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SOME REMARKS ON NEUTROSOPHIC T_0-SPACES

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Abstract. This paper is devoted to the study of the T_0 separation axiom for a neutrosophic topological space. We give a categorical study of these spaces in a special case of neutrosophic topological spaces called neutrosophic saturated topological spaces.

1. Introduction

As a generalization of the intuitionistic fuzzy set, the concepts of neutrosophy and neutrosophic set were introduced by Smarandache [17,18]. After that, in [15], Salama and Alblowi presented the definition of a neutrosophic topological space for the first time. These spaces are a generalization of intuitionistic fuzzy topological spaces.

The concept of neutrosophic topological spaces has been an area of study for many researchers. A lot of results have been produced by mathematicians. We can cite as examples the works of Smarandache [14,18] and Lupiáñez [9,10].

Many topological concepts, such as continuity, adherence, closure, interior, and separation axioms, have been generalized to neutrosophic topological spaces [10].

In this paper, we will study neutrosophic topological spaces from the point of view of category theory. We will work on a particular case of neutrosophic topological spaces called neutrosophic saturated topological spaces, which represent the objects of a category denoted by **SNTOP** and whose arrows are neutrosophic continuous maps. The collection of all the objects in this category which are neutrosophic T_0-spaces forms a subcategory of **SNTOP** denoted by $\mathbf{SNTOP_0}$.

The first part of this paper will be devoted to presenting all the important and necessary known definitions and remarks needed for our work.

In the next part, we will present the category **SNTOP** with its arrows and we will prove that the subcategory $\mathbf{SNTOP_0}$ is reflective in **SNTOP**. More precisely, this part is devoted to the construction of the T_0-reflection of a neutrosophic saturated topological space.

In the final paragraph, we will characterize arrows in **SNTOP** that are orthogonal to the subcategory $\mathbf{SNTOP_0}$.

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2. Preliminaries

Smarandache gave the definition of a neutrosophic set in [17], in which he linked the degrees of membership, non-membership and indeterminacy of every point in a given universe discourse. This definition allows topological spaces to be generalized to these sets. In 2012, Salama and Alblowi gave this generalization and defined neutrosophic topological spaces in [15].

Here, we present all the needed definitions and remarks related to neutrosophic topological spaces which will be used throughout this paper.

As mentioned in [16], Smarandache defines the non-standard real unit interval $]^-0,1^+[$ by adding to the interval [0,1] numbers of the form $1+\epsilon$ and $0-\epsilon$ such that ϵ is infinitesimal.

Neutrosophic sets are defined in [17] as follows:

Definition 2.1. A neutrosophic set (NS for short) A on the universe of discourse X is defined as

$$A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x)\rangle : x \in X\}$$

where $\mu_A, \sigma_A, \gamma_A : X \to]^-0,1^+[$ represent the degree of membership function (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$), and the degree of non-membership (namely $\gamma_A(x)$), respectively, of each element $x \in X$ to the set A and $^-0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^+$.

In the literature and regarding the philosophical point of view, $\mu_A(x)$, $\sigma_A(x)$, $4\gamma_A(x)$ take values from the real standard or non-standard subsets of $]^-0,1^+[$. However, it is difficult to use these values in the applications in engineering and real life scientific problems. *Hence, we will consider neutrosophic sets taking the value from the subsets of* $[0,1]$.

The collection of all neutrosophic sets over the universe X will be denoted by $\mathcal{N}(X)$.

Definition 2.2. Let $A, B \in \mathcal{N}(X)$. We say that A is a neutrosophic subset of B and denoted by $A \subseteq B$ (or we can say that B is a neutrosophic superset of A) if $\mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$.

