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ON DIFFERENTIATION OF A LEBESGUE INTEGRAL WITH RESPECT TO A PARAMETER

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Abstract. The aim of this paper is to discuss the absolute continuity of certain composite functions and differentiation of a Lebesgue integral with respect to a parameter. The results obtained are useful when analyzing strong solutions of partial differential equations with Carathéodory right-hand sides.

1. Introduction and notation

Differentiation under integral sign is one of the very old questions in calculus of real functions. For example, conditions sufficient to ensure that Leibniz's rule is applicable, i.e., that

$$\frac{\partial}{\partial y}\int_a^b f(x,y)\,\mathrm{d}x = \int_a^b \frac{\partial f(x,y)}{\partial y}\,\mathrm{d}x, \tag{1.1}$$

have been investigated already by Jordan, Harnack, de la Vallée-Poussin, Hardy, Young, and others (see, e. g., survey given in [2]). This rule and its generalizations play an important role in various parts of mathematics. In particular, we are interested in Carathéodory solutions to the partial differential inequality

$$\frac{\partial^2\gamma(t,x)}{\partial t\partial x} \ge p(t,x)\gamma(t,x) + q(t,x) \tag{1.2}$$

with non-negative coefficients p and q integrable on the rectangle $[a,b]\times[c,d]$ (see, [3, Proof of Corollary 3.2(b)]). It is known that such a solution is, e. g., the function

$$\gamma(t,x) = \int_a^t\int_c^x Z_{t,x}(s,\eta)q(s,\eta)\,\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in[a,b]\times[c,d],$$

where $Z_{t,x}$ denotes the so-called Riemann function of the corresponding characteristic initial value problem. However, Riemann functions can be explicitly written only in several simple cases and thus we need to find another solution to (1.2) which would be expressed effectively. By using a certain "two-dimensional analogy" of the well-known Cauchy formula for ODEs we arrive at the function

$$\gamma(t,x) = \int_a^t\int_c^x q(s,\eta)\,\mathrm{e}^{\int_s^t\int_\eta^x p(\xi_1,\xi_2)\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}\eta\mathrm{d}s \quad \text{for } (t,x)\in[a,b]\times[c,d]. \tag{1.3}$$

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We need to show that this function is absolutely continuous in the sense of Carathéodory[1] and that satisfies inequality (1.2) almost everywhere in $[a,b]\times[c,d]$. Let us mention that if the coefficients p and q are continuous, the problem indicated is not difficult. If p and q are discontinuous, the situation is much more complicated and we have not found any results applicable to this particular problem in the existing literature. In this paper, we adapt and extend known results in order to solve our problem. More precisely, we establish Theorem 2.7 guaranteeing the absolute continuity of the function

$$\lambda \mapsto \int_c^{\varphi(\lambda)} f(t,\lambda)\,\mathrm{d}t$$

and giving a formula for its derivative. Then, in Theorem 2.9, we investigate the question on the existence of partial derivatives of the function

$$(\lambda,\mu) \mapsto \int_a^{\lambda} h(t,\lambda,\mu)\,\mathrm{d}t. \tag{1.4}$$

The results obtained are applied to solve the above-mentioned problem (see Corollary 2.13) concerning partial differential inequality (1.2), because the function γ defined by relation (1.3) is a particular case of mapping (1.4).

The following notation will be used throughout the paper: $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the sets of all natural, rational, and real numbers, respectively, $\mathbb{R}^+=[0,+\infty[$, and for any $x\in\mathbb{R}$ we put $[x]_+=(|x|+x)/2$ and $[x]_-=(|x|-x)/2$. If $\Omega\subset\mathbb{R}^n$ is a measurable set then $\operatorname{meas}\Omega$ denotes the Lebesgue measure of Ω and $L(\Omega;\mathbb{R})$ stands for the space of Lebesgue integrable functions $p\colon\Omega\to\mathbb{R}$ endowed with the norm $\|p\|_L=\int_\Omega|p(x)|\,\mathrm{d}x$. Moreover, the partial derivatives of the function $u\colon\Omega\to\mathbb{R}$ at the point $x\in\Omega$ are denoted by

$$u'_{[k]}(x_1,\dots,x_n)=\frac{\partial u(x_1,\dots,x_n)}{\partial x_k} \qquad \text{for } k\in\{1,\dots,n\},$$

$$u''_{[k,\ell]}(x_1,\dots,x_n)=\frac{\partial^2 u(x_1,\dots,x_n)}{\partial x_k\partial x_\ell} \qquad \text{for } k,\ell\in\{1,\dots,n\}.$$

At last, $AC([\alpha,\beta];\mathbb{R})$ stands for the set of absolutely continuous functions on the interval $[\alpha,\beta]\subset\mathbb{R}$.

2. Main results

It is well known that combination of absolutely continuous functions might not be absolutely continuous. Therefore, before formulating of the main results (namely, Theorems 2.7 and 2.9) we present the following rather simple statement which we will need afterwards.

Proposition 2.1. *Let $\varphi\in AC([a,b];\mathbb{R})$ and $f\in AC([c,d];\mathbb{R})$, where $[c,d]=\varphi([a,b])$. Put*

$$F(t):=f\big(\varphi(t)\big) \quad \text{for } t\in[a,b]. \tag{2.1}$$

Then the following assertions are satisfied:

[1]This notion is defined in [1] (see also Lemma 3.1 below).

(a) *The relation*

$$F'(t) = f'\big(\varphi(t)\big)\varphi'(t) \quad \textit{for all } t \in \varphi^{-1}(E_1) \cap E_2$$

holds, where $E_1 = \{x \in [c,d] : \textit{there exists } f'(x)\}$ *and* $E_2 = \{t \in [a,b] : \textit{there exists } \varphi'(t)\}$.

(b) *If the function* φ *is monotone (not strictly, in general) then the function* F *is absolutely continuous.*

(c) *If the function* φ *is strictly monotone then*

$$F'(t) = f'\big(\varphi(t)\big)\varphi'(t) \quad \textit{for a.e. } t \in [a,b].^{2} \tag{2.2}$$

Remark 2.2. Let the function φ in Proposition 2.1 be strictly monotone. Then the set $\varphi^{-1}(E_1)$ in the part (a) is measurable (without any additional assumption) and $\operatorname{meas}\varphi^{-1}(E_1) = b - a$ if and only if the inverse function φ^{-1} is absolutely continuous (see, e. g., [4, Chapter IX, §3, Theorems 3 and 4]). Therefore, even in this case, part (c) does not follow, in general, from part (a), because the function φ^{-1} might not be absolutely continuous (see [5, Section 2]).

Corollary 2.3. *Let* $\varphi \in AC([a,b];\mathbb{R})$ *be a strictly monotone function and* $g \in L([c,d];\mathbb{R})$*, where* $[c,d] = \varphi([a,b])$*. Put*

$$F(t) = \int_c^{\varphi(t)} g(s)\,\mathrm{d}s \quad \textit{for } t \in [a,b]. \tag{2.3}$$

Then the function F *is absolutely continuous and*

$$F'(t) = g\big(\varphi(t)\big)\varphi'(t) \quad \textit{for a.e. } t \in [a,b].^{3} \tag{2.4}$$

Conditions guaranteeing that Leibniz's rule (1.1) for the Lebesgue integral is applicable at some particular point are well known. We mention here, for example, the following statement.

Proposition 2.4 ([2, Chapter V, Section 247])**.** *Let the function* $f\colon [c,d] \times [a,b] \to \mathbb{R}$ *satisfy the relations*

$$f(\cdot, x) \in L([c,d];\mathbb{R}) \qquad \textit{for all } x \in [a,b], \tag{2.5}$$

$$f(t, \cdot) \in AC([a,b];\mathbb{R}) \quad \textit{for a.e. } t \in [c,d], \tag{2.6}$$

and

$$f'_{[2]} \in L([c,d] \times [a,b];\mathbb{R}).^{4} \tag{2.7}$$

Moreover, let $\lambda_0 \in [a,b]$ *be such that*

$$\textit{the function } \int_c^d f'_{[2]}(t,\cdot)\,\mathrm{d}t\colon [a,b] \to \mathbb{R} \textit{ is continuous at the point } \lambda_0. \tag{2.8}$$

[2]In order to ensure that relation (2.2) is meaningful we put $f'(x) := \alpha(x)$ at those points $x \in [c,d]$, where the derivative of the function f does not exist, $\alpha\colon [c,d] \to \mathbb{R}$ beeing an arbitrary function. Observe that a choice of the function α has no influence on the value of the right-hand side of equality (2.2) (see Lemma 3.2 below).

[3]In order to ensure that relation (2.4) is meaningful we put $g(x) := \omega(x)$ at those points $x \in [c,d]$, where the function g is not defined, $\omega\colon [c,d] \to \mathbb{R}$ being an arbitrary function. Observe that a choice of the function ω has no influence on the value of the right-hand side of equality (2.4) (see Lemma 3.2 below).

[4]See Remark 2.5.

Put

$$F(\lambda) := \int_c^d f(t,\lambda)\,\mathrm{d}t \quad \text{for } \lambda \in [a,b]. \tag{2.9}$$

Then the function F is differentiable at the point λ_0 and

$$F'(\lambda_0) = \int_c^d f'_{[2]}(t,\lambda_0)\,\mathrm{d}t.$$

Remark 2.5. It follows from assumption (2.6) that there exists $f'_{[2]}(t,x)$ for all $(t,x) \in \Omega := \{(s,\eta) : s \in E,\ \eta \in A(s)\}$, where $E \subseteq [c,d]$ with $\operatorname{meas} E = d - c$ and, for any $s \in E$, we have $A(s) \subseteq [a,b]$ with $\operatorname{meas} A(s) = b - a$. Note that, in general, the set Ω might not be measurable. Clearly, in assumption (2.7) we require that the function $f'_{[2]}$ is defined (i.e., the partial derivative exists) almost everywhere in the rectangle $[c,d] \times [a,b]$. It is worth mentioning here that this assumption follows, e. g., from the existence of a function $g \in L([c,d] \times [a,b]; \mathbb{R})$ such that $f'_{[2]} \equiv g$ on Ω (see Lemma 3.5 below).

If we are not interested in differentiability of the function F at particular points, continuity assumption (2.8) in Proposition 2.4 can be omitted and thus we obtain the following result.

