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ON THE ULTRA-QUASI-TIGHT EXTENSIONS

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Abstract. In their previous paper [4], Künzi and Olela Otafudu constructed the ultra-quasi-metric hull of a T_0-ultra-quasi-metric space. In this article, we continue these studies by investigating the tightness and essentiality of extension maps in the category of ultra-quasi-metric spaces and nonexpansive maps. We show, for instance, that q-spherical completeness is preserved by a retraction map. Furthermore, we point out some categorical aspects of ultra-quasi-metrically injective hulls.

1. Introduction

In [2], Agyingi *et al.* investigated tight extensions in the category of T_0-quasi-metric spaces. Their results were used to study endpoints in T_0-quasi-metric spaces. Furthermore, Agyingi [1] introduced tight extensions in the category of ultra-quasi-metric spaces and nonexpansive maps by extending the results from [2] on the tight extensions from quasi-metric point of view to the framework of ultra-quasi-metric spaces.

In this article, we introduce the concept of tightness and essentiality of nonexpansive maps in the category of ultra-quasi-metric spaces and nonexpansive maps that we call *ultra-quasi-tight* and *ultra-quasi-essential*, respectively. We point out that the approach used in this article is different to the ultra-tree construction approach used in [1], but our findings extend the results from [3] and [6] on metric and quasi-metric settings, respectively. We establish, among other results, that ultra-quasi-tightness and ultra-quasi-essentiality of an extension of an ultra-quasi-metric space are equivalent. Comparable studies in the framework of T_0-quasi-metric spaces have been conducted before by Olela Otafudu and Mushaandja [6].

In addition, we show, for instance, that there exists a covariant functor from the category of T_0-ultra-quasi-metric spaces and nonexpansive maps into the category of ultra-quasi-metrically injective hulls on a T_0-ultra-quasi-metric space and nonexpansive maps.

2. Preliminaries

In the sequel, we shall consider $\sup A$ for some subsets $A \subseteq [0,\infty)$. We recall that $\sup A = 0$ if $A = \emptyset$. Let X be a set and $v : X \times X \to [0,\infty)$ be a function. Then, v is an *ultra-quasi-pseudometric* on X if

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(a) $v(x,x) = 0$ for all $x \in X$,
(b) $v(x,y) \leq \max\{v(x,z), v(z,y)\}$ for all $x, y, z \in X$.

If v is an ultra-quasi-pseudometric on X, then the pair (X, v) is called an *ultra-quasi-pseudometric space.*

If the function v satisfies the condition

(c) for any $x, y \in X, v(x,y) = 0 = v(y,x)$ implies $x = y$ instead of condition (a),

then v is called a T_0-ultra-quasi-metric on X and the pair (X, v) is called T_0*-ultra-quasi-metric space* (see for instance [4, 5]).

Furthermore, if v is an ultra-quasi-pseudometric on X, then the function $v^t : X \times X \to [0, \infty)$ defined by $v^t(x,y) = v(y,x)$, for all $x, y \in X$ is also an ultra-quasi-pseudometric on X and it is called the *conjugate ultra-quasi-pseudometric* of v.

Note that for any v ultra-quasi-pseudometric on X, the function v^s defined by $v^s(x,y) := \max\{v(x,y), v^t(x,y)\}$ is an ultra-pseudometric on X.

Example 2.1. ([4, Example 1]) Let the set $X = [0, \infty)$. If we endow X with the function n defined by

$$n(a,b) = \begin{cases} a & \text{if} \quad a > b \\ 0 & \text{if} \quad a \leq b \end{cases}$$

for all $x, y \in X$, then n is a T_0-ultra-quasi-metric on X. Furthermore, one sees that

$$n^s(a,b) = \begin{cases} \max\{a,b\} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

whenever $a, b \in X$.

Lemma 2.2. ([4, Lemma 1]) *Let (X, v) be an ultra-quasi-pseudometric space and $f : X \to [0, \infty)$ be a function. For any $x, y \in X$, we have*

$$n(f(x), f(y)) \leq v(x,y) \textit{ if and only if } f(x) \leq \max\{f(y), v(x,y)\}.$$

Proof. ($\Rightarrow$) Suppose that $f(x) > \max\{f(y), v(x,y)\}$. Then, $f(x) > f(y)$, so $n(f(x), f(y)) = f(x) \leq v(x,y)$ by hypothesis. Then, $f(x) \leq \max\{f(y), v(x,y)\} < f(x)$, which yields a contradiction.

($\Leftarrow$) If $n(f(x), f(y)) > v(x,y) \geq 0$, then $f(y) < f(x)$; the hypothesis gives $f(x) \leq v(x,y)$, giving the contradiction that $n(f(x), f(y)) = f(x) \leq v(x,y)$. □

We recall that a map $h : (X, v) \to (Y, w)$ between two ultra-quasi-pseudometric spaces (X, v) and (Y, w) is called *nonexpansive* provided $w(h(x), h(y)) \leq v(x,y)$ for all $x, y \in X$. The map $h : (X, v) \to (Y, w)$ is called an *isometry map* provided that $w(h(x), h(y)) = v(x,y)$ for all $x, y \in X$. Moreover, two ultra-quasi-pseudometric spaces (X, v) and (Y, w) will be called *isometric* provided that there exists a bijective isometric map $h : (X, v) \to (Y, w)$.

Corollary 2.3. ([4, Corollary 1]) *Let (X, v) be an ultra-quasi-pseudometric space. Then,*

(a) *the function* $f : (X, v) \to ([0, \infty), n)$ *is a nonexpansive map if and only if* $f(x) \leq \max\{f(y), v(x, y)\}$, *for all* $x, y \in X$;
(b) *the function* $f : (X, v) \to ([0, \infty), n^t)$ *is a nonexpansive map if and only if* $f(x) \leq \max\{f(y), v(y, x)\}$, *for all* $x, y \in X$.

3. Isbell-convex ultra-quasi-metric space

We start this section by recalling some useful concepts from [4] needed in the sequel.