Definition 2.3. [3] Let $\{A_i : i \in J\}$ be an arbitrary family of neutrosophic sets in X. Then,

(a) $\bigcap A_i = \{\langle x, \wedge\mu_{A_i}(x), \wedge\sigma_{A_i}(x), \vee\gamma_{A_i}(x)\rangle : x \in X\}$;
(b) $\bigcup A_i = \{\langle x, \vee\mu_{A_i}(x), \vee\sigma_{A_i}(x), \wedge\gamma_{A_i}(x)\rangle : x \in X\}$.

Definition 2.4. ([15]) The neutrosophic complement of $A \in \mathcal{N}(X)$ is denoted by A^c and defined by

$$A^c = \{\langle x, 1-\mu_A(x), 1-\sigma_A(x), 1-\gamma_A(x)\rangle : x \in X\}.$$

Definition 2.5. ([15]) The neutrosophic empty set is denoted by $\mathbf{0}_X$ and defined by

$$\mathbf{0}_X = \{\langle x, 0, 0, 1\rangle : x \in X\}.$$

The neutrosophic universal set is denoted by $\mathbf{1}_X$ and defined by

$$\mathbf{1}_X = \{\langle x, 1, 1, 0\rangle : x \in X\}.$$

Remarks 2.6. Let $A, B \in \mathcal{N}(X)$. Then,

(i) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
(ii) $A \cap A^c$ may not be equal to $\mathbf{0}_X$.
(iii) $A \cup A^c$ may not be equal to $\mathbf{1}_X$.

Definition 2.7. ([6]) If $\tau \subseteq \mathcal{N}(X)$, then τ is called a neutrosophic topology on X if

(i) $\mathbf{0}_X, \mathbf{1}_X \subseteq \tau$;
(ii) If $A, B \subseteq \tau$, then $A \cap B \subseteq \tau$;
(iii) The union of any number of neutrosophic sets of τ is also in τ.

The pair (X, τ) is called a neutrosophic topological space over X (in short, NTS). Moreover, the members of τ are said to be neutrosophic open sets in X (in short, NOS). If $A^c \in \tau$, then $A \in \mathcal{N}(X)$ is called a neutrosophic closed set in X (in short, NCS) and the family of all neutrosophic closed sets is denoted by τ^c.

Definition 2.8. Let (X, τ) be a neutrosophic topological space and $A \in \mathcal{N}(X)$. Then, the closure of A is the neutrosophic set $\overline{A}$ defined by

$$\overline{A} = \bigcap \{G | G \in \tau^c \text{ and } A \subseteq G\}.$$

Definition 2.9. Let X be a universe of discourse and $x \in X$. For $0 \leq r, t, s \leq 1$ such that $(r, t, s) \neq (0, 0, 1)$, the neutrosophic set defined as follows:

$$\{\langle x, r, t, s \rangle\} \cup \{\langle y, 0, 0, 1 \rangle; \text{ if } y \in X \setminus \{x\}\},$$

is called the neutrosophic point with support x and parameters r, t, s. It will be denoted by $x_{r,t,s}$.

It is obvious that every not empty neutrosophic set is the union of its neutrosophic points.

Now, we introduce the notions of image and preimage of neutrosophic sets.

Definition 2.10. [3] Let X and Y be two universes of discourses and $f : X \to Y$ be a map.

(i) If $B = \{\langle y, \mu_B(y), \sigma_B(y), \gamma_B(y) \rangle : y \in Y\} \in \mathcal{N}(Y)$, then the preimage of B under f, denoted by $f^{-1}(B)$, is the neutrosophic set in X defined by

$$f^{-1}(B) = \{\langle x, \mu_B(f(x)), \sigma_B(f(x)), \gamma_B(f(x)) \rangle : x \in X\};$$

(ii) If $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X\} \in \mathcal{N}(X)$, then the image of A under f, denoted by $f(A)$, is the neutrosophic set in Y defined by

$$f(A) = \{\langle y, \mu_{f(A)}(y), \sigma_{f(A)}(y), \gamma_{f(A)}(y) \rangle : y \in Y\},$$

such that

$$\mu_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$
$$\sigma_{f(A)}(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \sigma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$
$$\gamma_{f(A)}(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \gamma_A(x), & \text{if } f^{-1}(y) \neq \emptyset, \\ 1, & \text{otherwise.} \end{cases}$$

Definition 2.11. ([14]) Let (X, τ_1) and (Y, τ_2) be two NTSs, and let $f : X \to Y$ be a map. Then, f is said to be neutrosophic continuous if $f^{-1}(A) \in \tau_1, \forall A \in \tau_2$.

