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A DYNAMIC EFFECT ALGEBRAS WITH DUAL OPERATION

JAN PASEKA AND JIŘÍ JANDA

Abstract. Tense operators for MV-algebras were introduced by Diaconescu and Georgescu. Based on their definition Chajda and Kolařík presented the definition of tense operators for lattice effect algebras. Chajda and Paseka tackled the problem of axiomatizing tense operators on an effect algebra by introducing the notion of a partial dynamic effect algebra. They also gave representation theorems for dynamic effect algebras. We continue to extend their work for partial S-dynamic effect algebras i.e. in the case when tense operators satisfy required conditions also for the dual effect algebraic operation $\cdot$. We show that whenever tense operators are total our stronger notion coincides with their definition. We give also a representation theorem for partial S-dynamic effect algebras and its version for strict dynamic effect algebras.

1. Introduction

Tense operators for MV-algebras were introduced by Diaconescu and Georgescu in [4]. Based on their definition Chajda and Kolařík in [2] presented the definition of tense operators for lattice effect algebras. Chajda and Paseka then tackled the problem of axiomatizing tense operators on an effect algebra by introducing the notion of a partial dynamic effect algebra (see [3]).

The motivation for the study of tense operators under consideration is that they should "enable us to express the dimension of time in the logic of quantum mechanics".

To explain this idea, let us assume that E is an effect algebra associated as a "logic" with a (possibly quantum-mechanical) physical system $\mathcal{Q}$ under study, i.e. elements of E represent propositions about the system $\mathcal{Q}$. Of course, it is already well-known how to express the dimension of time for the quantum logic E of $\mathcal{Q}$, namely in terms of a one parameter group $\{\xi_t : t \in \mathbb{R}\}$ of continuous affine automorphisms acting on the state space S of E, i.e. the *dynamical group.* In quantum logic it is traditional to assume that S is a full set of states. This assumption enables one to realize E, by evaluation, as a sub-effect algebra of the complete MV-algebra $[0, 1]^S$.

The usual understanding is that, at any given moment, the system $\mathcal{Q}$ is in exactly one state $s \in S$, and the probability that a proposition $x \in E$ is true – if

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tested – when $\mathcal{Q}$ is in state s is given by $s(x)$. The statement that x is true when tested with the system in state s is often abbreviated as "x is true in state s".

Let us denote the corresponding proposition "x is always going to be the case" by $G(x)$. We expect that $s(G(x)) \leq \xi_t(s)(x)$ for all $t \geq 0$. Such a situation is described via frame (T, R) in our considerations such that $sR\xi_t(s)$. Similarly, let us denote the corresponding proposition "x has always been the case" by $H(x)$. Now, we expect that $s(H(x)) \leq \xi_t(s)(x)$ for all $t \leq 0$. We then write $\xi_t(s)Rs$.

Having the above considerations in mind we continue to extend the results from [3] in the setting of partial S-dynamic effect algebras, i.e. in the case when tense operators satisfy required conditions also for the dual effect algebraic operation $\cdot$. The motivation was a similar situation in a case of MV-algebras. As $+$ can be seen as an orthogonal disjunction, we can interpret the dual operation $\cdot$ as co-orthogonal conjunction. We show that whenever tense operators are total our stronger notion coincides with their definition. We give also a representation theorem for partial S-dynamic effect algebras and its version for strict dynamic effect algebras.

2. Preliminaries

At first, let us recall the concept of effect algebra (see [6] or [5]). By an *effect algebra* is meant a system $\mathcal{E} = (E; +, 0, 1)$ where 0 and 1 are distinguished elements of E, $0 \neq 1$, and $+$ is a partial binary operation on E satisfying the following axioms for $p, q, r \in E$:

- (E1) if $p + q$ is defined then $q + p$ is defined and $p + q = q + p$
- (E2) if $q + r$ is defined and $p + (q + r)$ is defined then $p + q$ and $(p + q) + r$ are defined and $p + (q + r) = (p + q) + r$
- (E3) for each $p \in E$ there exists a unique $p' \in E$ such that $p + p' = 1$; p' is called a *supplement* of p
- (E4) if $p + 1$ is defined then $p = 0$.

Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$, we can introduce the *induced order* $\leq$ on E as follows

$$a \leq b \quad \text{if for some } c \in E \quad c + a = b$$

(see, e.g. [5]). An effect algebra $\mathcal{E}$ is called a *lattice effect algebra* if $(E; \leq)$ is a lattice (with respect to the induced order). Evidently, $0 \leq a \leq 1$ for each $a \in E$ and $a \leq b$ implies $b' \leq a'$ for the supplements. Of course, $1 = 0'$ and $x'' = x$. If the underlying lattice $(E; \leq)$ is complete, we will call $\mathcal{E}$ a *complete lattice effect algebra.* It is worth noticing that $a + b$ exists in an effect algebra $\mathcal{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal). Dually, we have a partial operation $\cdot$ on E such that $a \cdot b$ exists in an effect algebra $\mathcal{E}$ if and only if $a' \leq b$ in which case $a \cdot b = (a' + b')'$. This allows us to equip E with a dual effect algebraic operation such that $\mathcal{E}^{op} = (E; \cdot, 1, 0)$ is again an effect algebra, ${}'^{\mathcal{E}^{op}} = {}'^{\mathcal{E}} = {}'$ and $\leq_{\mathcal{E}^{op}} = \leq^{op}$. A *morphism of effect algebras* is a map between them such that it preserves the partial operation $+$, the bottom and the top elements. In particular, $' : \mathcal{E} \to \mathcal{E}^{op}$ is a morphism of effect algebras. A map $s : E \to [0, 1]$ is called a *state* on $\mathcal{E}$ if $s(0) = 0$, $s(1) = 1$ and $s(x + y) = s(x) + s(y)$ whenever $x + y$ exists in $\mathcal{E}$.