Proposition 2.6. *Let $f\colon [c,d] \times [a,b] \to \mathbb{R}$ be a function satisfying relations* (2.5)–(2.7)*. Then the function F defined by formula* (2.9) *is absolutely continuous on the interval $[a,b]$ and*

$$F'(\lambda) = \int_c^d f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for a.e. } \lambda \in [a,b]. \tag{2.10}$$

If we add a variable upper boundary of the integral in (2.9), we obtain

Theorem 2.7. *Let the functions $\varphi \in AC([a,b];\mathbb{R})$ and $f\colon [c,d] \times [a,b] \to \mathbb{R}$ be such that relations* (2.5)–(2.7) *hold and $\varphi([a,b]) = [c,d]$. Put*

$$F(\lambda) := \int_c^{\varphi(\lambda)} f(t,\lambda)\,\mathrm{d}t \quad \text{for } \lambda \in [a,b]. \tag{2.11}$$

Then the following assertions are satisfied:

(a) *There exist sets $E_1 \subseteq [c,d]$ and $E_2 \subseteq [a,b]$ such that* $\operatorname{meas} E_1 = d - c$, $\operatorname{meas} E_2 = b - a$, *and*

$$F'(\lambda) = f\big(\varphi(\lambda),\lambda\big)\varphi'(\lambda) + \int_c^{\varphi(\lambda)} f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for all } \lambda \in \varphi^{-1}(E_1) \cap E_2.$$

(b) *If the function φ is monotone (not strictly, in general) then the function F is absolutely continuous on the interval $[a,b]$.*

(c) *If the function φ is strictly monotone then*

$$F'(\lambda) = f\big(\varphi(\lambda),\lambda\big)\varphi'(\lambda) + \int_c^{\varphi(\lambda)} f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for a.e. } \lambda \in [a,b].^5 \tag{2.12}$$

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[5] In order to ensure that relation (2.12) is meaningful we put $f(t,x) := \omega(t,x)$ at those points $(t,x) \in [c,d] \times [a,b]$, where the function f is not defined, $\omega\colon [c,d] \times [a,b] \to \mathbb{R}$ being an arbitrary function. Observe that a choice of the function ω has no influence on the value of the right-hand side of equality (2.12) (see Lemma 3.2 below).

Remark 2.8. Let the function φ in Theorem 2.7 is strictly monotone. Analogously to Remark 2.2 we can mention that relation (2.12) follows from part (a) if the inverse function φ^{-1} is absolutely continuous. In particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(s,t)\,\mathrm{d}s = f(t,t) + \int_a^t f'_{[2]}(s,t)\,\mathrm{d}s \quad \text{for a.e. } t\in[a,b]$$

whenever the function f satisfies relations (2.5)–(2.7) with $a=c$ and $b=d$.

As we have mentioned above, we need to show that the function γ defined by formula (1.3) is a Carathéodory solution to differential inequality (1.2). In particular, we have to show that the function γ is absolutely continuous on $[a,b]\times[c,d]$ in the sense of Carathéodory which, in view of Lemma 3.1, requires to derive formulas for partial derivatives of the function (1.4) with respect to each variable. For this purpose we establish the following statement which will be applied to prove Corollary 2.12 below.

Theorem 2.9. *Let $h\colon [a,b]\times[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the relations*

$$h(\cdot,x,z)\in L([a,b];\mathbb{R}) \qquad \textit{for all } (x,z)\in[a,b]\times[c,d], \tag{2.13}$$

$$h(t,\cdot,z)\in AC([a,b];\mathbb{R}) \quad \textit{for a.e. } t\in[a,b] \textit{ and all } z\in[c,d], \tag{2.14}$$

and

$$h'_{[2]}(\cdot,\cdot,z)\in L([a,b]\times[a,b];\mathbb{R}) \quad \textit{for all } z\in[c,d]^{6} \tag{2.15}$$

are satisfied. Put

$$H(\lambda,\mu) := \int_a^{\lambda} h(t,\lambda,\mu)\,\mathrm{d}t \quad \textit{for all } (\lambda,\mu)\in[a,b]\times[c,d]. \tag{2.16}$$

Then the following assertions are satisfied:

(a) *For any $\mu\in[c,d]$ fixed, we have $H(\cdot,\mu)\in AC([a,b];\mathbb{R})$ and*

$$H'_{[1]}(\lambda,\mu) = h(\lambda,\lambda,\mu) + \int_a^{\lambda} h'_{[2]}(t,\lambda,\mu)\,\mathrm{d}t \quad \textit{for a.e. } \lambda\in[a,b]. \tag{2.17}$$

(b) *Let, in addition to (2.13)–(2.15), there exist a number $k\in\{0,1\}$ such that*

$$(-1)^k h(t,x,\cdot)\colon [c,d]\to\mathbb{R} \textit{ is non-decreasing for all } x\in[a,b] \textit{ and a.e. } t\in[a,x], \tag{2.18}$$

$$(-1)^k h'_{[2]}(t,x,\cdot)\colon [c,d]\to\mathbb{R} \textit{ is non-decreasing for a.e. } (t,x)\in[a,b]\times[a,b],\ t\le x, \tag{2.19}$$

and

$$h(x,x,\cdot)\colon [c,d]\to\mathbb{R} \textit{ is continuous for a.e. } x\in[a,b], \tag{2.20}$$

$$\int_a^x h'_{[2]}(t,x,\cdot)\,\mathrm{d}t\colon [c,d]\to\mathbb{R} \textit{ is continuous for a.e. } x\in[a,b]. \tag{2.21}$$

[6] See Remark 2.10.

Then there exists a set $E_1 \subseteq [a,b]$ *such that* $\operatorname{meas} E_1 = b-a$ *and*

$$H'_{[1]}(\lambda,\mu) = h(\lambda,\lambda,\mu) + \int_a^\lambda h'_{[2]}(t,\lambda,\mu)\,\mathrm{d}t \quad \text{for all } \lambda \in E_1,\ \mu \in [c,d]. \tag{2.22}$$

(c) *Let, in addition to* (2.13)–(2.15) *and* (2.18)–(2.21), *for any* $x \in E_1$ *the function* h *satisfy*

$$h(x,x,\cdot) \in AC([c,d];\mathbb{R}), \tag{2.23}$$

$$h'_{[2]}(t,x,\cdot) \in AC([c,d];\mathbb{R}) \quad \text{for a.e. } t \in [a,x], \tag{2.24}$$

and

$$h''_{[2,3]}(\cdot,x,\cdot) \in L([a,x]\times[c,d];\mathbb{R}).^7 \tag{2.25}$$

Then, for any $\lambda \in E_1$ *fixed, we have* $H'_{[1]}(\lambda,\cdot) \in AC([c,d];\mathbb{R})$ *and*

$$H''_{[1,2]}(\lambda,\mu) = h'_{[3]}(\lambda,\lambda,\mu) + \int_a^\lambda h''_{[2,3]}(t,\lambda,\mu)\,\mathrm{d}t \quad \text{for all } \mu \in E_2(\lambda), \tag{2.26}$$

where $E_2(\lambda) \subseteq [c,d]$ *is such that* $\operatorname{meas} E_2(\lambda) = d-c$.

(d) *If, in addition to* (2.13)–(2.15), (2.18)–(2.21), *and* (2.23)–(2.25), *there is a function* $g \in L([a,b]\times[c,d];\mathbb{R})$ *such that*

$$g(x,z) = h'_{[3]}(x,x,z) + \int_a^x h''_{[2,3]}(t,x,z)\,\mathrm{d}t \qquad \text{for all } x \in E_1 \text{ and } z \in E_2(x), \tag{2.27}$$

then there exists $H''_{[1,2]}$ *almost everywhere on* $[a,b]\times[c,d]$ *and*

$$H''_{[1,2]}(\lambda,\mu) = g(\lambda,\mu) \quad \text{for a.e. } (\lambda,\mu) \in [a,b]\times[c,d]. \tag{2.28}$$

Remark 2.10. It follows from assumption (2.14) that, for any $z \in [c,d]$ fixed, there exists $h'_{[2]}(t,x,z)$ for all $(t,x) \in \Omega_z := \{(s,\eta) : s \in E_z,\ \eta \in B_z(s)\}$, where $E_z \subseteq [a,b]$ with $\operatorname{meas} E_z = b-a$ and, for any $s \in E_z$, we have $B_z(s) \subseteq [a,b]$ with $\operatorname{meas} B_z(s) = b-a$. Note that, in general, the set Ω_z might not be measurable. Clearly, in assumption (2.15) we require that, for every $z \in [c,d]$, the function $h'_{[2]}(\cdot,\cdot,z)$ is defined (i.e., the partial derivative exists) almost everywhere in the square $[a,b]\times[a,b]$. It is worth mentioning here that this assumption follows, e. g., from the existence of a function $g_z \in L([a,b]\times[a,b];\mathbb{R})$ such that $h'_{[2]}(\cdot,\cdot,z) \equiv g_z$ on Ω_z (see Lemma 3.5 below with $a=c$, $b=d$, and $f \equiv h(\cdot,\cdot,z)$).

Remark 2.11. Inclusion (2.25) is understood in the sense, which is analogous to that concerning inclusion (2.15) explained in Remark 2.10.

Now we apply Theorem 2.9 to the function γ defined by relation (1.3).

Corollary 2.12. *Let the function* $\gamma\colon [a,b]\times[c,d] \to \mathbb{R}$ *be defined by formula* (1.3), *where* $p,q \in L([a,b]\times[c,d];\mathbb{R}^+)$. *Then the following assertions are satisfied:*

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[7] See Remark 2.11.

(i) $\gamma(\cdot,x) \in AC([a,b];\mathbb{R})$ *for every* $x \in [c,d]$ *and the relation*

$$\gamma'_{[1]}(t,x) = \int_c^x q(t,\eta)\,\mathrm{d}\eta + \int_a^t \int_c^x q(s,\eta)\left(\int_\eta^x p(t,\xi_2)\,\mathrm{d}\xi_2\right) \mathrm{e}^{\int_s^t \int_\eta^x p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}\eta \mathrm{d}s \tag{2.29}$$

holds for a.e. $t \in [a,b]$ *and all* $x \in [c,d]$.

(ii) $\gamma(t,\cdot) \in AC([c,d];\mathbb{R})$ *for every* $t \in [a,b]$ *and the relation*

$$\gamma'_{[2]}(t,x) = \int_a^t q(s,x)\,\mathrm{d}s + \int_c^x \int_a^t q(s,\eta)\left(\int_s^t p(\xi_1,x)\,\mathrm{d}\xi_1\right) \mathrm{e}^{\int_s^t \int_\eta^x p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \mathrm{d}\eta \tag{2.30}$$

holds for all $t \in [a,b]$ *and a.e.* $x \in [c,d]$.

