{3.49}$$

We will show that β satisfies the assumptions of [1, Theorem 2.4]. Obviously, $\beta \in \widetilde{C}_0\big(\,]a,b[\,;\mathbb{R}\big)$ and

$$\beta(b) = 0, \qquad \beta(t) > 0 \qquad \text{for } t \in [a,b[\,. \tag{3.50}$$

Moreover, in view of (3.22), we have

$$\beta'(t) < 0 \qquad \text{for } t \in]a,b[\,. \tag{3.51}$$

Furthermore, (3.22) and (3.49) yield

$$\beta''(t) = \begin{cases} (t-a)^{-\nu_1}\rho_1(t)\beta'(t) & \text{for } t \in]a,c], \\ -(b-t)^{-\nu_2}\rho_2(t)\beta'(t) & \text{for } t \in]c,b[\,. \end{cases} \tag{3.52}$$

In view of (3.18) and (3.51), we have

$$\beta''(t) < 0 \qquad \text{for } t \in]a,c[\,, \qquad \beta''(t) > 0 \qquad \text{for } t \in]c,b[\,. \tag{3.53}$$

Finally, with respect to (3.20), or (3.21), from (3.52) we obtain

$$\begin{aligned} \beta'''(t) &= -\nu_1(t-a)^{-\nu_1-1}\rho_1(t)\beta'(t) - (t-a)^{-2\nu_1}\lambda_{11}\rho_1(t)\beta'(t) \\ &-(t-a)^{-2\nu_1}\lambda_{12}\beta'(t) \qquad \text{for } t \in]a,c], \end{aligned} \tag{3.54}$$

or

$$\begin{aligned} \beta'''(t) &= -\nu_2(b-t)^{-\nu_2-1}\rho_2(t)\beta'(t) - (b-t)^{-2\nu_2}\lambda_{21}\rho_2(t)\beta'(t) \\ &-(b-t)^{-2\nu_2}\lambda_{22}\beta'(t) \qquad \text{for } t \in]c,b[\,, \end{aligned} \tag{3.55}$$

respectively. Now using (3.52) in (3.54) and (3.55), we get

$$\beta'''(t) = -\left(\frac{\nu_1}{t-a} + \frac{\lambda_{11}}{(t-a)^{\nu_1}}\right)\beta''(t) - \frac{\lambda_{12}}{(t-a)^{2\nu_1}}\beta'(t) \qquad \text{for } t \in]a,c], \tag{3.56}$$

$$\beta'''(t) = \left(\frac{\nu_2}{b-t} + \frac{\lambda_{21}}{(b-t)^{\nu_2}}\right)\beta''(t) - \frac{\lambda_{22}}{(b-t)^{2\nu_2}}\beta'(t) \qquad \text{for } t \in]c,b[\,. \tag{3.57}$$

Note that, on account of (3.51) and (3.53), we have $\beta'''(t) \geq 0$ for $t \in]a,b[$ and, consequently, β'' is a nondecreasing function. Therefore,

$$\begin{aligned} \beta(t) &= -\int_t^b \beta'(s)\,\mathrm{d}s = -(b-t)\beta'(t) - \int_t^b (b-s)\beta''(s)\,\mathrm{d}s \\ &\leq -(b-t)\beta'(t) - \frac{(b-t)^2}{2}\beta''(t) \qquad \text{for } t \in]a,b[\,, \end{aligned}$$

and thus, (3.56) or (3.57) results in

$$\begin{aligned} \beta'''(t) &\geq -\left(\frac{\nu_1}{t-a} + \frac{\lambda_{11}}{(t-a)^{\nu_1}} - g(t)\frac{(b-t)^2}{2}\right)\beta''(t) \\ &-\left(\frac{\lambda_{12}}{(t-a)^{2\nu_1}} - g(t)(b-t)\right)\beta'(t) + g(t)\beta(t) \qquad \text{for a.e. } t \in]a,c], \end{aligned}$$

or

$$\begin{aligned} \beta'''(t) &\geq \left(\frac{\nu_2}{b-t} + \frac{\lambda_{21}}{(b-t)^{\nu_2}} + g(t)\frac{(b-t)^2}{2}\right)\beta''(t) \\ &-\left(\frac{\lambda_{22}}{(b-t)^{2\nu_2}} - g(t)(b-t)\right)\beta'(t) + g(t)\beta(t) \qquad \text{for a.e. } t \in]c,b[\,, \end{aligned}$$

respectively. In view of (2.12)–(2.15), (3.51), and (3.53), the latter two inequalities yield

$$\beta'''(t) \geq g(t)\frac{(t-\mu(t))^2}{2}\beta''(t) - g(t)(t-\mu(t))\beta'(t) + g(t)\beta(t) \qquad \text{for a.e. } t \in]a,b[\,. \tag{3.58}$$