A *morphism* $f : P_1 \to P_2$ *of bounded posets* is an order, top element and bottom element preserving map. Any morphism of effect algebras is a morphism of corresponding bounded posets. A morphism $f : P_1 \to P_2$ of bounded posets is *order reflecting* if ($f(a) \leq f(b)$ if and only if $a \leq b$) for all $a, b \in P_1$.

Observation 2.1 ([3]). *Let P_1, P_2 be bounded posets, T a set and $h_t : P_1 \to P_2, t \in T$ morphisms of bounded posets. The following conditions are equivalent:*

(i) $((\forall t \in T)\, h_t(a) \leq h_t(b)) \implies a \leq b$ *for any elements* $a, b \in P_1$*;*
(ii) *The map $h : P_1 \to P_2^T$ defined by $h(a) = (h_t(a))_{t\in T}$ for all $a \in P_1$ is order reflecting.*

We then say that $\{h_t : P_1 \to P_2; t \in T\}$ is a *full set of order preserving maps with respect to* P_2. Note that we may in this case identify P_1 with a subposet of P_2^T since h is an injective morphism of bounded posets.

3. S-tense operators

The second concept which will be used are so-called tense operators. They are in certain sense quantifiers which quantify over the time dimension of the logic under consideration. The semantical interpretation of these *tense operators* G and H is as follows. Consider a pair $(T, \leq)$ where T is a non-void set and $\leq$ is a partial order on T. Let $x \in T$ and $f(x)$ be a formula of a given logical calculus. We say that $G\big(f(t)\big)$ *is valid* if for any $s \geq t$ the formula $f(s)$ is valid. Analogously, $H\big(f(t)\big)$ is valid if $f(s)$ is valid for each $s \leq t$. Thus the unary operators G and H constitute an algebraic counterpart of the tense operations "it is always going to be the case that" and "it has always been the case that", respectively. These tense operators were firstly introduced as operators on Boolean algebras (see [1] for an overview). Chajda and Paseka introduced in [3] the notion of a partial dynamic effect algebra.

Definition 3.1. By a *partial dynamic effect algebra* is meant a triple $\mathcal{D} = (\mathcal{E}; G, H)$ such that $\mathcal{E} = (E; +, 0, 1)$ is an effect algebra and G, H are partial mappings of E into itself satisfying

(T1) $G(0) = 0$, $G(1) = 1$, $H(0) = 0$ and $H(1) = 1$
(T2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist and $H(x) \leq H(y)$ whenever $H(x), H(y)$ exist
(T3) if $x+y$ and $G(x), G(y), G(x+y)$ exist then $G(x)+G(y)$ exists and $G(x)+G(y) \leq G(x+y)$ and if $x+y$ and $H(x), H(y), H(x+y)$ exist then $H(x)+H(y)$ exists and $H(x)+H(y) \leq H(x+y)$
(T4) $x \leq GP(x)$ where $H(x')$ exists, $P(x) = H(x')'$ and $GP(x)$ exists, $x \leq HF(x)$ where $G(x')$ exists, $F(x) = G(x')'$ and $HF(x)$ exists.

Just defined G and H will be called *tense operators* of a partial dynamic effect algebra $\mathcal{D}$. If both G and H are total we will speak about a *dynamic effect algebra*.

We say that a partial map $G : E \to E$ is *contractive* (*transitive*) if $G(x) \leq x$ ($G(x) \leq G(G(x))$) for all $x \in M$ such that $G(x)$ is defined (for all $x \in M$ such that $G(x)$ and $G(G(x))$ are defined). A tense operator G that is both contractive and transitive is called a *conucleus*.

If $(\mathcal{E}_1; G_1, H_1)$ and $(\mathcal{E}_2; G_2, H_2)$ are partial dynamic effect algebras, then a *morphism of partial dynamic effect algebras* $f : (\mathcal{E}_1; G_1, H_1) \to (\mathcal{E}_2; G_2, H_2)$ is a morphism of effect algebras such that $f(G_1(x)) = G_2(f(x))$, for any $x \in E_1$ such $G_1(x)$ is defined and $f(H_1(y)) = H_2(f(y))$, for any $y \in E_1$ such $H_1(y)$ is defined. Moreover, whenever $\mathcal{D}$ satisfies also following condition

(T5) if $x \cdot y$, $G(x), G(y), G(x \cdot y)$ and $G(x) \cdot G(y)$ exist, then $G(x) \cdot G(y) \leq G(x \cdot y)$ and if $x \cdot y$, $H(x), H(y), H(x \cdot y)$ and $H(x) \cdot H(y)$ exist, then $H(x) \cdot H(y) \leq H(x \cdot y)$

we call it a *partial S-dynamic effect algebra.* G and H will be called *S-tense operators* of a partial S-dynamic effect algebra $\mathcal{D}$. And if both G and H are total we will speak about an *S-dynamic effect algebra.* A *morphism of partial S-dynamic effect algebras* is a morphism of corresponding partial dynamic effect algebras.