(iii) $\gamma'_{[1]}(t,\cdot) \in AC([c,d];\mathbb{R})$ *for a.e.* $t \in [a,b]$ *and the relation*

$$\gamma''_{[1,2]}(t,x) = q(t,x) + \int_a^t \int_c^x q(s,\eta) f(s,\eta,t,x)\, \mathrm{e}^{\int_s^t \int_\eta^x p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}\eta \mathrm{d}s \tag{2.31}$$

holds for a.e. $(t,x) \in [a,b] \times [c,d]$, *where*

$$f(s,\eta,t,x) := p(t,x) + \left(\int_s^t p(\xi_1,x)\,\mathrm{d}\xi_1\right)\left(\int_\eta^x p(t,\xi_2)\,\mathrm{d}\xi_2\right). \tag{2.32}$$

(iv) $\gamma'_{[2]}(\cdot,x) \in AC([a,b];\mathbb{R})$ *for a.e.* $x \in [c,d]$ *and the relation*

$$\gamma''_{[2,1]}(t,x) = q(t,x) + \int_c^x \int_a^t q(s,\eta) f(s,\eta,t,x)\, \mathrm{e}^{\int_\eta^x \int_s^t p(\xi_1,\xi_2)\,\mathrm{d}\xi_1 \mathrm{d}\xi_2}\,\mathrm{d}s \mathrm{d}\eta \tag{2.33}$$

holds for a.e. $(t,x) \in [a,b] \times [c,d]$, *where the function* f *is defined by formula* (2.32).

(v) $\gamma''_{[1,2]}, \gamma''_{[2,1]} \in L([a,b] \times [c,d];\mathbb{R})$ *and*

$$\gamma''_{[1,2]}(t,x) = \gamma''_{[2,1]}(t,x) \quad \text{for a.e. } (t,x) \in [a,b] \times [c,d]. \tag{2.34}$$

Corollary 2.13. *Let* $p,q \in L([a,b] \times [c,d];\mathbb{R}^+)$. *Then the function* γ *defined by relation* (1.3) *is a Carathéodory solution to differential inequality* (1.2).

3. Auxiliary statements

Lemma 3.1 ([6, Theorem 3.1])**.** *Let* $u\colon [a,b] \times [c,d] \to \mathbb{R}$ *be a function of two variables. Then the following assertions are equivalent:*

(1) *The function* u *is absolutely continuous on the rectangle* $[a,b] \times [c,d]$ *in the sense of Carathéodory.*[8]

(2) *The function* u *satisfies the relations:*
   (a) $u(\cdot,x) \in AC([a,b];\mathbb{R})$ *for every* $x \in [c,d]$ *and* $u(a,\cdot) \in AC([c,d];\mathbb{R})$,
   (b) $u'_{[1]}(t,\cdot) \in AC([c,d];\mathbb{R})$ *for a.e.* $t \in [a,b]$,
   (c) $u''_{[1,2]} \in L([a,b] \times [c,d];\mathbb{R})$.

(3) *The function* u *satisfies the relations:*

[8]This notion is defined in [1] (see also [6] and references therein).

(A) $u(t,\cdot) \in AC([c,d];\mathbb{R})$ *for every* $t \in [a,b]$ *and* $u(\cdot,c) \in AC([a,b];\mathbb{R})$,
(B) $u'_{[2]}(\cdot,x) \in AC([a,b];\mathbb{R})$ *for a.e.* $x \in [c,d]$,
(C) $u''_{[2,1]} \in L([a,b]\times[c,d];\mathbb{R})$.

Lemma 3.2 ([4, Chapter IX, §5, Lemma 2])**.** *Let* $\varphi \in AC([a,b];\mathbb{R})$ *be an increasing function and* $E \subseteq [\varphi(a),\varphi(b)]$ *be such that* $\operatorname{meas} E = 0$. *Then*

$$\operatorname{meas}\big\{t \in [a,b] : \varphi(t) \in E \text{ and the relation } \varphi'(t) = 0 \text{ does not hold}\big\} = 0.$$

Lemma 3.3 ([4, Chapter IX, §5, Theorem])**.** *Let* $\varphi \in AC([a,b];\mathbb{R})$ *be an increasing function and* $h \in L([\varphi(a),\varphi(b)];\mathbb{R})$. *Then*

$$\int_{\varphi(a)}^{\varphi(b)} h(x)\,\mathrm{d}x = \int_a^b h\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t.^{9} \tag{3.1}$$

Lemma 3.4 ([6, Proposition 3.5])**.** *Let* $g \in L([c,d]\times[a,b];\mathbb{R})$ *and*

$$G(t,x) := \int_a^x g(t,\eta)\,\mathrm{d}\eta \quad \text{for } t \in E,\ x \in [a,b], \tag{3.2}$$

where $E \subseteq [c,d]$ *with* $\operatorname{meas} E = d - c$. *Then*

$$G'_{[2]}(t,x) = g(t,x) \quad \text{for a.e. } (t,x) \in [c,d]\times[a,b].$$

Lemma 3.5. *Let the function* $f\colon [c,d]\times[a,b] \to \mathbb{R}$ *satisfy*

$$f(t,\cdot) \in AC([a,b];\mathbb{R}) \quad \text{for all } t \in E \subseteq [c,d],\ \operatorname{meas} E = d-c, \tag{3.3}$$

and there exist a function $g \in L([c,d]\times[a,b];\mathbb{R})$ *such that*

$$f'_{[2]}(t,x) = g(t,x) \quad \text{for all } t \in E \text{ and } x \in A(t), \tag{3.4}$$

where $A(t) \subseteq [a,b]$ *with* $\operatorname{meas} A(t) = b - a$. *Then the partial derivative* $f'_{[2]}$ *exits almost everywhere in* $[c,d]\times[a,b]$ *and*

$$f'_{[2]}(t,x) = g(t,x) \quad \text{for a.e. } (t,x) \in [c,d]\times[a,b]. \tag{3.5}$$

Proof. Assumptions (3.3) and (3.4) yield that

$$f(t,x) = f(t,a) + \int_a^x f'_{[2]}(t,\eta)\,\mathrm{d}\eta = f(t,a) + \int_a^x g(t,\eta)\,\mathrm{d}\eta \quad \text{for all } t \in E,\ x \in [a,b],$$

and thus desired relation (3.5) follows from Lemma 3.4. ☐

The next lemma is a direct generalisation of the result obtained by Tolstov in [7, §7] (see also [6, Proof of Proposition 3.5(i)]).

Lemma 3.6. *Let* $g\colon [c,d]\times[a,b] \to \mathbb{R}$ *be such that*

$$g(t,\cdot) \in L([a,b];\mathbb{R}^+) \quad \text{for a.e. } t \in [c,d] \tag{3.6}$$

and

$$\int_a^x g(\cdot,\eta)\,\mathrm{d}\eta \in L([c,d];\mathbb{R}^+) \quad \text{for all } x \in [a,b]. \tag{3.7}$$

---

[9] In order to ensure that relation (3.1) is meaningful we put $h(x) := \alpha(x)$ at those points $x \in [\varphi(a),\varphi(b)]$, where the function h is not defined, $\alpha\colon [\varphi(a),\varphi(b)] \to \mathbb{R}$ being an arbitrary function. Observe that a choice of the function α has no influence on the value of the right-hand side of equality (3.1) (see Lemma 3.2).

Put

$$G(t,x) := \int_c^t \left(\int_a^x g(s,\eta)\,\mathrm{d}\eta \right) \mathrm{d}s \quad \textit{for } (t,x) \in [c,d] \times [a,b]. \tag{3.8}$$

Then there exists a set $E \subseteq [c,d]$ *such that* $\operatorname{meas} E = d - c$ *and*

$$G'_{[1]}(t,x) = \int_a^x g(t,\eta)\,\mathrm{d}\eta \quad \textit{for all } t \in E \textit{ and } x \in [a,b]. \tag{3.9}$$

The following lemma concerns the so-called Carathéodory functions and gives a result which is well known (see, e. g., [1, §576]).

Lemma 3.7. *Let* $f\colon [a,b] \times [c,d] \times \mathbb{R}^2 \to \mathbb{R}$ *be such that*

$$f(\cdot,\cdot,\alpha,\beta)\colon [a,b] \times [c,d] \to \mathbb{R} \textit{ is measurable for all } (\alpha,\beta) \in \mathbb{R}^2, \tag{3.10}$$

$$f(x,z,\cdot,\cdot)\colon \mathbb{R}^2 \to \mathbb{R} \textit{ is continuous for a.e. } (x,z) \in [a,b] \times [c,d], \tag{3.11}$$

and let $u, v\colon [a,b]\times[c,d] \to \mathbb{R}$ *be measurable functions. Then the function* h *defined by the relation*

$$h(x,z) := f\big(x,z,u(x,z),v(x,z)\big) \tag{3.12}$$

is measurable on the rectangle $[a,b] \times [c,d]$.

At last, we formulate a lemma which can be found in Carathéodory's monograph [1] (see also [6, Lemma 3.1]).

Lemma 3.8. *Let* $g \in L([a,b] \times [c,d];\mathbb{R})$. *Then the function* G *defined by formula* (3.2) *is measurable on the rectangle* $[a,b] \times [c,d]$.

4. Proofs of main results

Proof of Proposition 2.1. (a) The assertion follows immediately from the rule for differentiation of composite functions.

(b) It can be proved easily by using the definition of absolutely continuous functions.

(c) Assume that the function φ is increasing (if it is decreasing, the proof is analogous). Then Lemma 3.3 yields that $f'\big(\varphi(\cdot)\big)\varphi'(\cdot) \in L([a,b];\mathbb{R})$ and

$$F(t) - F(a) = f\big(\varphi(t)\big) - f\big(\varphi(a)\big) = \int_{\varphi(a)}^{\varphi(t)} f'(x)\,\mathrm{d}x = \int_a^t f'\big(\varphi(s)\big)\varphi'(s)\,\mathrm{d}s$$

for all $t \in [a,b]$, which gives desired relation (2.2). □

Proof of Corollary 2.3. At first we put $g(x) := \omega(x)$ for those $x \in [c,d]$ in which the function g is not defined, where ω is the function from footnote in our corollary. Put

$$f(x) := \int_c^x g(s)\,\mathrm{d}s \quad \text{for } x \in [c,d].$$

It is clear $F(t) = f\big(\varphi(t)\big)$ for all $t \in [a,b]$, the function f is absolutely continuous, and $f'(x) = g(x)$ for all $x \in A$, where $A \subseteq [c,d]$ with $\operatorname{meas} A = d - c$.