Consider an ultra-quasi-metric space (X, v). Let $x \in X$ and $\epsilon \in [0, \infty)$. Then, the set $C_v(x, \epsilon) = \{z \in X : v(x, z) \leq \epsilon\}$ is a $\tau(v^t)$-closed ball of radius ϵ at x.

Let $(x_i)_{i \in I}$ be a family of points in X and let $(\epsilon_i)_{i \in I}$ and $(\delta_i)_{i \in I}$ be families of points in $[0, \infty)$. We say that the family of double balls $(C_v(x_i, \epsilon_i), C_{v^t}(x_i, \delta_i))_{i \in I}$ has the *mixed binary intersection property* provided $v(x_i, x_j) \leq \max\{\epsilon_i, \delta_j\}$, for all $i, j \in I$.

Furthermore, we say that (X, v) is *q-spherically complete* (or *Isbell-convex ultra-quasi-metric space* [4]) provided that each family of double balls

$$(C_v(x_i, \epsilon_i), C_{v^t}(x_i, \delta_i))_{i \in I},$$

possessing the mixed binary intersection property satisfies

$$\bigcap_{i \in I} [C_v(x_i, \epsilon_i) \cap C_{v^t}(x_i, \delta_i)] \neq 0.$$

For any $x, y \in X$ and $\epsilon, \delta \geq 0$, we know from [4, Lemma 9] that

$$C_v(x, \epsilon) \cap C_{v^t}(y, \delta) \neq \emptyset \text{ if and only if } v(x, y) \leq \max\{\epsilon, \delta\}.$$

Example 3.1. ([4, Example 2]) If we equip $[0, \infty)$ with the T_0-ultra-quasi-metric n in Example 2.1, then $([0, \infty), n)$ is an Isbell-convex ultra-quasi-metric space.

Definition 3.2. (Compare [6, Definition 5]) Let (X, v_X) and (Y, v_Y) be ultra-quasi-pseudometric spaces. A map $\phi : (X, v_X) \to (Y, v_Y)$ is called a *retraction* if ϕ is onto, nonexpansive and there exists an isometry $\varphi : (Y, v_Y) \to (X, v_X)$ such that $\phi \circ \varphi = Id_Y$.

Proposition 3.3. *Let* (X, v_X) *and* (Y, v_Y) *be two ultra-quasi-pseudometric spaces. If* (X, v_X) *is an Isbell-convex ultra-quasi-pseudometric space and the map* $\phi : (X, v_X) \to (Y, v_Y)$ *is a retraction, then* (Y, v_Y) *is an Isbell-convex ultra-quasi-pseudometric space too.*

Proof. Let $(C_{v_Y}(y_i, r_i), C_{v_Y^t}(y_i, s_i))_{i \in I}$ be a family of double balls in (Y, v_Y) having the mixed binary intersection property. We have to show that

$$\bigcap_{i \in I} C_{v_Y}(y_i, r_i) \cap C_{v_Y^t}(y_i, s_i) \neq \emptyset.$$

Since $\phi : (X, v_X) \to (Y, v_Y)$ is a retraction, then there exists an isometry $\varphi : (Y, v_Y) \to (X, v_X)$ such that $\phi \circ \varphi = Id_Y$. Then, for all $i, j \in I$, we have $v_Y(y_i, y_j) \leq \max\{r_i, s_j\}$ by the mixed binary intersection property. Furthermore,

$$v_Y(\varphi(y_i), \varphi(y_j)) = v_X(y_i, y_j) \leq \max\{r_i, s_j\}, \text{ for all } i, j \in I.$$

We have that

$$\bigcap_{i\in I}[C_{v_Y}(\varphi(y_i), r_i) \cap C_{v_Y^t}(\varphi(y_i), s_i)] \neq \emptyset.$$

Let $a \in \bigcap_{i\in I}[C_{v_Y}(\varphi(y_i), r_i) \cap C_{v_Y^t}(\varphi(y_i), s_i)]$. Since ϕ is a nonexpansive map, then for all $i \in I$, we have

$$v_Y(\phi(a), y_i) = v_Y(\phi(a), \phi(\varphi(y_i))) \leq v_X(a, \varphi(y_i)) \leq s_i,$$

and

$$v_Y(y_i, \phi(a)) = v_Y(\phi(\varphi(y_i)), \phi(a)) \leq v_X(\varphi(y_i), a) \leq r_i.$$

Hence, $\phi(a) \in C_{v_Y}(y_i, r_i) \cap C_{v_Y^t}(y_i, s_i)$, for all $i \in I$. Therefore,

$$\bigcap_{i\in I} C_{v_Y}(y_i, r_i) \cap C_{v_Y^t}(y_i, s_i) \neq \emptyset,$$

which completes the proof. □

Definition 3.4. (Compare [2, Definition 2]) Let (Y, v) be a T_0-ultra-quasi-metric space. If X is a subspace of (Y, v), then (Y, v) is said to be an ultra-tight extension of X if, for any ultra-quasi-pseudometric w on Y such that $w \leq v$ and w agrees with v on $X \times X$, we have $w = v$.

Let (X, v_X) be a T_0-ultra-quasi-metric space. The pair of functions $f = (f_1, f_2)$, where $f_i : X \to [0, \infty)(i = 1, 2)$, is called *strongly tight* [4] provided

$$v_X(x, y) \leq \max\{f_2(x), f_1(y)\} \text{ for all, } x, y \in X.$$

We say that a pair of functions $f = (f_1, f_2)$ is *extremal strongly tight* [4] (or *minimal*) among the strongly tight pairs of functions on (X, v_X) provided that it is a strongly tight pair if and only if for any strongly tight pair of function $g = (g_1, g_2)$ on (X, v_X) such that $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ for all $x \in X$ we have $g_1(x) = f_1(x)$ and $g_2(x) = f_2(x)$.

Let $\mathcal{UT}(X, v_X)$ denote the class of all strongly tight pairs of functions on (X, v_X). For each $f = (f_1, f_2)$ and $g = (g_1, g_2) \in \mathcal{UT}(X, v_X)$, we set

$$N_X(f, g) = \max\{\sup_{x\in X} n(f_1(x), g_1(x)), \sup_{x\in X} n(g_2(x), f_2(x))\}.$$

Then, N_X is an extended T_0-ultra-quasi-metric on $\mathcal{UT}(X, v_X)$.