Remark 3.2. Let us note that the partial mappings defined only for 0 and 1 by $G(0) = H(0) = 0$ and $G(1) = H(1) = 1$ are S-tense operators on every effect algebra. Hence, every effect algebra can be organized into partial S-dynamic effect algebra.

Example 3.3. In what follows we will present an example of a partial dynamic effect algebra which is not S-dynamic. Let $\mathcal{I} = ([0,1], +, 0)$ be an effect algebra on interval of real numbers where $+$ is the usual sum of real numbers restricted to those pairs $(x, y) \in [0,1]^2$ such that $x + y \leq 1$. Let $G, H : \mathcal{I} \to \mathcal{I}$ be partial maps defined as follows:

$$G(x) = H(x) = \begin{cases} 0 & \text{for } x = 0 \\ \frac{1}{16} & \text{for } x = \frac{1}{8} \\ \frac{1}{4} & \text{for } x = \frac{1}{4} \\ \frac{3}{4} & \text{for } x = \frac{3}{4} \\ \frac{15}{16} & \text{for } x = \frac{7}{8} \\ 1 & \text{for } x = 1 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that our operation "$\cdot$" is not the usual multiplication of real numbers, but as noted in the introduction the dual operation to $+$. Clearly G and H satisfy (T1), (T2), (T3) and (T4). But for the dual operation "$\cdot$" we have $G(\frac{7}{8} \cdot \frac{7}{8}) = G(1 - ((1 - \frac{7}{8}) + (1 - \frac{7}{8}))) = G(\frac{6}{8}) = \frac{3}{4} < \frac{7}{8} = 1 - ((1 - G(\frac{7}{8})) + (1 - G(\frac{7}{8}))) = G(\frac{7}{8}) \cdot G(\frac{7}{8})$.

The following result gives us new insight into mutual interrelation between the operators G and H.

Theorem 3.4 ([3, Theorem 9])**.** *Let $\mathcal{E}$ be an effect algebra, $G, H : E \to E$ mappings. Then the following conditions are equivalent:*

(1) *$(\mathcal{E}; G, H)$ is a dynamic effect algebra.*
(2) *G is a mapping of E into itself satisfying*
 (D1) *G has a left adjoint P such that $H = {}' \circ P \circ {}'$,*
 (D2) *if $x + y$ exists then $G(x) + G(y)$ exists and $G(x) + G(y) \leq G(x + y)$,*
 (D3) *if $x \cdot y$ exists then $P(x) \cdot P(y)$ exists and $P(x \cdot y) \leq P(x) \cdot P(y)$.*

And now we extend the above result for S-tense operators.

Theorem 3.5. *Let $\mathcal{E}$ be an effect algebra, $G, H : E \to E$ mappings. Then the following conditions are equivalent:*

(i) *$(\mathcal{E}; G, H)$ is an S-dynamic effect algebra.*

(ii) *G is a mapping of E into itself satisfying*

(D1) *G has a left adjoint P such that $H = {}' \circ P \circ {}'$,*

(D2) *if $x + y$ exists then $G(x) + G(y)$ exists and $G(x) + G(y) \leq G(x + y)$,*

(D3) *if $x \cdot y$ exists then $P(x) \cdot P(y)$ exists and $P(x \cdot y) \leq P(x) \cdot P(y)$.*

(D4) *if $x + y$ and $P(x) + P(y)$ exist then $P(x + y) \leq P(x) + P(y)$*

(D5) *if $x \cdot y$ and $G(x) \cdot G(y)$ exist then $G(x) \cdot G(y) \leq G(x \cdot y)$*

Proof. $\Longrightarrow$: By Theorem 3.4 we have (D1)–(D3). (D5) is clear from (T5). Let us assume that $P(x) + P(y)$ exists. Then $P(x) + P(y) = (P(x)' \cdot P(y)')' = (H(x') \cdot H(y'))'$. By (T5) $H(x') \cdot H(y') \leq H(x' \cdot y')$ and then $P(x) + P(y) = (H(x') \cdot H(y'))' \geq H(x' \cdot y')' = P(x + y)$ hence (D4) holds.
$\Longleftarrow$: Let (D1)–(D5) be satisfied. By Theorem 3.4, (T1)–(T4) holds. (T5) immediately follows from (D5) and totality of G. □

Remark 3.6. The preceding Theorem 3.5 allows us to work with S-dynamic effect algebras equipped with only one S-tense operator G satisfying conditions (D1)–(D6).