On the other hand, by using Proposition 2.1, we get a set $E \subseteq [a,b]$ such that $\operatorname{meas} E = b - a$ and

$$F'(t) = f'\big(\varphi(t)\big)\varphi'(t) \quad \text{for all } t \in E, \tag{4.1}$$

where we put $f'(x) := g(x)$ at those points $x \in [c,d]$ in which the derivative of the function f does not exist. Consequently, we have

$$F'(t) = g\big(\varphi(t)\big)\varphi'(t) \quad \text{for all } t \in E \cap \varphi^{-1}(A). \tag{4.2}$$

However, it follows from Lemma 3.2 that

$$\operatorname{meas}\Big\{t \in E : \varphi(t) \notin A \text{ and the relation } \varphi'(t) = 0 \text{ does not hold}\Big\} = 0$$

and thus equalities (4.1) and (4.2) yield the validity of desired relation (2.4). □

Proof of Proposition 2.6. By using assumption (2.5)–(2.7) and Fubini's theorem, we get

$$\begin{aligned}\int_a^\lambda \left(\int_c^d f'_{[2]}(t,x)\,\mathrm{d}t\right)\mathrm{d}x &= \int_c^d \left(\int_a^\lambda f'_{[2]}(t,x)\,\mathrm{d}x\right)\mathrm{d}t \\ &= \int_c^d \Big[f(t,\lambda) - f(t,a)\Big]\,\mathrm{d}t = F(\lambda) - F(a)\end{aligned}$$

for all $\lambda \in [a,b]$. Consequently, the function F is absolutely continuous and desired relation (2.10) holds because we have $\int_c^d f'_{[2]}(t,\cdot)\,\mathrm{d}t \in L([a,b];\mathbb{R})$. □

Proof of Theorem 2.7. (a) Let

$$H(\mu,\lambda) := \int_c^\mu f(t,\lambda)\,\mathrm{d}t \quad \text{for } (\mu,\lambda) \in [c,d] \times [a,b].$$

Then $F(\lambda) = H\big(\varphi(\lambda),\lambda\big)$ for all $\lambda \in [a,b]$ and, in view of assumptions (2.5)–(2.7), we get

$$\begin{aligned}H(\mu,\lambda) &= \int_c^\mu \left(\int_a^\lambda f'_{[2]}(t,x)\,\mathrm{d}x\right)\mathrm{d}t + \int_c^\mu f(t,a)\,\mathrm{d}t \\ &= \int_a^\lambda \left(\int_c^\mu f'_{[2]}(t,x)\,\mathrm{d}t\right)\mathrm{d}x + \int_c^\mu f(t,a)\,\mathrm{d}t \quad \text{for all } (\mu,\lambda) \in [c,d] \times [a,b].\end{aligned}$$

Therefore, Lemma 3.6 guarantees that there exists a set $E_1 \subseteq [c,d]$ with $\operatorname{meas} E_1 = d - c$ such that

$$H'_{[1]}(\mu,\lambda) = \int_a^\lambda f'_{[2]}(\mu,x)\,\mathrm{d}x + f(\mu,a) = f(\mu,\lambda) \quad \text{for all } \mu \in E_1,\ \lambda \in [a,b],$$

and that there is a set $E_2 \subseteq [a,b]$ such that $\operatorname{meas} E_2 = b - a$, there exists $\varphi'(\lambda)$ for every $\lambda \in E_2$, and

$$H'_{[2]}(\mu,\lambda) = \int_c^\mu f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for all } \mu \in [c,d],\ \lambda \in E_2.$$

Consequently, we obtain

$$\begin{aligned}F'(\lambda) &= H'_{[1]}\big(\varphi(\lambda),\lambda\big)\varphi'(\lambda) + H'_{[2]}\big(\varphi(\lambda),\lambda\big) \\ &= f\big(\varphi(\lambda),\lambda\big)\varphi'(\lambda) + \int_c^{\varphi(\lambda)} f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for all } \lambda \in \varphi^{-1}(E_1) \cap E_2.\end{aligned}$$

(b) Assume that the function φ is non-decreasing (if it is non-increasing, the proof is analogous) and let $\varepsilon > 0$ be arbitrary. Then, in view of assumptions (2.5) and (2.7), there exists $\omega > 0$ such that

$$\iint_E \big|f'_{[2]}(t,x)\big| \mathrm{d}t\mathrm{d}x < \frac{\varepsilon}{3} \quad \text{for all } E \subseteq [c,d]\times[a,b],\ \operatorname{meas} E < \omega \tag{4.3}$$

and

$$\int_I |f(t,a)|\,\mathrm{d}t < \frac{\varepsilon}{3} \quad \text{for all } I \subseteq [c,d],\ \operatorname{meas} I < \omega. \tag{4.4}$$

Moreover, there exist a number $0 < \delta \le \omega/(d-c)$ such that, for an arbitrary system $\big\{]a_k,b_k[\big\}_{k=1}^m$ of mutually disjoint subintervals of $[a,b]$ satisfying relation $\sum_{k=1}^m (b_k - a_k) < \delta$, we have

$$\sum_{k=1}^m \big|\varphi(b_k) - \varphi(a_k)\big| < \frac{\omega}{\max\{1, b-a\}}\,. \tag{4.5}$$

Now let $\big\{]a_k,b_k[\big\}_{k=1}^m$ be an arbitrary system of mutually disjoint subintervals of $[a,b]$ with property $\sum_{k=1}^m (b_k - a_k) < \delta$. Then inequality (4.5) holds, $\big\{[c,d]\times[a_k,b_k]\big\}_{k=1}^m$ and $\big\{[\varphi(a_k),\varphi(b_k)]\times[a,b]\big\}_{k=1}^m$ form systems of non-overlapping rectangles contained in $[c,d]\times[a,b]$, and $\big\{[\varphi(a_k),\varphi(b_k)]\big\}_{k=1}^m$ is a system of non-overlapping subintervals of $[c,d]$. According to assumptions (2.5)–(2.7), it is easy to verify that, for any $k = 1,\dots,m$, we have

$$\begin{aligned}
F(b_k) - F(a_k) &= \int_c^{\varphi(b_k)} f(t,b_k)\,\mathrm{d}t - \int_c^{\varphi(a_k)} f(t,a_k)\,\mathrm{d}t \\
&= \int_c^{\varphi(a_k)} \left(\int_{a_k}^{b_k} f'_{[2]}(t,x)\,\mathrm{d}x\right)\mathrm{d}t + \int_{\varphi(a_k)}^{\varphi(b_k)} f(t,b_k)\,\mathrm{d}t \\
&= \int_c^{\varphi(a_k)} \left(\int_{a_k}^{b_k} f'_{[2]}(t,x)\,\mathrm{d}x\right)\mathrm{d}t \\
&\quad + \int_{\varphi(a_k)}^{\varphi(b_k)} \left(\int_a^{b_k} f'_{[2]}(t,x)\,\mathrm{d}x\right)\mathrm{d}t + \int_{\varphi(a_k)}^{\varphi(b_k)} f(t,a)\,\mathrm{d}t
\end{aligned}$$

and thus, in view of relation (4.5), we get

$$\sum_{k=1}^m \big|F(b_k) - F(a_k)\big| \le \iint_{A_1} |f'_{[2]}(t,x)|\,\mathrm{d}t\mathrm{d}x + \iint_{A_2} |f'_{[2]}(t,x)|\,\mathrm{d}t\mathrm{d}x + \int_{A_3} |f(t,a)|\,\mathrm{d}t,$$

where

$$\begin{aligned}
\operatorname{meas} A_1 &= \sum_{k=1}^m (d-c)(b_k - a_k) < (d-c)\delta \le \omega, \\
\operatorname{meas} A_2 &= \sum_{k=1}^m \big(\varphi(b_k) - \varphi(a_k)\big)(b-a) < \frac{\omega(b-a)}{\max\{1,b-a\}} \le \omega, \\
\operatorname{meas} A_3 &= \sum_{k=1}^m \big(\varphi(b_k) - \varphi(a_k)\big) < \frac{\omega}{\max\{1,b-a\}} \le \omega.
\end{aligned}$$

Consequently, relations (4.3) and (4.4) yield that $\sum_{k=1}^m \big|F(b_k) - F(a_k)\big| < \varepsilon$ and thus the function F is absolutely continuous.

(c) Assume that the function φ is increasing (if it is decreasing, the proof is analogous). It follows from the assumptions imposed on φ and f that there exist $\varphi'(t)$ and $f'_{[2]}(t,x)$ for a.e. $t \in [a,b]$ and a.e. $(t,x) \in [c,d] \times [a,b]$, respectively. In order to ensure that all relations below are meaningful we put $\varphi'(t) := 0$ and $f'_{[2]}(t,x) := 0$ at those points in which the derivatives indicated do not exist. In such a way, the functions φ' and $f'_{[2]}$ are defined everywhere on $[a,b]$ and $[c,d] \times [a,b]$, respectively.

Let $E_1 \subseteq [a,b]$, $\operatorname{meas} E_1 = b-a$, be the set such that $f'_{[2]}(\cdot,x) \in L([c,d];\mathbb{R})$ for every $x \in E_1$. Put

$$h(\lambda,x) := \int_c^{\varphi(\lambda)} f'_{[2]}(t,x)\,\mathrm{d}t \quad \text{for all } \lambda \in [a,b] \text{ and } x \in E_1. \tag{4.6}$$

Clearly, we have

$$h(\lambda,\cdot) \in L([a,b];\mathbb{R}) \quad \text{for all } \lambda \in [a,b]. \tag{4.7}$$

Then, by using Fubini's theorem, we get

$$\begin{aligned} F(\lambda) &= \int_c^{\varphi(\lambda)} f(t,a)\,\mathrm{d}t + \int_c^{\varphi(\lambda)} \left(\int_a^{\lambda} f'_{[2]}(t,x)\,\mathrm{d}x \right) \mathrm{d}t \\ &= \int_c^{\varphi(\lambda)} f(t,a)\,\mathrm{d}t + \int_a^{\lambda} \left(\int_c^{\varphi(\lambda)} f'_{[2]}(t,x)\,\mathrm{d}t \right) \mathrm{d}x \\ &= \int_c^{\varphi(\lambda)} f(t,a)\,\mathrm{d}t + \int_a^{\lambda} h(\lambda,x)\,\mathrm{d}x \quad \text{for all } \lambda \in [a,b]. \end{aligned} \tag{4.8}$$

Moreover, Corollary 2.3 yields that

$$h(\cdot,x) \in AC([a,b];\mathbb{R}) \quad \text{for all } x \in E_1, \tag{4.9}$$

$$h'_{[1]}(\lambda,x) = f'_{[2]}\big(\varphi(\lambda),x\big)\varphi'(\lambda) \quad \text{for all } x \in E_1 \text{ and } \lambda \in A(x), \tag{4.10}$$

where $A(x) \subseteq [a,b]$ with $\operatorname{meas} A(x) = b-a$, and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_c^{\varphi(\lambda)} f(t,a)\,\mathrm{d}t = f\big(\varphi(\lambda),a\big)\varphi'(\lambda) \quad \text{for a.e. } \lambda \in [a,b]. \tag{4.11}$$

Now we put $f_1 :\equiv \big[f'_{[2]}\big]_+$, $f_2 :\equiv \big[f'_{[2]}\big]_-$, and

$$h_k(t,x) := f_k\big(\varphi(t),x\big)\varphi'(t) \quad \text{for all } (t,x) \in [a,b] \times [a,b],\ k = 1,2. \tag{4.12}$$