Remark 2.12. Let (X, τ_1) and (Y, τ_2) be two NTSs and let $f : X \to Y$ be a function. Then, the following statements are equivalent.

(1) f is a neutrosophic continuous map;
(2) $f^{-1}(A) \in \tau_1^c, \forall A \in \tau_2^c$.

Proposition 2.13. *Let $(X, \tau), (Y, \gamma), (Z, \sigma)$ be three neutrosophic topological spaces. If $f : (X, \tau) \to (Y, \gamma), g : (Y, \gamma) \to (Z, \sigma)$ are neutrosophic continuous, then $g \circ f : (X, \tau) \to (Z, \sigma)$ is also a neutrosophic continuous map.*

Proof. Let G be a neutrosophic open set in Z. We have

$$(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G)).$$

Since g is a neutrosophic continuous map, then $g^{-1}(G)$ is a neutrosophic open set in Y. Moreover, f is a neutrosophic continuous map. Therefore, $f^{-1}(g^{-1}(G))$ is a neutrosophic open set in X. □

Definition 2.14. [14] A map $f : (X, \tau_1) \to (Y, \tau_2)$ between two neutrosophic topological spaces is said to be

(i) neutrosophic open if $f(A) \in \tau_2, \forall A \in \tau_1$.
(ii) neutrosophic closed if $f(A) \in \tau_2^c, \forall A \in \tau_1^c$.

Definition 2.15. [12] A bijection $g : (X, \tau_1) \to (Y, \tau_2)$ is called a neutrosophic homeomorphism if g and g^{-1} are neutrosophic continuous.

3. Neutrosophic saturated topological space

Definition 3.1. A neutrosophic set A in the universe set X is said to be saturated if, for every neutrosophic point $x_{r,t,s}$, we have:

$$x_{r,t,s} \subseteq A \Rightarrow x_{1,1,0} \subseteq A.$$

Remarks 3.2. Let A be a neutrosophic set in the universe X.

(1) If A is saturated, then A^c is saturated.
(2) If A is saturated, then $A \cap A^c = \mathbf{0_X}$ and $A \cup A^c = \mathbf{1_X}$.

Definition 3.3. Let (X, τ) be a neutrosophic topological space. We say that τ is a neutrosophic saturated topology if A is saturated for every A in τ.
In this case, we say that (X, τ) is a neutrosophic saturated topological space.

Notation 3.4. We denote by **SNTOP** the category of all neutrosophic saturated topological spaces with neutrosophic continuous maps as arrows.

Definition 3.5. ([1]) A neutrosophic topological space (X, τ) is said to be a neutrosophic T_0-space if for every pair of neutrosophic points $x_{r,t,s}$ and $y_{i,j,k}$ in X, whose supports are different, there exists a neutrosophic open set A such that $x_{r,t,s} \subseteq A$ and $y_{i,j,k} \subseteq A^c$ (or $y_{i,j,k} \subseteq A$ and $x_{r,t,s} \subseteq A^c$).

The following theorem gives a characterization of neutrosophic T_0-spaces.