In what follows, we denote by $\nu_q(X, v_X)$ the class of minimal strongly tight pairs of functions on (X, v_X).

Moreover, we keep the same notation N_X for the restriction of N_X to $\nu_q(X, v_X) \times \nu_q(X, v_X)$. Then, N_X is a (real-valued) T_0-ultra-quasi-metric on $\nu_q(X, v_X)$ (see [4] for more details).

If the pair of functions $f = (f_1, f_2)$ is minimal strongly tight on (X, v_X), then

$$f_1(x) = \sup_{x\in X} n(v_X(y, x), f_2(y)),$$

and

$$f_2(x) = \sup_{x\in X} n(v_X(x, y), f_1(y)),$$

for any $x \in X$ (see [4, Corollary 4]).

For any $x \in X$, the pair of functions $f_x(y) = (v_X(x,y), v_X(y,x))$ for all $x \in X$ is minimal strongly tight on (X, v_X). The map e_X defined by $x \mapsto f_x$, for any $x \in X$ defines an isometric embedding of (X, v_X) into $(\nu_q(X, v_X), N_X)$ (see [4, Theorem 1]). The pair $(\nu_q(X, v_X), N_X)$ is called an *ultra-quasi-metrically injective hull* of (X, v_X). Note that the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space is q-spherically complete (or Isbell-convex ultra-quasi-metric space) and it is unique up to isometry.

The proof of the following result can be found in [1, Theorem 23].

Proposition 3.5. *Let (Y, v_Y) be a T_0-ultra-quasi-metric space. If X is a subspace of (Y, v_Y), then the following three conditions are equivalent:*

(a) *(Y, v_Y) is an ultra-quasi-tight extension of X;*
(b) *$v(y, y') = \sup\{v(x, x') : x, x' \in X, v(x, x') > v(x, y), v(x, x') > v(y', x')\}$ for all $y, y' \in Y$;*
(c) *$e_Y|_X(y)(x) = (v(y, x), v(x, y)), x \in X$, is minimal on X for all $y \in Y$ and the map $\phi : (Y, v) \to (\nu_q(X, v), N) : y \mapsto e_Y|_X$ is an isometric embedding.*

Let (Y, v) be an ultra-tight extension of X. From Proposition 3.5, one observes that if the map $v : Y \to (\nu_q(X, v), N)$ is defined by $v(y) = f_y$ for all $y \in Y$, then v is a unique isometric embedding (see [1]). Therefore, the ultra-quasi extension (Y, v) of X is seen as a subspace of the extension $(\nu_q(X, v), N)$ of X. Thus, $(\nu_q(X, v), N)$ is maximal among the T_0-ultra-quasi-metric ultra-quasi extensions of X.

Definition 3.6. Let (X, v_X) be an ultra-quasi-pseudometric space. Let $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ be families of points in X, and $(r_i)_{i\in I}, (s_i)_{i\in I}$ be families of positive real numbers. We say that the family $\mathcal{C} = (C_{v_X}(x_i, r_i), C_{v^t_X}(y_i, s_i))_{i\in I}$ of double balls in X meets potentially in X provided that there exists a T_0-ultra-quasi-metric ultra-quasi-metric extension (Y, v_Y) of (X, v_X) such that $\bigcap_{i\in I}(C_{v_Y}(x_i, r_i) \cap C_{v^t_Y}(y_i, s_i)) \neq \emptyset$.

Proposition 3.7. (Compare [2, Proposition 6]) *Let (X, v_X) be a T_0-ultra-quasi-metric space. If $\mathcal{C} = (C_{v_X}(x_i, r_i), C_{v^t_X}(x_i, s_i))_{i\in I}$ is a family of double balls in X, then the following conditions are equivalent:*

(a) *$\mathcal{C}$ meets potentially in X;*
(b) *For any $i, j \in I, C_{v_X}(x_i, r_i)$ meets potentially in X with any $C_{v_X}(x_j, r_j)$;*
(c) *$v_X(x_i, x_j) \leq \max\{r_i, s_j\}$, for all $i, j \in I$;*
(d) *there exists a minimal (strongly tight) function pair $h = (h_1, h_2)$ on X with $h_2(x_i) \leq r_i$ and $h_1(x_i) \leq s_i$ for all $i \in I$.*

Proof. We only prove (c) ⇒ (d) and (d) ⇒ (a), since (a) ⇒(b) ⇒(c) are straightforward.

(c) ⇒ (d) is obvious for $I = \emptyset$. For $I \neq \emptyset$, on $Y = \{x_i : i \in I\}$ and for all $y \in Y$, we define $g = (g_1, g_2)$ by $g_1(y) = \inf\{s_i : x_i = y\}$ and $g_2(y) = \inf\{r_i : x_i = y\}$.

Let $y_0 \in Y$. Then, we set

$$f_1(x) = \begin{cases} g_1(x) & \text{if } x \in Y \\ \max\{g_1(y_0), v_X(y_0, x)\} & \text{if } x \in X \setminus Y \end{cases}$$

and

$$f_2(x) = \begin{cases} g_2(x) & \text{if } x \in Y \\ \max\{v_X(x, y_0), g_2(y_0)\} & \text{if } x \in X \setminus Y. \end{cases}$$

It follows that $f_1(x_i) \leq s_i$ and $f_2(x_i) \leq r_i$ for all $i \in I$. Moreover, for any $x, x' \in X$, we have

$$v_X(x, x') \leq \max\{v_X(x, x_i), v_X(x_i, x')\} \leq \max\{f_2(x), f_1(x')\}.$$

Thus, $f = (f_1, f_2)$ is strongly tight on X. By Zorn's Lemma there exists a minimal strongly tight function pair $h = (h_1, h_2)$ on X such that $h_1(x) \leq f_1(x)$ and $h_2(x) \leq f_2(x)$, for all $x \in X$. Hence, $h \leq f$.

(d) ⇒ (a) Suppose that $h = (h_1, h_2)$ is a minimal strongly tight function pair on X such that $h_1(x_i) \leq s_i$ and $h_2(x_i) \leq r_i$ for all $i \in I$.