Relations (4.9) and (4.10) yield that $h_k(\cdot,x) \in L([a,b];\mathbb{R}^+)$ for all $x \in E_1$. Moreover, in view of assumption (2.7), we have $f_1, f_2 \in L([c,d] \times [a,b];\mathbb{R})$. Therefore, by virtue of Fubini's theorem and Lemma 3.2 one can show that $f_k\big(\varphi(t),\cdot\big)\varphi'(t) = h_k(t,\cdot) \in L([a,b];\mathbb{R}^+)$ for almost every $t \in [a,b]$. Unfortunately, it may happen that $h_k \notin L([a,b] \times [a,b];\mathbb{R}^+)$. However, we can show that on every rectangle contained in $[a,b] \times [a,b]$ there exist both iterated integrals and their values are equal to each other. Indeed, let $k \in \{1,2\}$ be fixed and $[a_1,b_1]\times[a_2,b_2]\subseteq[a,b]\times[a,b]$ be an arbitrary rectangle. Moreover, let

$$\Omega := \Big\{t\in[c,d] : f_k(t,\cdot)\in L([a,b];\mathbb{R})\Big\},\ w(t) := \begin{cases} \int_{a_2}^{b_2} f_k(t,x)\,\mathrm{d}x & \text{for } t\in\Omega,\\ 0 & \text{for } t\in[c,d]\setminus\Omega. \end{cases}$$

Then we have $\operatorname{meas}\Omega = d-c$ and $w\in L([c,d];\mathbb{R})$ and thus, Lemma 3.3 yields that $\int_{\varphi(a_1)}^{\varphi(b_1)} w(s)\,\mathrm{d}s = \int_{a_1}^{b_1} w\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t$. However, with respect to Lemma 3.2, one can verify that

$$w\big(\varphi(t)\big)\varphi'(t) = \int_{a_2}^{b_2} f_k\big(\varphi(t),x\big)\varphi'(t)\,\mathrm{d}x \quad \text{for a.e. } t\in[a,b],$$

which arrives at the equality

$$\int_{\varphi(a_1)}^{\varphi(b_1)}\left(\int_{a_2}^{b_2} f_k(s,x)\,\mathrm{d}x\right)\mathrm{d}s = \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} f_k\big(\varphi(t),x\big)\varphi'(t)\,\mathrm{d}x\right)\mathrm{d}t.$$

On the other hand, by using Lemma 3.3 we get

$$\int_{a_2}^{b_2}\left(\int_{\varphi(a_1)}^{\varphi(b_1)} f_k(s,x)\,\mathrm{d}s\right)\mathrm{d}x = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} f_k\big(\varphi(t),x\big)\varphi'(t)\,\mathrm{d}t\right)\mathrm{d}x.$$

Now comparing the last two relations we obtain the equality

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2} h_k(t,x)\,\mathrm{d}x\right)\mathrm{d}t = \int_{a_2}^{b_2}\left(\int_{a_1}^{b_1} h_k(t,x)\,\mathrm{d}t\right)\mathrm{d}x. \tag{4.13}$$

It means that the functions h_1 and h_2 satisfy all assumptions of Lemma 3.6 with $c=a$ and $d=b$ and thus there exists a set $E_2\subseteq[a,b]$ such that $\operatorname{meas}E_2 = b-a$ and

$$\frac{\partial}{\partial y}\int_a^y\left(\int_a^z h_k(t,x)\,\mathrm{d}x\right)\mathrm{d}t = \int_a^z h_k(y,x)\,\mathrm{d}x \qquad \text{for all } y\in E_2 \text{ and } z\in[a,b],\ k=1,2, \tag{4.14}$$

$$\frac{\partial}{\partial z}\int_a^z\left(\int_a^y h_k(t,x)\,\mathrm{d}t\right)\mathrm{d}x = \int_a^y h_k(t,z)\,\mathrm{d}t \qquad \text{for all } y\in[a,b] \text{ and } z\in E_2,\ k=1,2. \tag{4.15}$$

Moreover, by virtue of assumption (2.7) and Lemma 3.6, we can assume without loss of generality that E_2 is such that the relation

$$\frac{\partial}{\partial y}\int_a^y\left(\int_{z_1}^{z_2} |f'_{[2]}(s,x)|\,\mathrm{d}s\right)\mathrm{d}x = \int_{z_1}^{z_2} |f'_{[2]}(s,y)|\,\mathrm{d}s \qquad \text{for all } y\in E_2 \text{ and } z_1,z_2\in[c,d] \tag{4.16}$$

holds as well.

Let now $\lambda_0 \in E_1 \cap E_2$ be arbitrary. Then, by using relations (4.7), (4.9), (4.10), and (4.13), we get

$$\begin{aligned}
&\int_a^{\lambda} h(\lambda, x)\,\mathrm{d}x - \int_a^{\lambda_0} h(\lambda_0, x)\,\mathrm{d}x \\
&= \int_a^{\lambda_0} \left(\int_{\lambda_0}^{\lambda} h'_{[1]}(t, x)\,\mathrm{d}t \right) \mathrm{d}x + \int_{\lambda_0}^{\lambda} h(\lambda, x)\,\mathrm{d}x \\
&= \int_a^{\lambda_0} \left(\int_{\lambda_0}^{\lambda} h'_{[1]}(t, x)\,\mathrm{d}t \right) \mathrm{d}x + \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} h'_{[1]}(t, x)\,\mathrm{d}t \right) \mathrm{d}x \\
&\quad + \int_{\lambda_0}^{\lambda} \left(\int_a^{\lambda_0} h'_{[1]}(t, x)\,\mathrm{d}t \right) \mathrm{d}x \qquad (4.17) \\
&= \int_{\lambda_0}^{\lambda} \left(\int_a^{\lambda_0} h_1(t, x)\,\mathrm{d}x \right) \mathrm{d}t - \int_{\lambda_0}^{\lambda} \left(\int_a^{\lambda_0} h_2(t, x)\,\mathrm{d}x \right) \mathrm{d}t \\
&\quad + \int_{\lambda_0}^{\lambda} \left(\int_a^{\lambda_0} h_1(t, x)\,\mathrm{d}t \right) \mathrm{d}x - \int_{\lambda_0}^{\lambda} \left(\int_a^{\lambda_0} h_2(t, x)\,\mathrm{d}t \right) \mathrm{d}x \\
&\quad + \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} f'_{[2]}\big(\varphi(t), x\big)\varphi'(t)\,\mathrm{d}t \right) \mathrm{d}x
\end{aligned}$$

for all $\lambda \in [a, b]$. Observe that, in view of assumption (2.7) and Lemma 3.3, for any $\lambda, y \in [a, b]$ with the property $0 < (\lambda - \lambda_0)^2 \le (y - \lambda_0)(\lambda - \lambda_0)$ the relation

$$\begin{aligned}
&\left| \frac{1}{\lambda - \lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} f'_{[2]}\big(\varphi(t), x\big)\varphi'(t)\,\mathrm{d}t \right) \mathrm{d}x \right| \\
&= \left| \frac{1}{\lambda - \lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\varphi(\lambda_0)}^{\varphi(\lambda)} f'_{[2]}(s, x)\,\mathrm{d}s \right) \mathrm{d}x \right| \\
&\le \frac{\operatorname{sgn}(y - \lambda_0)}{\lambda - \lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\varphi(\lambda_0)}^{\varphi(y)} |f'_{[2]}(s, x)|\,\mathrm{d}s \right) \mathrm{d}x
\end{aligned}$$

holds and thus, by using equality (4.16), we get

$$\limsup_{\lambda \to \lambda_0+} \left| \frac{1}{\lambda - \lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} f'_{[2]}\big(\varphi(t), x\big)\varphi'(t)\,\mathrm{d}t \right) \mathrm{d}x \right| \le \int_{\varphi(\lambda_0)}^{\varphi(y)} |f'_{[2]}(s, \lambda_0)|\,\mathrm{d}s$$

for all $y \in [a, b]$, $y > \lambda_0$, and

$$\limsup_{\lambda \to \lambda_0-} \left| \frac{1}{\lambda - \lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} f'_{[2]}\big(\varphi(t), x\big)\varphi'(t)\,\mathrm{d}t \right) \mathrm{d}x \right| \le \int_{\varphi(y)}^{\varphi(\lambda_0)} |f'_{[2]}(s, \lambda_0)|\,\mathrm{d}s$$

for all $y \in [a,b]$, $y < \lambda_0$. Consequently, we have

$$\lim_{\lambda\to\lambda_0} \frac{1}{\lambda-\lambda_0} \int_{\lambda_0}^{\lambda} \left(\int_{\lambda_0}^{\lambda} f'_{[2]}\big(\varphi(t),x\big)\varphi'(t)\,\mathrm{d}t \right) \mathrm{d}x = 0. \tag{4.18}$$

Therefore, by virtue of conditions (4.12), (4.14), (4.15), and Lemma 3.3, it follows from relation (4.17) that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\lambda} \int_a^{\lambda} h(\lambda,x)\,\mathrm{d}x\bigg|_{\lambda=\lambda_0} &= \lim_{\lambda\to\lambda_0} \frac{1}{\lambda-\lambda_0} \left[\int_a^{\lambda} h(\lambda,x)\,\mathrm{d}x - \int_a^{\lambda_0} h(\lambda_0,x)\,\mathrm{d}x \right] \\
&= \int_a^{\lambda_0} h_1(\lambda_0,x)\,\mathrm{d}x - \int_a^{\lambda_0} h_2(\lambda_0,x)\,\mathrm{d}x \\
&\quad + \int_a^{\lambda_0} h_1(t,\lambda_0)\,\mathrm{d}t - \int_a^{\lambda_0} h_2(t,\lambda_0)\,\mathrm{d}t \\
&= \int_a^{\lambda_0} f'_{[2]}\big(\varphi(\lambda_0),x\big)\varphi'(\lambda_0)\,\mathrm{d}x + \int_a^{\lambda_0} f'_{[2]}\big(\varphi(t),\lambda_0\big)\varphi'(t)\,\mathrm{d}t \\
&= \int_a^{\lambda_0} f'_{[2]}\big(\varphi(\lambda_0),x\big)\varphi'(\lambda_0)\,\mathrm{d}x + \int_c^{\varphi(\lambda_0)} f'_{[2]}(s,\lambda_0)\,\mathrm{d}s.
\end{aligned}$$

These equalities and relations (4.8), (4.11) yield that

$$\begin{aligned}
F'(\lambda) = f\big(\varphi(\lambda),a\big)\varphi'(\lambda) &+ \int_a^{\lambda} f'_{[2]}\big(\varphi(\lambda),x\big)\varphi'(\lambda)\,\mathrm{d}x \\
&+ \int_c^{\varphi(\lambda)} f'_{[2]}(t,\lambda)\,\mathrm{d}t \quad \text{for all } \lambda \in A,
\end{aligned} \tag{4.19}$$

where $A \subseteq [a,b]$, $\operatorname{meas} A = b-a$. It remains to show that the relation

$$f\big(\varphi(\lambda),a\big)\varphi'(\lambda) + \int_a^{\lambda} f'_{[2]}\big(\varphi(\lambda),x\big)\varphi'(\lambda)\,\mathrm{d}x = f\big(\varphi(\lambda),\lambda\big)\varphi'(\lambda) \tag{4.20}$$

holds for a.e. $\lambda \in A$. Indeed, let

$$E_3 = \big\{t \in [c,d] : f(t,\cdot) \in AC([a,b];\mathbb{R})\big\}.$$

Then, in view of assumption (2.6), we have $\operatorname{meas} E_3 = d-c$ and

$$f(t,a) + \int_a^{y} f'_{[2]}(t,x)\,\mathrm{d}x = f(t,y) \quad \text{for all } y \in [a,b] \text{ and } t \in E_3. \tag{4.21}$$