Theorem 3.6. *Let (X, τ) be a neutrosophic topological space. Then, the following statements are equivalent:*

(i) (X, τ) *is a neutrosophic* T_0*-space;*

(ii) *For each* $x \neq y$*, there exists* $H \in \tau$ *such that*

$$\begin{cases} \bigcup_{r,t,s} x_{r,t,s} \subseteq H \text{ and } \bigcup_{i,j,k} y_{i,j,k} \subseteq H^c \text{ or,} \\ \bigcup_{r,t,s} x_{r,t,s} \subseteq H^c \text{ and } \bigcup_{i,j,k} y_{i,j,k} \subseteq H; \end{cases}$$

(iii) *For every* $x \neq y$*, there exists* $H \in \tau$ *such that*

$$\begin{cases} x_{1,1,0} \subseteq H \text{ and } y_{1,1,0} \subseteq H^c \text{ or,} \\ x_{1,1,0} \subseteq H^c \text{ and } y_{1,1,0} \subseteq H. \end{cases}$$

Proof. (i) $\Rightarrow$ (iii) We apply the definition of neutrosophic T_0-spaces to the neutrospohic points $x_{1,1,0}$ and $y_{1,1,0}$.

(iii) $\Rightarrow$ (ii) Let $x_{r,t,s}, y_{i,j,k}$ be two neutrosophic points such that $x \neq y$. Then, there exists $H \in \tau$ such that, for example, $x_{1,1,0} \subseteq H$ and $y_{1,1,0} \subseteq H^c$ [by the hypothesis]. Notice that $x_{r,t,s} \subseteq x_{1,1,0}, \forall r, t, s$ from the fact that $r, t \leq 1$ and $0 \leq s$. We can see that $x_{r,t,s} \subseteq H$ and $y_{i,j,k} \subseteq H^c$. So,

$$\bigcup_{r,t,s} x_{r,t,s} \subseteq H \quad \text{and} \quad \bigcup_{i,j,k} y_{i,j,k} \subseteq H^c.$$

(ii) $\Rightarrow$ (i) Straightforward. □

In the following, we will be interested in a neutrosophic saturated topological space.

Now, we denote by $\mathbf{SNTOP_0}$ the full subcategory of a neutrosophic T_0-space in **SNTOP**.

Our goal is to prove that $\mathbf{SNTOP_0}$ is reflective in **SNTOP**.

Let (X, τ) be a neutrosophic saturated topological space. We define on X the binary relation $\sim$ by:

$$x \sim y \text{ if and only if } \overline{x_{1,1,0}} = \overline{y_{1,1,0}}.$$

It is obvious that $\sim$ is an equivalence relation. We denote by $X/\sim$ the quotient set and by ρ_X the canonical surjection from X onto $X/\sim, x \mapsto \rho_X(x) = \bar{x}$.

Remarks 3.7.

(1) $\widetilde{\tau} = \{A \in \mathcal{N}(X/\sim) \text{ such that } \rho_X^{-1}(A) \in \tau\}$ defines a saturated topology on $X/\sim$.

(2) ρ_X is a neutrosophic continuous map.

Proposition 3.8. $\rho_X : (X, \tau) \to (X/\sim, \widetilde{\tau})$ *is a neutrosophic open map.*

Proof. First, $\mathbf{0_X}$ is included in every soft set. Let $A \in \tau$. We have to show that $\rho_X^{-1}(\rho_X(A)) = A$. It is clear that $A \subseteq \rho_X^{-1}(\rho_X(A))$. Conversely, if $\mathbf{0_X} \neq x_{r,t,s} \subseteq \rho_X^{-1}(\rho_X(A))$, then there exists $\mathbf{0_X} \neq y_{i,j,k} \subseteq A$ such that $\rho_X(x) = \rho_X(y)$ so that $\overline{x_{1,1,0}} = \overline{y_{1,1,0}}$.

If $x_{r,t,s} \nsubseteq A$, then $x_{r,t,s} \subseteq A^c$ (since A is saturated), and thus $\overline{x_{1,1,0}} \subseteq A^c$, which is impossible because $y_{1,1,0} \subseteq \overline{x_{1,1,0}}$ and $y_{1,1,0} \nsubseteq A^c$.