If for some $x \in X$, $h = (v_X(x, .), v_X(., x))$, then

$$x \in \bigcap_{i \in I} (C_{v_X}(x_i, r_i) \cap C_{v_X^t}(x_i, s_i)).$$

Hence, the family $\mathcal{C}$ meets potentially in X.

If for some $x \in X, h \neq (v_X(x, .), v_X(., x))$, then we extend X to a space Y by adding one point y_0 to X. Furthermore, we define a T_0-ultra-quasi-metric v_Y on Y which extends v_X by $v_Y(x, y_0) = h_2(x)$ and $v_Y(y_0, x) = h_1(x)$ for all $x \in X$ and $v_Y(y_0, y_0) = 0$. By using the fact that $h = (h_1, h_2)$ is a contracting function pair and, by the strong tightness of $h = (h_1, h_2)$, it is readily checked that v_Y satisfies the strong triangle inequality on Y. Moreover, since $h_1(x)$ or $h_2(x)$ is positive, v_Y is a T_0-ultra-quasi-metric on Y. Therefore,

$$y_0 \in \bigcap_{i \in I} (C_{v_Y}(x_i, r_i) \cap C_{v_Y^t}(x_i, s_i)),$$

which completes the proof. □

4. Ultra-quasi-tight extension

We introduce the concepts of ultra-quasi-tightness and ultra-quasi-essentiality of an extension, and we show that these two concepts are equivalent.

Definition 4.1. Let (X, v_X) and (Y, v_Y) be T_0-ultra-quasi-metric spaces and $\alpha : (X, v_X) \to (Y, v_Y)$ be an extension of (X, v_X). Then,

(a) the map α is said to be *ultra-quasi-tight* provided that for any T_0-ultra-quasi-metric v on Y, which satisfies $v(y_1, y_2) \leq v_Y(y_1, y_2)$ for all $y_1, y_2 \in Y$ and $v(\alpha(x_1), \alpha(x_2)) = v_X(x_1, x_2)$ for all $x_1, x_2 \in X$, we have that $v = v_Y$.

(b) the map α is said to be *ultra-quasi-essential* provided that for any nonexpansive map $\varphi : (Y, v_Y) \to (Z, v_Z)$, for which $\varphi \circ \alpha : (X, v_X) \to (Z, v_Z)$ is an extension of (X, v_X), we have that φ is an extension of (Y, v_Y).

Theorem 4.2. *Let (X, v_X) and (Y, v_Y) be T_0-ultra-quasi-metric spaces and $\alpha : (X, v_X) \to (Y, v_Y)$ be an extension of (X, v_X). Then, α is ultra-quasi-tight if and only if α is ultra-quasi-essential.*

Proof. ($\Leftarrow$) Suppose that the extension map $\alpha : (X, v_X) \to (Y, v_Y)$ is ultra-quasi-essential. Let v be a T_0-ultra-quasi-metric on Y such that

$$v(y_1, y_2) \leq v_Y(y_1, y_2), \text{ for all } y_1, y_2 \in Y$$

and

$$v(\alpha(x_1), \alpha(x_2)) = v_X(x_1, x_2), \text{ for all } x_1, x_2 \in X.$$

Since the identity map $id_Y : (Y, v_Y) \to (Y, v_Y)$ is nonexpansive and α is an isometry, we have that $id_Y \circ \alpha$ is an isometry by ultra-quasi-essentiality of α and $v = v_Y$. Hence, α is ultra-quasi-tight.

($\Rightarrow$) Suppose that the extension map $\alpha : (X, v_X) \to (Y, v_Y)$ is ultra-quasi-tight and let $\varphi : (Y, v_Y) \to (Z, v_Z)$ be an isometry, where (Z, v_Z) is a T_0-ultra-quasi-metric space. In order to show that α is ultra-quasi-essential, let us consider v to be a T_0-ultra-quasi-metric on Y defined by

$$v(y, y') = \max\{k v_Y(y, y'), (1 - k) v_Z(\varphi(y), \varphi(y'))\}$$

for all $y, y' \in Y$ and $0 < k < 1$. For any $y, y' \in Y$, we have

$$\begin{aligned} v(y, y') &= \max\{k v_Y(y, y'), (1 - k) v_Z(\varphi(y), \varphi(y'))\} \\ &\leq \max\{k v_Y(y, y'), (1 - k) v_Y(y, y')\} \quad (\varphi \text{ is nonexpansive }) \\ &\leq v_Y(y, y'). \end{aligned}$$

Thus,

$$v(y, y') \leq v_Y(y, y') \text{ for all } y, y' \in Y.$$

We claim that $v(\alpha(x), \alpha(x')) = v_X(x, x')$ for all $x, x' \in X$. Suppose that

$$v(\alpha(x), \alpha(x')) \neq v_X(x, x').$$

Case 1. If $v(\alpha(x), \alpha(x')) > v_X(x, x')$. Then, by the definition of T_0-ultra-quasi-metric v, we have

$$\begin{aligned} v(\alpha(x), \alpha(x')) &= \max\{k v_Y(\alpha(x), \alpha(x')), (1 - k) v_Z(\varphi(\alpha(x), \varphi(\alpha(x')))\} \\ &= \max\{k v_X(x, x'), (1 - k) v_X(x, x')\} \quad (\alpha \text{ and } \varphi \circ \alpha \text{ are isometries}) \\ &\leq v_X(x, x') \quad \text{(a contradiction)}. \end{aligned}$$

Case 2. If $v(\alpha(x), \alpha(x')) < v_X(x, x')$, then let $y = \alpha(x), y' = \alpha(x')$ because α is an isometry. In addition, since α is an isometry, we have $v_Y(y, y') = v_Y(\alpha(x), \alpha(x')) = v_X(x, x') > v(\alpha(x), \alpha(x')) = v(y_1, y_2)$ - this contradicts (1).