Theorem 3.7. *Let $(\mathcal{E}, G, H)$ be a dynamic effect algebra. Then $(\mathcal{E}, G, H)$ is an S-dynamic effect algebra.*

Proof. According to Theorem 3.4 and Theorem 3.5 we have to show that (D1)–(D3) implies (D4) and (D5). Let $(\mathcal{E}, G, H)$ be a dynamic effect algebra and $x, y \in E$ such that $x \cdot y$ and $G(x) \cdot G(y)$ are defined. By (D1), $PG(x) \leq x$ and $PG(y) \leq y$. Using the preceding observation and (D3) we get that $P(G(x) \cdot G(y)) \leq P(G(x)) \cdot P(G(y)) \leq x \cdot y$. Since P is a left adjoint we obtain $G(x) \cdot G(y) \leq G(x \cdot y)$. Now let us verify (D4). Suppose that $x, y \in E$ such that $x + y$ and $P(x) + P(y)$ are defined. We have by (T4) $x \leq GP(x)$ and $y \leq GP(y)$. Using (D2) we realize that $x + y \leq GP(x) + GP(y) \leq G(P(x) + P(y))$. Hence $P(x + y) \leq P(x) + P(y)$. □

In what follows we want to provide a meaningful procedure giving S-tense operators on every effect algebra which will be in accordance with an intuitive idea of time dependency.

Lemma 3.8. *Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra. Let $a_i, b_i \in E$ for $i \in I$ and let $\bigwedge\{a_i; i \in I\}$, $\bigwedge\{b_i; i \in I\}$, $\bigwedge\{a_i; i \in I\} \cdot \bigwedge\{b_i; i \in I\}$ be defined. Then also $a_i \cdot b_i$ is defined for all $i \in I$ and whenever $\bigwedge\{a_i \cdot b_i; i \in I\}$ exists, then*

$$\bigwedge\{a_i; i \in I\} \cdot \bigwedge\{b_i; i \in I\} \leq \bigwedge\{a_i \cdot b_i; i \in I\}.$$

Proof. Since $\bigwedge\{a_i; i \in I\} \cdot \bigwedge\{b_i; i \in I\}$ is defined, then also $a_i \cdot b_i$ exists for every $i \in I$. For every $i \in I$ we have

$$\bigwedge\{a_i; i \in I\} \cdot \bigwedge\{b_i; i \in I\} \leq a_i \cdot b_i$$

and then

$$\bigwedge\{a_i; i \in I\} \cdot \bigwedge\{b_i; i \in I\} \leq \bigwedge\{a_i \cdot b_i; i \in I\}.$$

□

By a *frame* (see, e.g. [4]) is meant a couple (T, R) where T is a non-void set and R is a binary relation on T. Furthermore, we will assume that for all $x \in T$ there are $y, z \in T$ such that zRx and xRy. The set T is considered to be a *time scale*, the relation R expresses a relationship "to be before" and "to be after". Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$ and a non-void set T, we can produce the direct power $\mathcal{E}^T = (E^T; +, o, j)$ where the operation $+$ is defined and evaluated on $p, q \in E^T$ componentwise, i.e. $p + q$ is defined if $p(t) + q(t)$ is defined for each $t \in T$ and then $(p+q)(t) = p(t) + q(t)$. By duality $p \cdot q$ is defined iff $p(t) \cdot q(t) = (p(t)' + q(t)')'$ is defined for each $t \in T$. Moreover, o, j are such elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$.

Theorem 3.9. *Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra and and let (T, R) be a frame. Then for every complete lattice M where E is a subposet of M we can define partial mappings G, H of E^T into itself as follows: For all $p \in E^T$, $G(p)$ is defined iff for all $s \in T$, $\bigwedge_M \{p(t); sRt\} \in E$ in which case*

$$G(p)(s) = \bigwedge\nolimits_M \{p(t); sRt\} = \bigwedge\nolimits_E \{p(t); sRt\}$$

and for all $p \in E^T$, $H(p)$ is defined iff for all $s \in T$ $\bigwedge_M \{p(t); tRs\} \in E$ in which case

$$H(p)(s) = \bigwedge\nolimits_M \{p(t); tRs\} = \bigwedge\nolimits_E \{p(t); tRs\}.$$

If this is the case, then G, H are S-tense operators on $\mathcal{E}^T$, i.e. $\mathcal{D} = (\mathcal{E}^T; G, H)$ is a partial S-dynamic effect algebra. Moreover, the following holds:

(a) *If R is reflexive then both G and H are contractive.*
(b) *If R is transitive then both G and H are transitive.*
(c) *If R is both reflexive and transitive then both G and H are conuclei.*

Proof. By [3, Theorem 6], (T1)–(T4) holds. Prove (T5). Assume that $p, q \in E^T$, $p \cdot q$ exists and $G(p), G(q), G(p \cdot q), G(p) \cdot G(q)$ exist. Hence, $p(t) \cdot q(t)$ exists for each $t \in T$. By Lemma 3.8 also $G(p)(s) \cdot G(q)(s) = \bigwedge_E \{p(t); sRt\} \cdot \bigwedge_E \{q(t); sRt\}$ $\leq \bigwedge_E \{p(t) \cdot q(t); sRt\} = G(p \cdot q)(s)$ for each $s \in T$. Thus $G(p) \cdot G(q) \leq G(p \cdot q)$. Analogously we can show $H(p) \cdot H(q) \leq H(p \cdot q)$.