Put

$$B_1 := \big\{\lambda \in A : \varphi(\lambda) \in E_3\big\},$$

$$B_2 := \big\{\lambda \in A : \varphi(\lambda) \notin E_3 \text{ and the relation } \varphi'(\lambda)=0 \text{ holds}\big\},$$

and

$$B_3 := \big\{\lambda \in A : \varphi(\lambda) \notin E_3 \text{ and the relation } \varphi'(\lambda)=0 \text{ does not hold}\big\}.$$

Then $B_1 \cup B_2 \cup B_3 = A$ and, by using Lemma 3.2, we get $\operatorname{meas} B_3 = 0$. Moreover, it is clear that, in view of (4.21), the relation (4.20) is satisfied for every $\lambda \in$

$B_1 \cup B_2$. Consequently, relation (4.20) holds almost everywhere on A and thus (4.19) guarantees the validity of desired relation (2.12). □

Proof of Theorem 2.9. We first extend the function h outside of $[a,b] \times [a,b] \times [c,d]$ by setting $h(t,x,z) := 0$.

(a) For any $\mu \in [c,d]$ fixed, the assumptions of Theorem 2.7 are satisfied with $a = c$, $b = d$, $f(\cdot,\cdot) \equiv h(\cdot,\cdot,\mu)$, and $\varphi = \mathrm{id}_{[a,b]}$ and thus the assertion follows immediately from Theorem 2.7(b),(c).

(b) We can assume without loss of generality that $k = 0$. According to assumptions (2.20) and (2.21), we can find a set $\Omega_1 \subseteq\,]a,b[$ of the measure $b-a$ such that

$$h(x,x,\cdot)\colon [c,d] \to \mathbb{R} \text{ is continuous for all } x \in \Omega_1 \tag{4.22}$$

and

$$\int_a^x h'_{[2]}(t,x,\cdot)\,\mathrm{d}t\colon [c,d] \to \mathbb{R} \text{ is continuous for all } x \in \Omega_1. \tag{4.23}$$

It follows from the assertion (a) that, for any $\mu \in [c,d]$, there exists a set $A(\mu) \subseteq [a,b]$ such that $\operatorname{meas} A(\mu) = b-a$ and

$$H'_{[1]}(\lambda,\mu) = h(\lambda,\lambda,\mu) + \int_a^\lambda h'_{[2]}(t,\lambda,\mu)\,\mathrm{d}t \quad \text{for all } \mu \in [c,d],\ \lambda \in A(\mu). \tag{4.24}$$

Put $\Omega_2 = \cap_{\mu\in B} A(\mu)$, where $B = \big([c,d] \cap \mathbb{Q}\big) \cup \{c,d\}$. Since the set B is countable, the set Ω_2 is measurable and $\operatorname{meas}\Omega_2 = b-a$. Clearly, condition (4.24) yields that

$$H'_{[1]}(\lambda,\mu) = h(\lambda,\lambda,\mu) + \int_a^\lambda h'_{[2]}(t,\lambda,\mu)\,\mathrm{d}t \quad \text{for all } \lambda \in \Omega_2,\ \mu \in B. \tag{4.25}$$

Now let $\lambda_0 \in \Omega_1 \cap \Omega_2$ be arbitrary point and $\{\ell_n\}_{n=1}^{+\infty}$ be an arbitrary sequence of non-zero real numbers such that

$$\lim_{n\to+\infty} \ell_n = 0. \tag{4.26}$$

Put

$$g_n(\mu) := \frac{1}{\ell_n}\left[\int_a^{\lambda_0+\ell_n} h(t,\lambda_0+\ell_n,\mu)\,\mathrm{d}t - \int_a^{\lambda_0} h(t,\lambda_0,\mu)\,\mathrm{d}t\right] \qquad \text{for all } \mu \in [c,d],\ n \in \mathbb{N}. \tag{4.27}$$

According to relations (4.25)–(4.27), we obtain

$$\lim_{n\to+\infty} g_n(\mu) = h(\lambda_0,\lambda_0,\mu) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu)\,\mathrm{d}t \quad \text{for all } \mu \in B. \tag{4.28}$$

Observe that, in view of assumptions (2.13)–(2.15), for any $\mu \in [c,d]$ we have

$$g_n(\mu) = \frac{1}{\ell_n}\left[\int_{\lambda_0}^{\lambda_0+\ell_n} h(t,\lambda_0+\ell_n,\mu)\,\mathrm{d}t + \int_a^{\lambda_0}\int_{\lambda_0}^{\lambda_0+\ell_n} h'_{[2]}(t,x,\mu)\,\mathrm{d}x\mathrm{d}t\right] \text{ if } \ell_n > 0$$

and

$$g_n(\mu) = \frac{1}{|\ell_n|}\left[\int_{\lambda_0-|\ell_n|}^{\lambda_0} h(t,\lambda_0,\mu)\,\mathrm{d}t + \int_a^{\lambda_0-|\ell_n|}\int_{\lambda_0-|\ell_n|}^{\lambda_0} h'_{[2]}(t,x,\mu)\,\mathrm{d}x\mathrm{d}t\right] \text{ if } \ell_n < 0.$$

Therefore, assumptions (2.18) and (2.19) yield that the functions g_n ($n \in \mathbb{N}$) are non-decreasing on $[c,d]$.

We will show that relation (4.28) holds for every $\mu \in [c,d]$. Indeed, let $\mu_0 \in [c,d]$ and $\varepsilon > 0$ be arbitrary. Then, in view of relations (4.22) and (4.23), there exist $\mu_1, \mu_2 \in B$ such that $\mu_1 \leq \mu_0 \leq \mu_2$ and

$$\begin{aligned}\Bigg| h(\lambda_0,\lambda_0,\mu_0) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t \\ - h(\lambda_0,\lambda_0,\mu_m) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_m)\,\mathrm{d}t\Bigg| < \frac{\varepsilon}{2} \quad \text{for } m = 1,2.\end{aligned} \tag{4.29}$$

Moreover, by virtue of limit (4.28), there exists $n_0 \in \mathbb{N}$ such that

$$\left| g_n(\mu_m) - h(\lambda_0,\lambda_0,\mu_m) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_m)\,\mathrm{d}t\right| < \frac{\varepsilon}{2} \quad \text{for } n \geq n_0,\ m = 1,2. \tag{4.30}$$

Now, by using relations (4.29), (4.30), and the monotonicity of the functions g_n, we obtain

$$\begin{aligned} g_n(\mu_0) - h(\lambda_0,\lambda_0,\mu_0) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t \\ \leq g_n(\mu_2) - h(\lambda_0,\lambda_0,\mu_2) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_2)\,\mathrm{d}t \\ + h(\lambda_0,\lambda_0,\mu_2) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_2)\,\mathrm{d}t \\ - h(\lambda_0,\lambda_0,\mu_0) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } n \geq n_0 \end{aligned}$$

and

$$\begin{aligned} h(\lambda_0,\lambda_0,\mu_0) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t - g_n(\mu_0) \\ \leq h(\lambda_0,\lambda_0,\mu_1) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_1)\,\mathrm{d}t - g_n(\mu_1) \\ - h(\lambda_0,\lambda_0,\mu_1) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_1)\,\mathrm{d}t \\ + h(\lambda_0,\lambda_0,\mu_0) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for } n \geq n_0 \end{aligned}$$

and thus

$$\left|g_n(\mu_0) - h(\lambda_0,\lambda_0,\mu_0) - \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu_0)\,\mathrm{d}t\right| < \varepsilon \quad \text{for } n \geq n_0.$$

Consequently, in view of arbitrariness of μ_0 and ε, the relation

$$\lim_{n\to+\infty} g_n(\mu) = h(\lambda_0,\lambda_0,\mu) + \int_a^{\lambda_0} h'_{[2]}(t,\lambda_0,\mu)\,\mathrm{d}t \quad \text{for all } \mu \in [c,d]$$

holds. Since λ_0 and $\{\ell_n\}_{n=1}^{+\infty}$ were also arbitrary and $\operatorname{meas}\Omega_1 \cap \Omega_2 = b-a$, the last relation guarantees the validity of desired equality (2.22) with $E_1 = \Omega_1 \cap \Omega_2$.

(c) For any $\lambda \in E_1$ fixed, the assumptions of Proposition 2.6 are satisfied with $f(\cdot,\cdot) \equiv h(\cdot,\lambda,\cdot)$ on $[a,\lambda]\times[c,d]$ and thus the assertion follows from Proposition 2.6.

(d) It follows immediately from Lemma 3.5 with $f \equiv H'_{[1]}$ on $[a,b]\times[c,d]$. □

Now we establish a technical lemma in order to simplify the proof of Corollary 2.12.

Lemma 4.1. *Let $p,q \in L([a,b]\times[c,d];\mathbb{R}^+)$ and*

$$h(t,x,z) := \int_c^z q(t,s)\,\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\mathrm{d}s \qquad \text{for all } t \in E \text{ and } (x,z) \in [a,b]\times[c,d], \tag{4.31}$$

where $E \subseteq [a,b]$ with $\operatorname{meas} E = b-a$. Then the function h satisfies relations (2.14), (2.15), and there exists a set $\Omega \subseteq [a,b]\times[a,b]$ such that $\operatorname{meas}\Omega = (b-a)^2$ and

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \qquad \text{for all } (t,x) \in \Omega,\ t \leq x,\ \text{and all } z \in [c,d]. \tag{4.32}$$

Proof. Let $t \in E$ and $z \in [c,d]$ be arbitrary. We put

$$f_{t,z}(s,x) := q(t,s)\,\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1} \quad \text{for a.e. } s \in [c,d] \text{ and all } x \in [a,b].$$

Clearly, the function $f_{t,z}$ satisfies conditions (2.5), (2.6), and

$$f_{t,z}{}'_{[2]}(s,x) = q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1} \qquad \text{for a.e. } s \in [c,d] \text{ and all } x \in A(s), \tag{4.33}$$

where $A(s) \subseteq [a,b]$ with $\operatorname{meas} A(s) = b-a$. With the function $f_{t,z}$ we associate the function

$$f^0_{t,z}(s,x) := q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}.$$

Clearly, the function $f^0_{t,z}$ is defined almost everywhere on the rectangle $[c,d]\times[a,b]$. According to the assumptions $p,q \in L([a,b]\times[c,d];\mathbb{R}^+)$ and Lemma 3.8, we see that the function $f^0_{t,z}$ is measurable on the rectangle $[c,d]\times[a,b]$. Moreover, we have

$$|f^0_{t,z}(s,x)| \le q(t,s)\left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\|p\|_L} \quad \text{for a.e. } (s,x)\in[c,d]\times[a,b]$$

and thus $f^0_{t,z}\in L([c,d]\times[a,b];\mathbb{R})$. Hence, in view of equality (4.33), the function ${f_{t,z}}'_{[2]}$ satisfies condition (2.7) (see Lemma 3.5 with $g\equiv f^0_{t,z}$). Consequently, Proposition 2.6 yields the validity of relation (2.14) and

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \qquad (4.34)$$
$$\text{for all } t\in E,\ z\in[c,d],\ x\in B(t,z),$$

where $B(t,z)\subseteq[a,b]$ with $\operatorname{meas}B(t,z)=b-a$.