Hence, $v(\alpha(x), \alpha(x')) = v_X(x, x') = v_Y(\alpha(x), \alpha(x'))$ for all $x, x' \in X$. Furthermore, we have $v(y, y') = v_Y(y, y')$ for all $y, y' \in Y$ by the ultra-quasi-tightness of α. It follows that for all $y, y' \in Y$,

$$\begin{aligned} v_Z(\varphi(y), \varphi(y')) &= v_Z(\varphi(\alpha(x)), \varphi(\alpha(x'))) \\ &= v_X(x, x') = v_Y(\alpha(x), \alpha(x')) = v_Y(y, y'). \end{aligned}$$

Thus, φ is an isometry, and hence α is ultra-quasi-essential. □

Theorem 4.3. *For any T_0-ultra-quasi-metric spaces (X, v_X) and (Y, v_Y), if the map $\alpha : (X, v_X) \to (Y, v_Y)$ is an extension of (X, v_X), then the following conditions are equivalent:*

(a) $\alpha : (X, v_X) \to (Y, v_Y)$ *is ultra-quasi-tight;*

(b) $\alpha : (X, v_X) \to (Y, v_Y)$ *is ultra-quasi-essential;*
(c) $v_Y(y, y') = \sup\{v_X(x, x') : x, x' \in X, v_X(x, x') > v_Y(\alpha(x), y), v_X(x, x') > v_Y(y', \alpha(x'))\}$, *for all* $y, y' \in Y$*;*
(d) $e_Y|_X(y)(x) = (v_X(y, x), v_X(x, y)), x \in X$, *is minimal on* X *for all* $y \in Y$ *and the map* $\phi : (Y, v_Y) \to (\nu_q(X, v_X), N_X) : y \mapsto e_Y|_X$ *is an isometric embedding.*

Proof. The proof follows from [1, Theorem 23] and Theorem 4.2. □

Proposition 4.4. *Let* (X, v) *and* (Y, v_Y) *be* T_0*-ultra-quasi-metric spaces. If* $\alpha : (X, v) \to (Y, v_Y)$ *is an ultra-quasi-tight extension, then there exists a unique ultra-quasi-tight extension* $e_Y : (Y, v_Y) \to (\nu_q(X, v), N_X)$ *such that the triangle*

$$\begin{array}{ccc} (X, v) & \xrightarrow{\ \alpha\ } & (Y, v_Y) \\ & \searrow^{e_X} & \downarrow{\scriptstyle e_Y} \\ & & (\nu_q(X, v), N_X) \end{array}$$

commutes.

Proof. Indeed, we have that $(\nu_q(X, v), N_X)$ is ultra-quasi-metric injective by [4, Theorem 2] as $(\nu_q(X, v), N_X)$ is a q-spherically complete T_0-ultra-quasi-metric space. Then, this guarantees the existence of the map e_Y and the commutativity of the diagram.

From Theorem 4.3, we have that the map $\alpha : (X, v) \to (Y, v_Y)$ is an ultra-quasi-essential as it is ultra-quasi-tight and $e_Y \circ \alpha = e_X$ is an isometry. It follows that $e_Y : (Y, v_Y) \to (\nu_q(X, v), N_X)$ is an isometry.

Suppose that $e'_Y : (Y, v_Y) \to (\nu_q(X, v), N_X)$ is another isometry such that $e'_Y \circ \alpha = e_X$. We have to show that $e_Y = e'_Y$.

Let $y \in Y$ and $x \in X$. We have

$$\begin{aligned} (e_Y(y))_1(x) &= N_X(e_Y(y), e_X(x)) = N_X(e_Y(y), e_Y(\alpha(x))) = v_Y(y, \alpha(x)) \\ &= v_Y(e'_Y(y), e'_Y(\alpha(x))) = N_X(e'_Y(y), e_X(x)) = (e'_Y(y))_1(x). \end{aligned}$$

Hence, $(e_Y(y))_1(x) = (e'_Y(y))_1(x)$, for all $x \in X$.

Furthermore, one shows by duality that $(e_Y(y))_2(x) = (e'_Y(y))_2(x)$, for all $x \in X$. So $e_Y = e_X$, which ends the proof. □

Definition 4.5. Let (Y, v_Y) be a T_0-ultra-quasi-metric space. Then, (Y, v_Y) is called *ultra-quasi-metrically injective* provided that whenever (X, v_X) is a T_0-ultra-quasi-metric space, any subspace A of (X, v_X) and any nonexpansive map $\varphi : A \to (Y, v_Y)$, φ can be extended to a nonexpansive map $\phi : (X, v_X) \to (Y, v_Y)$.

Definition 4.6. Let (X, v_X) and (Y, v_Y) be T_0-ultra-quasi-metric spaces. In addition, suppose $\alpha : (X, v_X) \to (Y, v_Y)$ is an extension of (X, v_X). Then, $\alpha(X)$ is called

(a) an *ultra-quasi-metrically injective hull* of (X, v_X) provided that (Y, v_Y) is q-spherically complete and α is ultra-quasi-tight,
(b) an *ultra-quasi-metrically injection* of (X, v_X) provided that (Y, v_Y) is ultra-quasi-metrically injective and α is ultra-quasi-metric-essential.

The following is a consequence of Theorem 4.3 and [4, Theorem 2].

Proposition 4.7. *Let $(X, v_X)$and (Y, v_Y) be T_0-ultra-quasi-metric spaces. If $\alpha : (X, v_X) \to (Y, v_Y)$ is an extension of (X, v_X), then (Y, v_Y) is an ultra-quasi-metrically injective hull of (X, v_X) if and only if $\alpha(X)$ is an ultra-quasi-metrically injection of (X, v_X).*

Theorem 4.8. *Let (X, v_X) and (Y, v_Y) be T_0-ultra-quasi-metric spaces. If $\alpha : (X, v_X) \to (Y, v_Y)$ is an ultra-quasi-tight extension of (X, v_X), then there exists an isomorphism $\varphi : (\nu_q(X, v_X), N_X) \to (\nu_q(Y, v_Y), N_Y)$.*

Proof. Let $\alpha : (X, v_X) \to (Y, v_Y)$ be an ultra-quasi-tight extension of (X, v_X). If $e'_Y : (Y, v_Y) \to (\nu_q(Y, v_Y), N_Y)$ is an extension, then by Proposition 4.4, there exists a unique ultra-quasi-extension $e_Y : (Y, v_Y) \to (\nu_q(X, v_Y), N_X)$ such that the triangle

$$\begin{array}{ccc} (X, v_X) & \xrightarrow{\ \alpha\ } & (Y, v_Y) \\ & \searrow^{e_X} & \downarrow{\scriptstyle e_Y} \\ & & (\nu_q(X, v_X), N_X) \end{array}$$

commutes.