It is enough to proceed for G only. Let us check (a). Assume that $p \in E^T$ and $G(p)$ is defined. Since R is reflexive then from tRt we obtain that $G(p)(t) = \bigwedge \{p(t) | t \in T, sRt\} \leq p(t)$. Let us proceed similarly for (b). Assume that $p \in E^T$ and both $G(p)$ and $G(G(p))$ are defined. We have, for all $s \in T$,

$$\begin{aligned} G(p)(s) &= \bigwedge \{p(u) | u \in T, sRu\} \leq \bigwedge \{p(u) | t, u \in T, sRt, tRu\} \\ &= \bigwedge \{\bigwedge \{p(u) | u \in T, tRu\} | t \in T, sRt\} \\ &= \bigwedge \{G(p)(t) | t \in T, sRt\} = G(G(p))(s) \end{aligned}$$

since $\{u \in T | t \in T, sRt, tRu\} \subseteq \{u \in T | sRu\}$ by transitivity. The validity of (c) follows immediately from (a) and (b). □

Remark 3.10. If the relation R on a non-void set T is a quasiorder, i.e. our frame is $(T, \leq)$ with $\leq$ reflexive and transitive, then $s \leq t$ expresses the fact that t "follows" s and s "is before" t. Then, for the operators H and G as defined in Theorem 3.9, $H(p)$ can be interpreted as "a history" of an element $p \in E^T$ and $G(p)$ is "a future" of p. More precisely, $H(p)$ says that *p was true in past with at* *least the same degree as p is in present* and $G(p)$ says that p *will be true in future with at least the same degree as it is now.*

Corollary 3.11. *Let $\mathcal{M} = (M; +, 0, 1)$ be a complete lattice effect algebra and let (T, R) be a frame. Define mappings $\widehat{G}, \widehat{H}$ of M^T into itself as follows: For all $p \in M^T$,*

$$\widehat{G}(p)(x) = \bigwedge\nolimits_M \{p(y); xRy\}$$

and

$$\widehat{H}(p)(x) = \bigwedge\nolimits_M \{p(y); yRx\}.$$

Then $\widehat{G}, \widehat{H}$ are S-tense operators on $\mathcal{M}^T$.

4. Representation of partial S-dynamic effect algebras

In Theorem 3.9, we presented a construction of natural S-tense operators when an effect algebra and a frame are given. However, we can ask, for a given S-dynamic effect algebra $(\mathcal{E}; G, H)$, whether there exist a frame (T, R) and an effect algebra $\mathcal{M} = (M; +, 0, 1)$ such that the S-tense operators G, H can be derived by this construction in the effect algebra $\mathcal{M}$ where $\mathcal{E}$ is embedded into the power algebra $\mathcal{M}^T$. Hence, we ask, if every element p of E is in the form $(p(t))_{t\in T}$ in M^T, $G(p)(s) = \bigwedge_M \{p(t); sRt\}$ and $H(p)(s) = \bigwedge_M \{p(t); tRs\}$. If such a representation exists then one can recognize the time variability of elements of E expressed as time dependent functions $p : T \to M$.

From Corollary 3.11 we immediately see that $(\mathcal{M}^T; \widehat{G}, \widehat{H})$ is automatically an S-dynamic effect algebra. This yields that any representable partial dynamic effect algebra should be also S-dynamic.

Proposition 4.1 ([3, Proposition 14])**.** *Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra with a full set S of morphisms into a complete lattice effect algebra $\mathcal{L}$, $M = L^S$. Then the map $i^S_{\mathcal{E}} : E \to L^S$ given by $i^S_{\mathcal{E}}(x)(s) = s(x)$ for all $x \in E$ and all $s \in S$ is an order reflecting morphism of effect algebras such that $i^S_{\mathcal{E}}(E)$ is a sub-effect algebra of $\mathcal{M} = \mathcal{L}^S$.*

It follows from the above proposition that any effect algebra with a full set S of states can be represented as a sub-effect algebra of $[0, 1]^S$. So we immediately obtain from Theorem 3.9 and from Proposition 4.1.

Corollary 4.2. *Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra with a full set S of morphisms into a complete lattice effect algebra $\mathcal{L}$, $M = L^S$ and let (T, R) be a frame. Define partial mappings G, H of E^T into itself as follows: For all $p \in E^T$, $G(p)$ is defined iff for all $x \in T$ $\bigwedge_M \{p(y); xRy\} \in E$ in which case*

$$G(p)(x) = \bigwedge\nolimits_M \{p(y); xRy\}$$

and for all $p \in E^T$, $H(p)$ is defined iff for all $x \in T$ $\bigwedge_M \{p(y); yRx\} \in E$ in which case

$$H(p)(x) = \bigwedge\nolimits_M \{p(y); yRx\}.$$

Then G, H are S-tense operators on $\mathcal{E}^T$, i.e. $\mathcal{D} = (\mathcal{E}^T; G, H)$ is a partial S-dynamic effect algebra and $i^{S,T}_{\mathcal{E}} : \mathcal{E}^T \to \mathcal{M}^T$ defined by $i^{S,T}_{\mathcal{E}}((x_t)_{t\in T}) = ((i^S_{\mathcal{E}}(x_t))_{t\in T})$ is an order reflecting morphism of effect algebras into the S-dynamic complete lattice effect algebra $(\mathcal{M}^T; \widehat{G}, \widehat{H})$ given by Theorem 3.9.