Hence, $x_{r,t,s} \subseteq A$ and we deduce that $\rho_X^{-1}(\rho_X(A)) = A$. □

Proposition 3.9. *Let (X, τ) be a neutrosophic saturated topological space. Then the following statements are equivalent:*

(i) *(X, τ) is a neutrosophic T_0-space;*

(ii) $\overline{x_{1,1,0}} = \overline{y_{1,1,0}} \Rightarrow x = y$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $x \neq y$. Since (X, τ) is a neutrosophic T_0-space, there exists $A \in \tau$ such that, for example, $x_{1,1,0} \subseteq A$ and $y_{1,1,0} \subseteq A^c$. Then, $\overline{y_{1,1,0}} \subseteq A^c$. As A is saturated, $x_{1,1,0} \nsubseteq A^c$. Therefore, $x_{1,1,0} \nsubseteq \overline{y_{1,1,0}}$.

(ii) $\Rightarrow$ (i) Let $x \neq y$. By *(ii)*, there exists $F \in \tau^c$ such that, for example, $x_{1,1,0} \subseteq F$ and $y_{1,1,0} \nsubseteq F$. And since F is saturated, $y_{1,1,0} \subseteq F^c$, which implies that (X, τ) is a neutrosophic T_0-space. ☐

Theorem 3.10. *$({}^{X}\!/\!\sim, \widetilde{\tau})$ is a neutrosophic T_0-space.*

Proof. Let $\overline{x} \neq \overline{y}$. Then, there exists $O \in \tau$ such that, for example, $x_{1,1,0} \subseteq O$ and $y_{1,1,0} \subseteq O^c$, which implies that $\overline{x}_{1,1,0} \subseteq \rho_X(O)$ and $\overline{y}_{1,1,0} \subseteq \rho_X(O^c) = (\rho_X(O))^c$. Since ρ_X is neutrosophic open, then $\rho_X(O)$ is neutrosophic open. Therefore, $({}^{X}\!/\!\sim, \widetilde{\tau})$ is a neutrosophic T_0-space. ☐

It is interesting to note that reflective subcategories appear throughout mathematics, for example, in algebra's free group and free ring functors, topology's different compactification functors, and analysis' completion functors. See, for example, [11, p.90]. Recall from [11, p.89] that a subcategory D of a category C is termed reflective (in C) if the inclusion functor $I : D \longrightarrow C$ has a left adjoint functor $F : C \longrightarrow D$; i.e., if, for each object A of C, there exists an object $F(A)$ of D and a morphism $\rho_A : A \longrightarrow F(A)$ in C such that, for each object X in D and each morphism $f : A \longrightarrow X$ in C, there exists a unique morphism $f : F(A) \longrightarrow X$ in D such that $\widetilde{f} \circ \rho_A = f$.

Now, we are in a position to present the principal result of the paper.

For this, using the characterization given by MacLane in [11], we will prove that, for every $(X, \tau) \in \mathbf{SNTOP}$, there exists an object $({}^{X}\!/\!\sim, \widetilde{\tau}) \in \mathbf{SNTOP_0}$ and a morphism $\rho_X : (X, \tau) \to ({}^{X}\!/\!\sim, \widetilde{\tau})$ such that, for each $(Y, \gamma) \in \mathbf{SNTOP_0}$ and each neutrosophic continuous map f from $(X, \tau) \to (Y, \gamma)$, there exists a unique neutrosophic continuous map $\tilde{f} : ({}^{X}\!/\!\sim, \widetilde{\tau}) \to (Y, \gamma)$ rending the following diagram commutative.

$$\begin{tikzcd}
(X,\tau) \arrow[rr, "\rho_X"] \arrow[dr, "f"'] & & ({}^{X}\!/\!\sim, \widetilde{\tau}) \arrow[dl, "\tilde{f}"] \\
& (Y,\gamma) &
\end{tikzcd}$$

Theorem 3.11. $\mathbf{SNTOP_0}$ *is reflective in* $\mathbf{SNTOP}$.