Since $e_Y \circ \alpha = e_X$ is ultra-quasi-tight and e_Y is ultra-quasi-tight, there exists a unique ultra-quasi-tight extension $g : (\nu_q(X, v_X), N_X) \to (\nu_q(Y, v_Y), N_Y)$. By the maximality of e'_Y, we have

$$\begin{array}{ccc} (Y, v_Y) & \xrightarrow{\ e_Y\ } & (\nu_q(X, v_X), N_X) \\ & \searrow^{e'_Y} & \downarrow{\scriptstyle g} \\ & & (\nu_q(Y, v_Y), N_Y) \end{array}$$

commutes.

Furthermore, by the ultra-quasi-tightness of $e'_Y = g \circ e_Y$, it follows that g is ultra-quasi-tight, since $(\nu_q(X, v_X), N_X)$ is unique up to isometry by [4, Proposition 7(b)]. Hence, g is an isomorphism. □

The following lemma is an ultra-quasi-metric version of [6, Lemma 15]. Therefore, we leave its proof to the reader.

Lemma 4.9. *Let $(X, v_X)$and (Y, v_Y) be T_0-ultra-quasi-metric spaces. An extension $\alpha : (X, v_X) \to (Y, v_Y)$ of (X, v_X) is ultra-quasi-tight if whenever $y, y' \in Y$, we have $v_Y(y, y') = \sup\{v_X(x, x') : x, x' \in X, v_X(x, x') > v_Y(\alpha(x), y), v_X(x, x') > v_Y(y', \alpha(x))\}$.*

Remark 4.10. From Lemma 4.9, it is easy to see that for any T_0-ultra-quasi-metric space (X, v), the isometry $e_X : (X, v) \to (\nu_q(X, v), N_X)$ is an ultra-quasi-tight extension of (X, v). We have

$$N_X(f, g) = \sup\{v(x, x') : x, x' \in X, v(x, x') > f_2(x) \text{ and } v(x, x') > g_1(x')\},$$

from [4, Lemma 8]. Moreover, $N_X(f, g) = \sup\{v(x, x') : x, x' \in X, v(x, x') > N_X(e_X(x), f) \text{ and } v(x, x') > N_X(g, e_X(x'))\}$ for any $f = (f_1, f_2), g = (g_1, g_2) \in$

$\nu_q(X,v)$. Furthermore, $e_X(X)$ is an ultra-quasi-metrically injective hull and ultra-quasi-metrically injective because $(\nu_q(X,v),N_X)$ is an ultra-quasi-metrically injective hull. In addition, e_X is a maximal ultra-tight extension of (X,v) by Proposition 4.4.

Remark 4.11. The maximal ultra-quasi-tight extension of any T_0-ultra-quasi-metric space (X,v_X) is unique up to isomorphism.

5. A functor between ultra-quasi-metrically injective hulls

Let (X,v) and (Y,w) be T_0-ultra-quasi-metric spaces and $\varphi:(X,v)\to(Y,w)$ be a nonexpansive map. For any $y\in Y$ and $(f_1,f_2)\in(\nu_q(X,v),N_X)$, we define a pair of functions $f_\varphi=((f_\varphi)_1,(f_\varphi)_2)$ by

$$(f_\varphi)_1(y):=\inf_{x\in X}\max\{w(\varphi(x),y),f_1(x)\}$$

and

$$(f_\varphi)_2(y):=\inf_{x\in X}\max\{w(y,\varphi(x)),f_2(x)\}.$$

It is easy to see that the functions $(f_\varphi)_1:Y\to[0,\infty)$ and $(f_\varphi)_2:Y\to[0,\infty)$ are well defined.

Proposition 5.1. *Let (X,v) and (Y,w) be T_0-ultra-quasi-metric spaces and $\varphi:(X,v)\to(Y,w)$ be a nonexpansive map. Then, we have:*

(a) *The function pair $f_\varphi=((f_\varphi)_1,(f_\varphi)_2)$ is strongly tight on (Y,w) whenever $f=(f_1,f_2)\in\nu_q(X,v)$.*
(b) *If $x\in X$, then $(f_\varphi)_1(\varphi(x))=0=(f_\varphi)_2(\varphi(x))$.*
(c) *The functions $(f_\varphi)_1:(Y,w)\to([0,\infty),n^t)$ and $(f_\varphi)_2:(Y,w)\to([0,\infty),n)$ are nonexpansive.*

Proof. It is easy to prove (b). Therefore, we only prove (a) and (c).
(a) Let $y,y'\in Y$. Then,

$$\begin{aligned}
&\max\{(f_\varphi)_2(y),(f_\varphi)_1(y')\}\\
&=\max\left(\inf_{x\in X}\max\{w(y,\varphi(x)),f_2(x)\},\ \inf_{x'\in X}\max\{w(\varphi(x'),y'),f_1(x')\}\right)\\
&\geq\inf_{x,x'\in X}\max\{w(y,\varphi(x)),w(\varphi(x'),y'),f_2(x),f_1(x')\}\\
&\geq\inf_{x,x'}\max\{w(y,\varphi(x)),v(x,x'),w(\varphi(x'),y')\}\quad(f\text{ is strongly tight})\\
&\geq\inf_{x,x'}\max\{w(y,\varphi(x)),w(\varphi(x),\varphi(x')),w(\varphi(x'),y')\}\quad(\varphi\text{ nonexpansive})\\
&=w(y,y')\quad(\text{by strong triangle inequality}).
\end{aligned}$$

Thus, $f_\varphi=((f_\varphi)_1,(f_\varphi)_2)$ is strongly tight on (Y,w).

(c) Let us prove that $(f_\varphi)_2:Y\to[0,\infty)$ is nonexpansive and the proof for $(f_\varphi)_1:(Y,w)\to([0,\infty),n^t)$ can be obtained by similar arguments.