Now we will show that the function h satisfies condition (2.15). Indeed, for any $z\in[c,d]$ fixed we put

$$\varphi_z(x,t,s) := q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}.$$

Clearly, the function φ_z is defined almost everywhere on the set $[a,b]\times[a,b]\times[c,d]$. By using the assumptions $p,q\in L([a,b]\times[c,d];\mathbb{R}^+)$ and Lemma 3.8, we easily get the measurability of the function φ_z on the set $[a,b]\times[a,b]\times[c,d]$. Moreover, it is clear that

$$|\varphi_z(x,t,s)| \le q(t,s)\left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\|p\|_L} \quad \text{for a.e. } (x,t,s)\in[a,b]\times[a,b]\times[c,d]$$

and thus $\varphi_z\in L([a,b]\times[a,b]\times[c,d];\mathbb{R})$. Hence, Fubini's theorem yields that, for any $z\in[c,d]$, the function $\int_c^z\varphi_z(\cdot,\cdot,s)\,\mathrm{d}s$ is integrable on $[a,b]\times[a,b]$ which, together with equality (4.34), ensures the validity of condition (2.15) (see Lemma 3.5 with $a=c$, $b=d$, and $g(\cdot,\cdot)\equiv\int_c^z\varphi_z(\cdot,\cdot,s)\,\mathrm{d}s$) and

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \qquad (4.35)$$
$$\text{for all } z\in[c,d] \text{ and } (t,x)\in C(z),$$

where $C(z)\subseteq E\times[a,b]$ with $\operatorname{meas}C(z)=(b-a)^2$. Put $\Omega=\cap_{z\in D}C(z)$, where $D=\big([c,d]\cap\mathbb{Q}\big)\cup\{c,d\}$. Since the set D is countable, the set Ω is measurable and $\operatorname{meas}\Omega=(b-a)^2$. Clearly, condition (4.35) yields that

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \qquad (4.36)$$
$$\text{for all } (t,x)\in\Omega \text{ and } z\in D.$$

Now let $(t_0, x_0) \in \Omega$, $t_0 \le x_0$, be arbitrary point and $\{\ell_n\}_{n=1}^{+\infty}$ be an arbitrary sequence of non-zero real numbers such that relation (4.26) holds. Put

$$g_n(z) := \frac{1}{\ell_n} \int_c^z q(t_0, s)\, \mathrm{e}^{\int_{t_0}^{x_0} \int_s^z p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1} \left[\mathrm{e}^{\int_{x_0}^{x_0+\ell_n} \int_s^z p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1} - 1 \right] \mathrm{d}s$$

$$\text{for all } z \in [c, d],\ n \in \mathbb{N}. \tag{4.37}$$

According to relations (4.26), (4.36), and (4.37), we obtain

$$\lim_{n \to +\infty} g_n(z) = \int_c^z q(t_0, s) \left(\int_s^z p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^z p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \tag{4.38}$$

for all $z \in D$. Note also that the functions g_n ($n \in \mathbb{N}$) are non-decreasing on $[c, d]$, because the functions p and q are non-negative and $t_0 \le x_0$.

We will show that relation (4.38) holds for every $z \in [c, d]$. Indeed, let $z_0 \in [c, d]$ and $\varepsilon > 0$ be arbitrary. By using the inequality

$$\mathrm{e}^{y_2} - \mathrm{e}^{y_1} \le \mathrm{e}^{y_2}(y_2 - y_1) \quad \text{for all } y_1, y_2 \in \mathbb{R},\ y_1 \le y_2, \tag{4.39}$$

it can be easily verified that

$$\begin{aligned}
\Bigg| \int_c^{z_0} & q(t_0, s) \left(\int_s^{z_0} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \\
&- \int_c^{z} q(t_0, s) \left(\int_s^{z} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z} p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \Bigg| \\
\le{}& \left(\int_c^d p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\|p\|_L} \left| \int_z^{z_0} q(t_0, s)\, \mathrm{d}s \right| \\
&+ \left(\int_c^d q(t_0, s)\, \mathrm{d}s \right) \mathrm{e}^{\|p\|_L} \left| \int_z^{z_0} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right| \\
&+ \left(\int_c^d q(t_0, s)\, \mathrm{d}s \right) \left(\int_c^d p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\|p\|_L} \left| \int_a^b \int_z^{z_0} p(\xi_1, \xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1 \right|
\end{aligned}$$

for all $z \in [c, d]$ and thus there exist $z_1, z_2 \in D$ such that $z_1 \le z_0 \le z_2$ and

$$\begin{aligned}
\Bigg| \int_c^{z_0} & q(t_0, s) \left(\int_s^{z_0} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \\
&- \int_c^{z_m} q(t_0, s) \left(\int_s^{z_m} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_m} p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \Bigg| < \frac{\varepsilon}{2}
\end{aligned} \tag{4.40}$$

for $m = 1, 2$. Moreover, by virtue of limit (4.38), there exists $n_0 \in \mathbb{N}$ such that

$$\left| g_n(z_m) - \int_c^{z_m} q(t_0, s) \left(\int_s^{z_m} p(x_0, \xi_2)\, \mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_m} p(\xi_1,\xi_2)\, \mathrm{d}\xi_2 \mathrm{d}\xi_1}\, \mathrm{d}s \right| < \frac{\varepsilon}{2}$$

$$\text{for } n \ge n_0,\ m = 1, 2. \tag{4.41}$$

Now, by using inequalities (4.40), (4.41), and the monotonicity of g_n, for every $n \geq n_0$ we obtain

$$\begin{aligned} g_n(z_0) - \int_c^{z_0} q(t_0,s) \left(\int_s^{z_0} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ \leq g_n(z_2) - \int_c^{z_2} q(t_0,s) \left(\int_s^{z_2} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_2} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ + \int_c^{z_2} q(t_0,s) \left(\int_s^{z_2} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_2} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ - \int_c^{z_0} q(t_0,s) \left(\int_s^{z_0} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

and

$$\begin{aligned} \int_c^{z_0} q(t_0,s) \left(\int_s^{z_0} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s - g_n(z_0) \\ \leq \int_c^{z_1} q(t_0,s) \left(\int_s^{z_1} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_1} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s - g_n(z_1) \\ + \int_c^{z_0} q(t_0,s) \left(\int_s^{z_0} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ - \int_c^{z_1} q(t_0,s) \left(\int_s^{z_1} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_1} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and thus we have

$$\left| g_n(z_0) - \int_c^{z_0} q(t_0,s) \left(\int_s^{z_0} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z_0} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \right| < \varepsilon \text{ for } n \geq n_0.$$

Consequently, in view of arbitrariness of z_0 and ε, the relation (4.38) holds for all $z \in [c,d]$. Since the sequence $\{\ell_n\}_{n=1}^{+\infty}$ was also arbitrary, we have proved that

$$h'_{[2]}(t_0,x_0,z) = \int_c^{z} q(t_0,s) \left(\int_s^{z} p(x_0,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_{t_0}^{x_0} \int_s^{z} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s$$

for all $z \in [c,d]$. Mention on arbitrariness of the point (t_0,x_0) completes the proof. □

Proof of Corollary 2.12. Clearly

$$\gamma(\lambda,\mu) = \int_a^{\lambda} h(t,\lambda,\mu)\,\mathrm{d}t \quad \text{for all } (\lambda,\mu) \in [a,b] \times [c,d],$$

where the function h is defined by formula (4.31) with $E \subseteq [a,b]$, $\operatorname{meas} E = b-a$.

(i) We first mention that condition (2.13) holds. It follows from Lemma 4.1 that the function h also satisfies conditions (2.14), (2.15), and (4.32), where $\Omega \subseteq [a,b] \times [a,b]$ is such that $\operatorname{meas} \Omega = (b-a)^2$. Consequently, the assumptions of Theorem 2.9(a) are fulfilled and thus $\gamma(\cdot,\mu) \in AC([a,b];\mathbb{R})$ for every $\mu \in [c,d]$.

Now observe that conditions (2.18), (2.19) with $k = 0$ and (2.20) are satisfied because we assume $p, q \in L([a,b] \times [c,d]; \mathbb{R}^+)$. Moreover, in view of condition (4.32), there exists a set $A \subseteq [a,b]$ such that $\operatorname{meas} A = b - a$ and

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s) \left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s$$
$$\text{for all } x \in A,\ t \in B(x), \text{ and } z \in [c,d],$$

where $B(x) \subseteq [a,x]$ is such that $\operatorname{meas} B(x) = x - a$. Therefore, for any $x \in A$ fixed we have

$$\begin{aligned} h'_{[2]}(t,x,z_2) - h'_{[2]}(t,x,z_1) &= \int_{z_1}^{z_2} q(t,s) \left(\int_s^{z_2} p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_t^x \int_s^{z_2} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ &\quad + \int_c^{z_1} q(t,s) \left(\int_{z_1}^{z_2} p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_t^x \int_s^{z_2} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s \\ &\quad + \int_c^{z_1} q(t,s) \left(\int_s^{z_1} p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \\ &\qquad \times \left[\mathrm{e}^{\int_t^x \int_s^{z_2} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1} - \mathrm{e}^{\int_t^x \int_s^{z_1} p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1} \right] \mathrm{d}s \end{aligned}$$

for a.e. $t \in [a,x]$ and all $z_1, z_2 \in [c,d]$. Therefore, by using inequality (4.39), for every $c \le z_1 \le z_2 \le d$ we get

$$\begin{aligned} \left| \int_a^x h'_{[2]}(t,x,z_2)\,\mathrm{d}t - \int_a^x h'_{[2]}(t,x,z_1)\,\mathrm{d}t \right| &\le \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\|p\|_L} \left(\int_a^b \int_{z_1}^{z_2} q(t,s)\,\mathrm{d}s\mathrm{d}t \right) \\ &\quad + \|q\|_L\, \mathrm{e}^{\|p\|_L} \left(\int_{z_1}^{z_2} p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \\ &\quad + \|q\|_L\, \mathrm{e}^{\|p\|_L} \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2 \right) \left(\int_a^b \int_{z_1}^{z_2} p(t,s)\,\mathrm{d}s\mathrm{d}t \right). \end{aligned}$$

Consequently, relation (2.21) holds and thus, according to Theorem 2.9(b), there exists a set $E_1 \subseteq A$ such that $\operatorname{meas} E_1 = b - a$ and

$$\begin{aligned} \gamma'_{[1]}(\lambda,\mu) &= \int_c^\mu q(\lambda,s)\,\mathrm{d}s \\ &\quad + \int_a^\lambda \int_c^\mu q(t,s) \left(\int_s^\mu p(\lambda,\xi_2)\,\mathrm{d}\xi_2 \right) \mathrm{e}^{\int_t^\lambda \int_s^\mu p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s\mathrm{d}t \end{aligned}$$

for all $\lambda \in E_1$ and $\mu \in [c,d]$.