The previous corollary allows us to introduce, for any effect algebra $\mathcal{E}$ with an effect-algebraic extension of $+$-operation onto the MacNeille completion $\mathrm{MC}(E)$ and for any frame (T, R), a partial S-dynamic effect algebraic structure on $\mathcal{E}^T$ that can be fully reconstructed from $(\mathrm{MC}(\mathcal{E})^T; \widehat{G}, \widehat{H})$.

Definition 4.3. Let $\mathcal{E}, \mathcal{F}$ be effect algebras. A map $s : E \to F$ is called a *semi-S-morphism from $\mathcal{E}$ into $\mathcal{F}$* if it is order preserving, $s(0) = 0$, $s(1) = 1$, $s(x) \leq s(y)'$ and $s(x)+s(y) \leq s(x+y)$ whenever $x+y$ exists and also $s(x) \geq s(y)'$ and $s(x) \cdot s(y) \leq s(x \cdot y)$ whenever $x \cdot y$ exists. If $F = [0, 1]$ then s is called a *semi-S-state*.

Remark 4.4. Whenever we have a dynamic effect algebra $(\mathcal{E}, G, H)$, then G, H are by Theorem 3.7 also S-tense operators. This does not hold for the notion of semi-S-morphism (semi-S-states) where the inequality for the dual operation "$\cdot$" does not follow from inequality for operation "$+$".

Lemma 4.5. *Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra with a full set S of semi-S-morphisms into a complete lattice effect algebra $\mathcal{L}$, $M = L^S$. Then the map $i^S_{\mathcal{E}} : E \to M$ given by $i^S_{\mathcal{E}}(x)(s) = s(x)$ for all $x \in E$ and all $s \in S$ is an order reflecting semi-S-morphisms into M.*

Proof. According to [3, Lemma 17] we have $(i^S_{\mathcal{E}}(x) + i^S_{\mathcal{E}}(y))(s) \leq i^S_{\mathcal{E}}(x + y)(s)$ for all $s \in S$. Since S is full we have from Observation 2.1 that $i^S_{\mathcal{E}} : E \to M$ is an order reflecting morphism of bounded posets. Let $x \cdot y$ exists in $\mathcal{E}$. Then $(i^S_{\mathcal{E}}(x) \cdot i^S_{\mathcal{E}}(y))(s) = i^S_{\mathcal{E}}(x)(s) \cdot i^S_{\mathcal{E}}(y)(s) = s(x) \cdot s(y) \leq s(x \cdot y) = i^S_{\mathcal{E}}(x \cdot y)(s)$ for all $s \in S$. Hence $i^S_{\mathcal{E}}(x) \cdot i^S_{\mathcal{E}}(y) \leq i^S_{\mathcal{E}}(x \cdot y)$. □

Lemma 4.6 ([3])**.** *Let $(\mathcal{E}; G, H)$ be an S-dynamic effect algebra, $\mathcal{F}$ be an effect algebra and $s : E \to F$ a semi-S-morphism from $\mathcal{E}$ into $\mathcal{F}$. Then $s \circ G$ and $s \circ H$ are semi-S-morphisms from $\mathcal{E}$ into $\mathcal{F}$.*

Proof. We can use the same argument as in [3, Lemma 18]. □

Lemma 4.7. *Let $(\mathcal{E}; G, H)$ be a dynamic effect algebra with a set T of semi-S-morphisms into a complete lattice effect algebra $\mathcal{L}$. Let us put $R_G = \{(s, t) \in T \times T; (\forall x \in E)(s(G(x)) \leq t(x))\}$. Then*

(i) *If G is contractive then R_G is reflexive.*
(ii) *If G is transitive then R_G is transitive.*

Proof. (i): $G(x) \leq x$ yields $s(G(x)) \leq s(x)$ for all $x \in M$ and all $s \in T$. Hence sR_Gs.

(ii): Let $s, t, u \in T$, sR_Gt and tR_Gu. Let $x \in M$. Then $s(G(x)) \leq s(G(G(x))) \leq t(G(x)) \leq u(x)$. Hence sR_Gu.

□

Now, let us prove a representation theorem for dynamic effect algebras with a full set of semi-S-morphisms.

Theorem 4.8 (Weak representation theorem for dynamic effect algebras)**.** *For any dynamic effect algebra $(\mathcal{E}; G, H)$ with a full set S of semi-S-morphisms into a complete lattice effect algebra $\mathcal{L}$, there exists a set T of semi-S-morphisms containing S such that $i^T_{\mathcal{E}}$ is an order reflecting semi-S-morphism into the complete* *lattice S-dynamic effect algebra* $(\mathcal{M};\widehat{G},\widehat{H})$, $M = L^T$ *and a frame* (T, R_G), *such that, for all* $s,t \in T$, $(s,t) \in R_G$ *iff* $(\forall x \in E)(s(G(x)) \le t(x))$. *Further, for all* $x \in E$ *and for all* $s \in T$, $s(G(x)) = \bigwedge_L \{t(x); sR_G t\}$.