Let $y \in Y$. Then,

$$\begin{aligned}(f_\varphi)_2(y) &= \inf_{x\in X} \max\{w(y,\varphi(x)), f_2(x)\}\\ &\le \inf_{x\in X} \max\{w(y,y'), w(y',\varphi(x)), f_2(x)\}\\ &\le \max(w(y,y'), \inf_{x\in X} \max\{w(y',\varphi(x)), f_2(x)\})\\ &= \max\{w(y,y'), (f_\varphi)_2(y')\} \text{ (by definition of } (f_\varphi)_2).\end{aligned}$$

Hence, $(f_\varphi)_2$ is nonexpansive. □

Proposition 5.2. *Let (X, v) and (Y, w) be T_0-ultra-quasi-metric spaces and $\varphi : (X, v) \to (Y, w)$ be a nonexpansive map. Then, the function pair $f_\varphi = ((f_\varphi)_1, (f_\varphi)_2)$ is minimal strongly tight on (Y, w) whenever $f = (f_1, f_2) \in \nu_q(X, v)$.*

Proof. Let $f = (f_1, f_2) \in \nu_q(X, v)$. Suppose that the function pair $f_\varphi = ((f_\varphi)_1, (f_\varphi)_2)$ is not minimal strongly tight on (Y, w).

Let $g = (g_1, g_2)$ be a strongly tight function pair such that

$$(g_1, g_2) < ((f_\varphi)_1, (f_\varphi)_2).$$

Then, there exists $y_0 \in Y$ such that $g_1(y_0) \le (f_\varphi)_1(y_0)$ and $g_2(y_0) \le (f_\varphi)_2(y_0)$.

Suppose $g_2(y_0) < (f_\varphi)_2(y_0)$. The case $g_1(y_0) < (f_\varphi)_1(y_0)$ follows similarly. For any $x \in X$, since $(f_\varphi)_2$ is nonexpasive bz Proposition 5.1(c), it follows that

$$\begin{aligned}(f_\varphi)_2(y_0) &\le \max\{w(y_0,\varphi(x)), (f_\varphi)_2(\varphi(x))\}\\ &\le \max\{g_2(y_0), g_1(\varphi(x)), (f_\varphi)_2(\varphi(x))\} \quad \text{(by the strong tightness of } g)\\ &\le \max\{g_2(y_0), (f_\varphi)_1(\varphi(x)), (f_\varphi)_2(\varphi(x))\}\\ &= g_2(y_0) \quad \text{(by Proposition 5.1(b))}.\end{aligned}$$

We reach a contradiction. Hence, $f_\varphi = ((f_\varphi)_1, (f_\varphi)_2)$ is a minimal strongly tight function pair on (Y, w). □

Lemma 5.3. *Let (X, v) and (Y, w) be T_0-ultra-quasi-metric spaces and $\varphi : (X, v) \to (Y, w)$ be a nonexpansive map. Then,*

$$N_Y(f_\varphi, g_\varphi) \le N_X(f, g) \text{ whenever } f = (f_1, f_2), g = (g_1, g_2) \in \nu_q(X, v).$$

Proof. Let $y \in Y$ and $f = (f_1, f_2), g = (g_1, g_2) \in \nu_q(X, v)$. We just consider the case $n((f_\varphi)_1(y), (g_\varphi)_1(y)) = (f_\varphi)_1(y)$. If $n((f_\varphi)_1(y), (g_\varphi)_1(y)) = 0$, there is nothing to prove.

Then, we have

$$\begin{aligned}n((f_\varphi)_1(y), (g_\varphi)_1(y)) = (f_\varphi)_1(y) &= \inf_{x\in X} \max\{w(\varphi(x), y), f_1(x)\}\\ &\le \max\{w(y,y), f_1(x)\} \text{ (taking } \varphi(x) = y)\\ &= f_1(x)\\ &\le \sup_{x\in X} n(f_1(x), g_1(x)).\end{aligned}$$

Hence,

$$\sup_{y\in Y} n((f_\varphi)_1(y), (g_\varphi)_1(y)) \le \sup_{x\in X} n(f_1(x), g_1(x)). \tag{5.1}$$

Furthermore, by duality, one shows that

$$\sup_{y\in Y} n((g_\varphi)_2(y),(f_\varphi)_2(y)) \leq \sup_{x\in X} n(g_2(x),f_2(x)). \tag{5.2}$$

Combining inequalities (5.1) and (5.2), we have

$$N_Y(f_\varphi,g_\varphi) \leq N_X(f,g) \text{ whenever } f=(f_1,f_2), g=(g_1,g_2)\in \nu_q(X,v).$$

□

Proposition 5.4. *Let (X,u), (Y,v) and (Z,w) be T_0-ultra-quasi-metric spaces. If the maps $\varphi : (X,u)\to(Y,v)$ and $\psi : (Y,v)\to(Z,w)$ are nonexpansive, then we have $f_{\psi\circ\varphi}=(f_\varphi)_\psi$, where the function pair $f_{\psi\circ\varphi}$ is defined by $f_{\psi\circ\varphi} := ((f_{\psi\circ\varphi})_1,(f_{\psi\circ\varphi})_2)$ whenever $f=(f_1,f_2)\in\nu_q(X,v)$.*

Proof. Let $f=(f_1,f_2)\in\nu_q(X,v)$. We only prove that $(f_{\psi\circ\varphi})_1(z)=((f_\varphi)_\psi)_1(z)$ whenever $z\in Z$ and the proof of $(f_{\psi\circ\varphi})_2(z)=((f_\varphi)_\psi)_2(z)$ whenever $z\in Z$ follows by similar arguments.