Proof. It is sufficient to prove that for any neutrosophic saturated topological space (X, τ), $({}^{X}\!/\!\sim, \widetilde{\tau})$ is the T_0-reflection of (X, τ).

- *Uniqueness:* It is unique by $\tilde{f}(\rho_X(x)) = f(x)$.
- *$\tilde{f}$ is well defined:* Let $x, y \in X$. If $f(x) \neq f(y)$, then there exists $A \in \gamma$ such that, for example, $(f(x))_{1,1,0} \subseteq A$ and $(f(y))_{1,1,0} \subseteq A^c$. By the neutrosophic continuity of f, we have $f^{-1}(A) \in \tau$. And thus, $x_{1,1,0} \subseteq f^{-1}(A)$ and $y_{1,1,0} \subseteq f^{-1}(A^c) = (f^{-1}(A))^c$, which implies that $\bar{x} \neq \bar{y}$.

- $\tilde{f}$ *is a neutrosophic continuous map:* Let $B \in \gamma$. We have $\rho_X^{-1}(\widetilde{f}^{-1}(B)) = (\widetilde{f} \circ \rho_X)^{-1}(B) = f^{-1}(B) \in \tau$. Then, $\widetilde{f}^{-1}(B) \in \widetilde{\tau}$, which implies that $\widetilde{f}$ is a neutrosophic continuous map.

Finally, $({}^{X}/_{\sim}, \widetilde{\tau})$ is the neutrosophic T_0-reflection of (X, τ).

□

Example 3.12. Let $X = \{a, b, c\}$ and $\tau = \{\mathbf{0}_X, \mathbf{1}_X, U, V\}$, where

$$U = \{\langle a, 1, 1, 0\rangle, \langle b, 1, 1, 0\rangle, \langle c, 0, 0, 1\rangle\}$$

and

$$V = \{\langle a, 0, 0, 1\rangle, \langle b, 0, 0, 1\rangle, \langle c, 1, 1, 0\rangle\}.$$

(X, τ) is clearly a neutrosophic saturated topological space, but it is not a neutrosophic T_0-space. Using the relation "$\sim$" defined above, we have $\overline{a_{1,1,0}} = \overline{b_{1,1,0}}$, then ${}^{X}/_{\sim} = \{\bar{a}, \bar{c}\}$ and $\widetilde{\tau} = \{\mathbf{0}_{X/\sim}, \mathbf{1}_{X/\sim}, \widetilde{U}, \widetilde{V}\}$, where

$$\widetilde{U} = \{\langle \bar{a}, 1, 1, 0\rangle, \langle \bar{c}, 0, 0, 1\rangle\}$$

and

$$\widetilde{V} = \{\langle \bar{a}, 0, 0, 1\rangle, \langle \bar{c}, 1, 1, 0\rangle\}.$$

$({}^{X}/_{\sim}, \widetilde{\tau})$ is the T_0-reflection of (X, τ).

4. The class of morphisms in **SNTOP** orthogonal to all neutrosophic T_0-spaces in **SNTOP**

A morphism $f : A \longrightarrow B$ and an object X in a category C are called orthogonal [4] if the mapping $hom_C(f; X) : hom_C(B; X) \longrightarrow hom_C(A; X)$ that takes g to gf is bijective. For a class of morphisms σ (resp., a class of objects D), we denote by $\sigma^{\perp}$ the class of objects orthogonal to every f in σ (resp., by $D^{\perp}$ the class of morphisms orthogonal to all X in D) [4].

The orthogonality class of morphisms $D^{\perp}$ associated with a reflective subcategory D of a category C satisfies the following identity $D^{\perp\perp} = D$ [2, proposition 2.6]. Thus, it is of interest to give explicitly the class $D^{\perp}$. Note also that, if $I : D \longrightarrow C$ is the inclusion functor and $F : C \longrightarrow D$ is a left adjoint functor of I, then the class $D^{\perp}$ is the collection of all morphisms of C rendered invertible by the functor F [2, proposition 2.3].