On the other hand,

$$\int\limits_{\mu(t)}^{t} \beta'(s)\,\mathrm{d}s = (t-\mu(t))\beta'(t) - \int\limits_{\mu(t)}^{t} (s-\mu(t))\beta''(s)\,\mathrm{d}s$$
$$\geq (t-\mu(t))\beta'(t) - \frac{(t-\mu(t))^2}{2}\beta''(t) \qquad \text{for a.e. } t\in[a,b]. \tag{3.59}$$

Now using (3.59) in (3.58), we obtain

$$\beta'''(t) \geq -g(t)\int\limits_{\mu(t)}^{t} \beta'(s)\,\mathrm{d}s + g(t)\beta(t) = g(t)\beta(\mu(t)) \qquad \text{for a.e. } t\in[a,b]. \tag{3.60}$$

Consequently, (3.50), (3.51), and (3.60) imply that all the assumptions of [1, Theorem 2.4] are fulfilled. □

Proof of Theorem 2.9. If $g \equiv 0$, then the conclusion of theorem follows from [1, Remarks 1.4 and 2.3]. Therefore, we assume that

$$\int\limits_a^b g(s)\,\mathrm{d}s > 0. \tag{3.61}$$

Put

$$\beta(t) = \frac{1}{(b-a)^2}\Bigg((b-t)(t-a)\int\limits_t^b (b-s)(s-a)g(s)\,\mathrm{d}s$$
$$-\frac{(t-a)^2}{2}\int\limits_t^b (b-s)^2 g(s)\,\mathrm{d}s + \frac{(b-t)^2}{2}\int\limits_a^t (s-a)^2 g(s)\,\mathrm{d}s\Bigg) \qquad \text{for } t\in[a,b]. \tag{3.62}$$

We will show that β satisfies the assumptions of [1, Theorem 2.5] with ℓ defined by (1.4). It can be easily verified that

$$\beta'''(t) = g(t) \qquad \text{for a.e. } t\in[a,b], \tag{3.63}$$
$$\beta(a) = 0, \qquad \beta'(b) = 0, \qquad \beta(b) = 0. \tag{3.64}$$

Defining

$$\gamma(t) = \beta(a+b-t) \qquad \text{for } t\in[a,b],$$

from (3.63) and (3.64) we obtain

$$\gamma'''(t) = -g(a+b-t) \qquad \text{for a.e. } t\in[a,b],$$
$$\gamma(a) = 0, \qquad \gamma'(a) = 0, \qquad \gamma(b) = 0.$$

Therefore, according to [1, Remark 2.3, Theorem 2.10] and the inequality (3.61), we have $\gamma(t) > 0$ for $t \in\,]a,b[$ and, consequently,

$$\beta(t) > 0 \qquad \text{for } t\in\,]a,b[\,. \tag{3.65}$$

Furthermore, (2.16) and (3.62) imply (3.44), which, when used in (3.63), yields (3.45). Finally, $\beta \in \widetilde{C}_0\big(]a,b[\,;\mathbb{R}\big)$ and (3.45), (3.64), and (3.65) imply that all the assumptions of [1, Theorem 2.5] are fulfilled. □

Proof of Corollary 2.10. Define β by (3.62) and put

$$\gamma(t) = \beta(a+b-t) \qquad \text{for } t \in [a,b]. \tag{3.66}$$

Then γ satisfies (3.2) with

$$p(t) = g(a+b-t) \qquad \text{for a.e. } t \in [a,b]. \tag{3.67}$$

Analogously to the proof of Corollary 2.2, in view of (2.17) and (3.67), it can be easily verified that (3.14) holds where $t_0 \in]a,b[$ is such that (3.9) is satisfied. Thus, in view of (3.66), we have (3.44) and, consequently, (2.16) is fulfilled. □

Proof of Theorem 2.11. Put

$$\beta(t) = (b-t)^{1+\frac{\sqrt{3}}{3}}(t-a)^{1-\frac{\sqrt{3}}{3}} \qquad \text{for } t \in [a,b]. \tag{3.68}$$

Obviously, $\beta \in \widetilde{C}_0\big(]a,b[\,;\mathbb{R}\big)$, $\beta(t) > 0$ for $t \in]a,b[$,

$$\beta(a) = 0, \qquad \beta'(b) = 0, \qquad \beta(b) = 0,$$

and

$$\beta'''(t) = \frac{2\sqrt{3}(b-a)^3}{9(b-t)^3(t-a)^3}(b-t)^{1+\frac{\sqrt{3}}{3}}(t-a)^{1-\frac{\sqrt{3}}{3}} \qquad \text{for a.e. } t \in [a,b].$$

Using (2.18) in the latter equality, in view of (3.68), we get

$$\beta'''(t) \geq g(t)\beta(\mu(t)) \qquad \text{for a.e. } t \in [a,b].$$

Thus, all the assumptions of [1, Theorem 2.5] are fulfilled. □

References

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