For any $z\in Z$, we have

$$\begin{aligned}
&((f_\varphi)_\psi)_1(z) = \inf_{y\in Y}\max\{w(\psi(y),z),(f_\varphi)_1(y)\}\\
&= \inf_{x\in X, y\in Y}\max\{w(\psi(y),z), v(\varphi(x),y), f_1(x)\}\\
&\geq \inf_{x\in X, y\in Y}\max\{w(\psi(y),z), w(\psi(\varphi(x)),\psi(y)), f_1(x)\} \quad \text{(since } \psi \text{ is nonexpansive)}\\
&\geq \inf_{x\in X}\max\{w(\psi(\varphi(x)),z), f_1(x)\} \quad \text{(by strong triangle inequality).}
\end{aligned}$$

Thus,

$$((f_\varphi)_\psi)_1(z) \geq (f_{\psi\circ\varphi})_1(z) \quad \text{for all } z\in Z. \tag{5.3}$$

Moreover, for any $z\in Z$, we have

$$\begin{aligned}
((f_\varphi)_\psi)_1(z) &= \inf_{y\in Y}\max\{w(\psi(y),z),(f_\varphi)_1(y)\}\\
&\leq \inf_{x\in X}\max\{w(\psi(\varphi(x))),(f_\varphi)_1(\varphi(x))\} \quad (\text{taking } y=\varphi(x))\\
&\leq \inf_{x\in X} w(\psi(\varphi(x)))\,\text{since}\,(f_\varphi)_1(\varphi(x))=0\\
&\leq \inf_{x\in X}\max\{w(\psi(\varphi(x))), f_1(x)\}\,\text{since}\, f_1(x)\geq 0.
\end{aligned}$$

Hence,

$$((f_\varphi)_\psi)_1(z) \leq (f_{\psi\circ\varphi})_1(z) \quad \text{for all } z\in Z. \tag{5.4}$$

By combining inequalities (5.3) and (5.4), we have the desired equality. □

Remark 5.5. Let (X,v) be a T_0-quasi-metric space. The the identity map $\mathrm{Id}_X : (X,v)\to(X,v)$ is a nonexpansive map. For any $f=(f_1,f_2)\in\nu_q(X,v)$ and $x\in X$, we have

$$\begin{aligned}
(f_{\mathrm{Id}_X})_1(x) &= \inf_{x'\in X}\max\{v(\mathrm{Id}_x(x'),x), f_1(x')\}\\
&= f_1(x) \text{ (by taking } x=x').
\end{aligned}$$

Similarly, $(f_{\mathrm{Id}_X})_2(x)=f_2(x)$ whenever $x\in X$. Therefore, $f_{\mathrm{Id}_X}=f$ whenever $f=(f_1,f_2)\in\nu_q(X,v)$.

In the following, UQMS denotes the category of T_0-ultra-quasi-metric spaces with nonexpansive maps and IUQMS denotes the category of ultra-quasi-metrically injective hulls on a T_0-ultra-quasi-metric space with nonexpansive maps.

For any two objects (X, v) and (Y, w) of UQMS and for any nonexpansive map $\varphi : (X, v) \to (Y, w)$, we define ν_q : UQMS $\to$ IUQMS by

$$\nu_q(\varphi)(f) := f_\varphi \text{ whenever } f = (f_1, f_2) \in \nu_q(X, v).$$

The following result is a consequence of Proposition 5.1, Proposition 5.2, Lemma 5.3, Proposition 5.4 and Remark 5.5.

Proposition 5.6. *Let (X, v) and (Y, w) be T_0-ultra-quasi-metric spaces and φ : $(X, v) \to (Y, w)$ be a nonexpansive map. Then, for any $f = (f_1, f_2) \in \nu_q(X, v)$, we have that $\nu_q(\varphi)(f)$ defined above is a covariant functor from UQMS into IUQMS.*

Theorem 5.7. *Let (X, v) and (Y, w) be T_0-ultra-quasi-metric spaces and φ : $(X, v) \to (Y, w)$ be a nonexpansive map. Then, the following diagram is commutative.*

$$\begin{array}{ccc} (X, v) & \xrightarrow{e_X} & \nu_q(X, v) \\ \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \nu_q(\varphi)} \\ (Y, w) & \xrightarrow{e_Y} & \nu_q(Y, w) \end{array}$$

Proof. Indeed, we show that $(\nu_q(\varphi) \circ e_X)(a) = (e_Y \circ \varphi)(a)$ for any $a \in X$. Since

$$(e_Y \circ \varphi)(a) = e_Y(\varphi(a)) = f_{\varphi(a)}$$

and

$$(\nu_q(\varphi) \circ e_X)(a) = (\nu_q(\varphi))(f_a) = (f_a)_\varphi$$

whenever $a \in X$. It follows that we only need to show that

$$f_{\varphi(a)}(y) = (f_a)_\varphi(y) \text{ for any } y \in Y \text{ and } a \in X.$$

Indeed, for any $y \in Y$, we have

$$\begin{aligned} ((f_a)_\varphi)_1(y) &= \inf_{b \in X} \max[w(\varphi(b), y), (f_a)_1(b)] \\ &= \inf_{b \in X} \max[w(\varphi(b), y), v(a, b)] \\ &\geq \inf_{b \in X} \max[w(\varphi(b), y), w(\varphi(a), \varphi(b))] \\ &= w(\varphi(a), y). \end{aligned}$$

Thus,

$$((f_a)_\varphi)_1(y) \geq (f_{\varphi(a)})_1(y) \text{ whenever } y \in Y. \tag{5.5}$$

Moreover,

$$\begin{aligned} ((f_a)_\varphi)_1(y) &= \inf_{b \in X} \max[w(\varphi(b), y), (f_a)_1(b) \\ &= \inf_{b \in X} \max[w(\varphi(b), y), v(a, b)] \\ &\leq w(\varphi(a), y) \quad \text{(by taking } b = a). \end{aligned}$$

So,

$$((f_a)_\varphi)_1(y) \leq (f_{\varphi(a)})_1(y) \quad \text{whenever } y \in Y. \tag{5.6}$$

From (5.5) and (5.6), we have $((f_a)_\varphi)_1(y) = (f_{\varphi(a)})_1(y)$ whenever $y \in Y$. By duality, one shows that

$$((f_a)_\varphi)_2(y) = (f_{\varphi(a)})_2(y) \text{ whenever } y \in Y.$$

Therefore,

$$f_{\varphi(a)}(y) = (f_a)_\varphi(y) \quad \text{for any } y \in Y \text{ and } a \in X.$$

□

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