(ii) Since we can change the order of the integrations in relation (1.3), the assertion follows immediately from the above-proved part (i) by changing the role of the variables t and x.

(iii) Let E_1 be the set appearing in the proof of part (i) and $x \in E_1$ be an arbitrary point. Then we have

$$h(x,x,z) = \int_c^z q(x,s)\,\mathrm{d}s \quad \text{for all } z \in [c,d]$$

and

$$h'_{[2]}(t,x,z) = \int_c^z q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s$$
$$\text{for all } t \in B(x) \text{ and } z \in [c,d],$$

where $B(x) \subseteq [a,x]$ with $\operatorname{meas} B(x) = x - a$. Clearly, condition (2.23) holds.

Let $t \in B(x)$ be arbitrary. We put

$$f_{t,x}(s,z) := q(t,s)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}$$
$$\text{for a.e. } s \in [c,d] \text{ and all } z \in [c,d].$$

Then the function $f_{t,x}$ satisfies conditions (2.5), (2.6) (in which $a = c$, $b = d$), and

$$\begin{aligned} {f_{t,x}}'_{[2]}(s,z) = q(t,s)\Big[&p(x,z) \\ &+ \left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\Big]\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1} \end{aligned}$$
$$\text{for a.e. } s \in [c,d] \text{ and all } x \in C(s), \tag{4.42}$$

where $C(s) \subseteq [c,d]$ with $\operatorname{meas} C(s) = d - c$. With the function $f_{t,x}$ we associate the function

$$\begin{aligned} f^0_{t,x}(s,z) := q(t,s)\Big[&p(x,z) \\ &+ \left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\Big]\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}. \end{aligned}$$

Clearly, the function $f^0_{t,x}$ is defined almost everywhere on the square $[c,d]\times[c,d]$. According to the assumptions $p, q \in L([a,b]\times[c,d];\mathbb{R}^+)$ and Lemma 3.8, we see that the function $f^0_{t,x}$ is measurable on the square $[c,d]\times[c,d]$. Moreover, we have

$$|f^0_{t,x}(s,z)| \le q(t,s)\left[p(x,z) + \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_a^b p(\xi_1,z)\,\mathrm{d}\xi_1\right)\right]\mathrm{e}^{\|p\|_L}$$
$$\text{for a.e. } (s,z) \in [c,d]\times[c,d]$$

and thus $f^0_{t,x} \in L([c,d]\times[c,d];\mathbb{R})$. Hence, in view of equality (4.42), the function $f_{t,x}$ satisfies condition (2.7) (see Lemma 3.5 with $a = c$, $b = d$, and $g \equiv f^0_{t,x}$). Consequently, Theorem 2.7 (with $a = c$, $b = d$, and $\varphi = \mathrm{id}_{[c,d]}$) yields the validity of relation (2.24) and

$$h''_{[2,3]}(t,x,z) = \int_c^z q(t,s)\Big[p(x,z) + \left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\Big]\,\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \quad \text{for all } t\in B(x) \text{ and } z\in D(t,x), \tag{4.43}$$

where $D(t,x) \subseteq [c,d]$ with $\operatorname{meas} D(t,x) = d-c$.

Now we will show that the function h satisfies condition (2.25). Indeed, we put

$$g_x(t,z) := \int_c^z q(t,s)\Big[p(x,z) + \left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\Big]\,\mathrm{e}^{\int_t^x\int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s.$$

Clearly, the function g_x is defined almost everywhere on the rectangle $[a,x]\times[c,d]$. Observe that

$$\begin{aligned} g_x(t,z) &= p(x,z)\,\mathrm{e}^{\int_t^x\int_c^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\int_c^z q(t,s)\,\mathrm{e}^{-\int_t^x\int_c^s p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \\ &\quad + \left(\int_c^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\mathrm{e}^{\int_t^x\int_c^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1} \\ &\qquad\qquad \times\int_c^z q(t,s)\,\mathrm{e}^{-\int_t^x\int_c^s p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \\ &\quad - \left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\mathrm{e}^{\int_t^x\int_c^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1} \\ &\qquad\qquad \times\int_c^z q(t,s)\left(\int_c^s p(x,\xi_2)\,\mathrm{d}\xi_2\right)\mathrm{e}^{-\int_t^x\int_c^s p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}\,\mathrm{d}s \end{aligned}$$

for a.e. $(t,z)\in[a,x]\times[c,d]$ whence, by using the assumptions $p,q\in L([a,b]\times[c,d];\mathbb{R}^+)$ and Lemma 3.8, we get the measurability of the function g_x on the rectangle $[a,x]\times[c,d]$. Moreover, it is clear that

$$|g_x(t,z)| \le \Big[p(x,z) + \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_a^b p(\xi_1,z)\,\mathrm{d}\xi_1\right)\Big] \times \mathrm{e}^{\|p\|_L}\int_c^d q(t,s)\,\mathrm{d}s \quad \text{for a.e. } (t,z)\in[a,x]\times[c,d]$$

and thus $g_x \in L([a,x]\times[c,d];\mathbb{R})$. Hence, in view of equality (4.43), we see that condition (2.25) holds (see Lemma 3.5 with $b=x$ and $f(\cdot,\cdot)\equiv h'_{[2]}(\cdot,x,\cdot)$). Consequently, Theorem 2.9(c) yields that $\gamma'_{[1]}(\lambda,\cdot)\in AC([c,d];\mathbb{R})$ for every $\lambda\in E_1$

and

$$\gamma''_{[1,2]}(\lambda,\mu) = q(\lambda,\mu) + \int_a^\lambda \int_c^\mu q(t,s) \left[p(\lambda,\mu) + \left(\int_t^\lambda p(\xi_1,\mu)\,\mathrm{d}\xi_1\right)\left(\int_s^\mu p(\lambda,\xi_2)\,\mathrm{d}\xi_2\right)\right] \times \mathrm{e}^{\int_t^\lambda \int_s^\mu p(\xi_1,\xi_2)\,\mathrm{d}\xi_2 \mathrm{d}\xi_1}\,\mathrm{d}s\mathrm{d}t \quad \text{for all } \lambda \in E_1,\ \mu \in E_2(\lambda), \tag{4.44}$$

where $E_2(\lambda) \subseteq [c,d]$ is such that $\operatorname{meas} E_2(\lambda) = d-c$.

Finally we put

$$\varphi(t,s,x,z) := q(t,s)\left[p(x,z) + \left(\int_t^x p(\xi_1,z)\,\mathrm{d}\xi_1\right)\left(\int_s^z p(x,\xi_2)\,\mathrm{d}\xi_2\right)\right]\mathrm{e}^{\int_t^x \int_s^z p(\xi_1,\xi_2)\,\mathrm{d}\xi_2\mathrm{d}\xi_1}.$$

Clearly, the function φ is defined almost everywhere on the set $[a,b]\times[c,d]\times[a,b]\times[c,d]$. By using the assumptions $p,q \in L([a,b]\times[c,d];\mathbb{R}^+)$ and Lemma 3.8, it is easy to verify that the function φ is measurable on the set $[a,b]\times[c,d]\times[a,b]\times[c,d]$. Moreover, we have

$$|\varphi(t,s,x,z)| \le q(t,s)\left[p(x,z) + \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_a^b p(\xi_1,z)\,\mathrm{d}\xi_1\right)\right]\mathrm{e}^{\|p\|_L} \quad \text{for a.e. } (t,s,x,z) \in [a,b]\times[c,d]\times[a,b]\times[c,d]$$

and thus $\varphi \in L([a,b]\times[c,d]\times[a,b]\times[c,d];\mathbb{R})$. Now we extend the function φ outside of the set $[a,b]\times[c,d]\times[a,b]\times[c,d]$ by setting $\varphi(t,s,x,z) := 0$ and we put

$$f(x,z,\alpha,\beta) := \int_a^\alpha \int_c^\beta \varphi(t,s,x,z)\,\mathrm{d}s\mathrm{d}t \quad \text{for a.e. } (x,z) \in [a,b]\times[c,d] \text{ and all } (\alpha,\beta) \in \mathbb{R}^2.$$

Then conditions (3.10) and (3.11) are satisfied and

$$|f(x,z,x,z)| \le \mathrm{e}^{\|p\|_L}\,\|q\|_L\left[p(x,z) + \left(\int_c^d p(x,\xi_2)\,\mathrm{d}\xi_2\right)\left(\int_a^b p(\xi_1,z)\,\mathrm{d}\xi_1\right)\right]$$

for a.e. $(x,z) \in [a,b]\times[c,d]$. Consequently, Lemma 3.7 yields the integrability of the function

$$g(x,z) := q(x,z) + f(x,z,x,z) \quad \text{for a.e. } (x,z) \in [a,b]\times[c,d]. \tag{4.45}$$

If we set $g(x,z) := \gamma''_{[1,2]}(x,z)$ at those points $(x,z) \in \{(t,s) : t \in E_1,\ s \in E_2(t)\}$ in which g is not defined[10] then, in view of equality (4.44), the function g satisfies condition (2.27). Therefore, Theorem 2.9(d) yields that the partial derivative $\gamma''_{[1,2]}$ exists almost everywhere in the rectangle $[a,b]\times[c,d]$ and that desired relation (2.31) holds for a.e. $(t,x) \in [a,b]\times[c,d]$.

[10] The set of such points has the measure equal to zero and thus the function g remains integrable on $[a,b]\times[c,d]$.

(iv) Since we can change the order of the integrations in relation (1.3), the assertion follows immediately from the above-proved part (iii) by changing the role of the variables t and x.

(v) It follows from the integrability of the function g defined by formula (4.45) and the above-proved equalities (2.31) and (2.33). □

Proof of Corollary 2.13. According to Corollary 2.12, we get from Lemma 3.1 absolute continuity of the function γ in the sense of Carathéodory. Since the functions p and q are non-negative, it follows from equality (2.32) that the function γ is a solution to differential inequality (1.2). □

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