Moreover, if G *is contractive (transitive, a conucleus) then* $\widehat{G}$ *is contractive (transitive, a conucleus).*

Proof. Let T be the smallest set of of semi-S-morphisms into $\mathcal{L}$ containing S such that $s \in T$ implies that $s \circ G, s \circ H \in T$. Then we have to put $R_G = \{(s,t) \in T \times T; (\forall x \in E)(s(G(x)) \le t(x))\}$. Recall first that since $\mathcal{L}$ is a complete lattice effect algebra we have from Corollary 3.11 that $(M;\widehat{G},\widehat{H})$ is an S-dynamic effect algebra. Here $\widehat{G}(m)(s) = \bigwedge_L\{m(t); sR_G t\}$ and $\widehat{H}(m)(s) = \bigwedge_L\{m(t); tR_G s\}$ for all $m \in M$ and for all $s \in T$. Moreover, since T contains S it is again a full set of semi-S-morphisms. Therefore by Lemma 4.5 $i^T_{\mathcal{E}}$ is an order reflecting semi-S-morphisms into $\mathcal{M}$. It is enough to verify that for all $x \in E$ and for all $s \in T$, $s(G(x)) = \bigwedge_L\{t(x); sR_G t\}$. But this is immediate since, for any $s \in T$, we have a semi-S-morphism $t_s = s \circ G \in T$ such that $s(G(x)) = t_s(x)$ for all $x \in E$. Therefore, $s(G(x)) = t_s(x) \ge \bigwedge_L\{t(x); sR_G t\} \ge s(G(x))$ which yields that $i^T_{\mathcal{E}}(G(x))(s) = \bigwedge_L\{t(x); sR_G t\}$. Then $i^T_{\mathcal{E}}(G(x)) = \widehat{G}(i^T_{\mathcal{E}}(x))$.

The remaining part of the theorem follows immediately from Lemma 4.7 and Theorem 3.9. □

Corollary 4.9. *Let* $(\mathcal{E}; G, H)$ *be a dynamic effect algebra such that an extension of* $+$*-operation onto the MacNeille completion* $\mathrm{MC}(E)$ *of* E *exists. Then there is a countable frame* (T, R) *such that* $G = \widehat{G}_{|E}$ *with* $(\mathrm{MC}(\mathcal{E})^T;\widehat{G},\widehat{H})$ *given as in Theorem 3.9.*

Proof. Let $i_E : \mathcal{E} \to \mathrm{MC}(\mathcal{E})$ be the corresponding effect-algebraic embedding. Then T is the smallest set of semi-S-morphisms containing i_E that is closed under composition with G and H. Evidently $T = \{i_E \circ G^{n_1} \circ H^{m_1} \cdots \circ G^{n_k} \circ H^{m_k}; k, n_1, m_1, \dots, n_k, m_k \in \mathbb{N}_0\}$ is countable and we have an order reflecting semi-S-morphism $i^T_{\mathcal{E}} : \mathcal{E} \to \mathrm{MC}(\mathcal{E})^T$ of effect algebras. □

We show a positive example of this construction. The counterexample can be found in [3, Example 13].

Example 4.10. Let $\mathcal{I}_{\mathbb{Q}} = ([0,1], +, 0) \cap \mathbb{Q}$ be an effect algebra on the interval of rational numbers between 0 and 1 where $+$ is the usual sum and it is defined whenever $x + y \le 1$. Let $G : \mathcal{I}_{\mathbb{Q}} \to \mathcal{I}_{\mathbb{Q}}$ be a map defined by $G(x) = \frac{1}{2}x$ for all $0 \le x \le \frac{2}{3}$ and $G(x) = 2x - 1$ for $\frac{2}{3} \le x \le 1$. Then G clearly satisfies (T1)–(T3) and has a left adjoint P. Since G is a bijection, $P = G^{-1}$. Therefore $P(x) = 2x$ for all $0 \le x \le \frac{1}{3}$ and $P(x) = \frac{1}{2}x + \frac{1}{2}$ for $\frac{1}{3} \le x \le 1$. It is not hard to show that $G(x) = (P(x'))'$ for all $x \in \mathcal{I}_{\mathbb{Q}}$ hence we can define $H(x) = (P(x'))' = G(x)$ for all $x \in \mathcal{I}_{\mathbb{Q}}$ and $(\mathcal{I}_{\mathbb{Q}}, G, H)$ is a dynamic effect algebra.

We can see that $\mathrm{MC}(\mathcal{I}_{\mathbb{Q}}) = \mathcal{I}$, $T = \{i_{I_{\mathbb{Q}}} \circ G^n, n \in \mathbb{N}_0\}$ and $(i_{I_{\mathbb{Q}}} \circ G^n) R_G (i_{I_{\mathbb{Q}}} \circ G^m)$ iff $m \le n+1$. An effect algebraic embedding $i^T_{\mathcal{I}_{\mathbb{Q}}} : \mathcal{I}_{\mathbb{Q}} \to \mathcal{I}^T$ is given by $i^T_{\mathcal{I}_{\mathbb{Q}}}(x)(i_{I_{\mathbb{Q}}} \circ G^n) = i_{I_{\mathbb{Q}}} \circ G^n(x)$ for all $x \in \mathcal{I}_{\mathbb{Q}}$. The elements of T can be identified with the elements of $\mathbb{N}_0$ by the bijection $i_{I_{\mathbb{Q}}} \circ G^n \mapsto n$ for all $n \in N_0$. Hence $R_G \subseteq \mathbb{N}_0 \times \mathbb{N}_0$,

$nR_G m$ iff $m \leq n+1$. We may identify the elements of $\mathcal{I}^T$ as sequences of real numbers from the unit interval. Hence $i^T_{\mathcal{I}_{\mathbb{Q}}}(x) \mapsto (x, G(x), G^2(x), \ldots, G^n(x), \ldots)$. Using the relation R_G we have $\widehat{G}(a_0, a_1, \ldots) = (a_0 \wedge a_1, a_0 \wedge a_1 \wedge a_2, \ldots, \bigwedge_{j \leq n+1} a_j, \ldots)$ for all $a_i \in I$, $i \in \mathbb{N}_0$. Since $(i_{I_{\mathbb{Q}}} \circ G^n)(x) \geq (i_{I_{\mathbb{Q}}} \circ G^m)(x)$ for all $x \in \mathcal{I}_{\mathbb{Q}}$ if and only if $n \leq m$, for the elements embedded by $i^T_{\mathcal{I}_{\mathbb{Q}}}$ we have $\widehat{G}(x, G(x), \ldots, G^n(x), \ldots) = (G(x), G^2(x), \ldots, G^{n+1}(x), \ldots) = i^T_{\mathcal{I}_{\mathbb{Q}}}(G(x))$ for all $x \in \mathcal{I}_{\mathbb{Q}}$.

To get a stronger version of representation theorem as in [1] for Boolean algebras and [3] for dynamic effect algebras we have to impose more on S-tense operators G and H.

Definition 4.11. A dynamic effect algebra $(\mathcal{E}; G, H)$ is called *strict* if the following condition is satisfied:

(T3S) if $x, y \in E$ and $x + y$ exists then $G(x) + G(y)$ and $H(x) + H(y)$ exist, $G(x) + G(y) = G(x + y)$ and $H(x) + H(y) = H(x + y)$.

Recall that the basic difference between Theorem 4.8 and the following Theorem 4.12 is that from Theorem 4.8 we are not able directly recognize the S-tense operator H but Theorem 4.12 gives us this possibility. Hence any strict dynamic effect algebra $(\mathcal{E}; G, H)$ with a full set of morphisms from $\mathcal{E}$ into a complete lattice effect algebra $\mathcal{L}$ can be fully reconstructed as a subobject of a power $\mathcal{M}$ of copies of L equipped with naturally introduced S-tense operators $\widehat{G}, \widehat{H}$.

Theorem 4.12 (Strong representation theorem for S-dynamic effect algebras)**.** *For any strict dynamic effect algebra $(\mathcal{E}; G, H)$ with a full set S of morphisms from $\mathcal{E}$ into a complete lattice effect algebra $\mathcal{L}$, there exists a set T of morphisms from $\mathcal{E}$ into $\mathcal{L}$ containing S such that $\mathcal{E}$ is a sub-effect algebra of the complete lattice S-dynamic effect algebra $(\mathcal{M}; \widehat{G}, \widehat{H})$, $M = L^T$ and a frame (T, R), such that, for all $s, t \in T$, $(s, t) \in R$ iff $(\forall x \in E)(s(G(x)) \leq t(x))$. Further, for all $x \in E$ and for all $s \in T$, $s(G(x)) = \bigwedge_L \{t(x); sRt\}$ and $s(H(x)) = \bigwedge_L \{t(x); tRs\}$.*

Moreover, if G is contractive (transitive, a conucleus) then both $\widehat{G}$ and $\widehat{H}$ are contractive (transitive, conuclei).

Proof. By Theorem 3.9 we know that $(\mathcal{M}; \widehat{G}, \widehat{H})$ is an S-dynamic effect algebra. The rest of the theorem follows from [3, Theorem 29], Theorem 3.9 and Lemma 4.7. □

Remark 4.13. Let us note that for a strict dynamic effect algebra $(\mathcal{E}; G, H)$ with a full set of states S we can take as $\mathcal{L}$ the canonical effect algebra $\mathcal{L} = ([0, 1]; +, 0, 1)$ on the unit interval $[0, 1]$ of reals and each state s in S is in fact a morphism from $\mathcal{E}$ into $\mathcal{L}$. By Theorem 4.12, $\mathcal{E}$ is isomorphic to a sub-effect algebra of $\mathcal{L}^T$ and the relation R of the frame (T, R) is fully determined by the states as pointed in the theorem. If only semi-S-states are considered then a similar representation in a weak form is given by Theorem 4.8 for any S-dynamic effect algebra.

Corollary 4.14 ([3])**.** *Let $(\mathcal{E}; G, H)$ be a strict dynamic effect algebra such that an extension of $+$-operation onto the MacNeille completion* $\mathrm{MC}(E)$ *of E exists.*

Then there is a countable frame (T, R) *such that* $G = \widehat{G}_{|E}$ *and* $H = \widehat{H}_{|E}$ *with* $(\mathrm{MC}(\mathcal{E})^T; \widehat{G}, \widehat{H})$ *given as in Theorem 3.9.*

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Jan Paseka, Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2 2, 611 37 Brno, Czech Republic
e-mail: paseka@math.muni.cz

Jiří Janda, Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2 2, 611 37 Brno, Czech Republic
e-mail: 98599@mail.muni.cz