This section is focused on the study of the orthogonal class $\mathbf{SNTOP_0}^{\perp}$.

Grothendieck and Dieudonné introduced the notion of quasihomeomorphism between topological spaces in [5] to solve some algebraic topology problems. Recall that a continuous map $q : (X, \tau) \to (Y, \gamma)$ is said to be a quasihomeomorphism if $U \mapsto q^{-1}(U)$ defines a bijection $\mathcal{O}(Y) \to \mathcal{O}(X)$, where $\mathcal{O}(X)$ is the set of all open subsets of the space X.

Definition 4.1. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a neutrosophic continuous map between two neutrosophic saturated topological spaces. f is said to be a neutrosophic-quasihomeomorphism (in short, N-quasihomeomorphism) if $U \mapsto f^{-1}(U)$ defines a bijection between τ_1 (resp. τ_1^c) and τ_2 (resp. τ_2^c).

Proposition 4.2. $\rho_X : (X, \tau) \to ({}^{X}/_{\sim}, \widetilde{\tau})$ *is a N-quasihomeomorphism.*

Proof. It follows from $\rho_X^{-1}(\rho_X(A)) = A$. □

Proposition 4.3. *If $f : X \to Y$, $g : Y \to Z$ are neutrosophic continuous maps between saturated NTSs and two of the maps f, g, $g \circ f$ are N-quasihomeomorphisms, then so is the third one.*

Proof. 1. Suppose that f and g are two N-quasihomeomorphisms. For any neutrosophic saturated open set U of X, let V be the unique neutrosophic saturated open set of Y such that $U = f^{-1}(V)$ and let W be the unique neutrosophic saturated open set of Z such that $V = g^{-1}(W)$. It is clear that W is the unique neutrosophic saturated open set of Z such that $U = (g \circ f)^{-1}(W)$. We conclude that $g \circ f$ is a N-quasihomeomorphism.

2. Suppose that g and $g \circ f$ are N-quasihomeomorphisms. Let U be a neutrosophic saturated open set in X. Since $g \circ f$ is a N-quasihomeomorphism, there exists a unique neutrosophic saturated open set W in Z such that $U = (g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Now, $V = g^{-1}(W)$ is a neutrosophic saturated open set of Y satisfying $U = f^{-1}(V)$. Let us show that V is the unique neutrosophic saturated open set of Y such that $U = f^{-1}(V)$. Indeed, let V' be a neutrosophic saturated open set of Y such that $U = f^{-1}(V')$. Then, there exists a unique neutrosophic saturated open set W' of Z such that $V' = g^{-1}(W')$. So,

$$(g \circ f)^{-1}(W) = U = f^{-1}(V') = f^{-1}(g^{-1}(W')) = (g \circ f)^{-1}(W').$$

Finally, $W = W'$, and consequently $V = g^{-1}(W) = g^{-1}(W') = V'$.

3. Suppose that f and $g \circ f$ are N-quasihomeomorphisms. If V is a neutrosophic saturated open set in Y , $f^{-1}(V)$ is a neutrosophic saturated open set in X. Then, there exists a unique neutrosophic saturated open set W in Z such that

$$(g \circ f)^{-1}(W) = f^{-1}(V).$$

It is easy to show that W is the unique neutrosophic saturated open set in Z such that $V = g^{-1}(W)$. Therefore, g is a N-quasihomeomorphism. □

Given (X, τ_1) is a neutrosophic saturated topological space, the construction of its T_0-reflection denoted by $({}^{X}\!/\!\sim, \widetilde{\tau_1})$ satisfies some categorical properties. For each neutrosophic saturated topological space (Y, τ_2) and each neutrosophic continuous map f from (X, τ_1) to (Y, τ_2), there exists a unique neutrosophic continuous map $\tilde{f} : ({}^{X}\!/\!\sim, \widetilde{\tau_1}) \to ({}^{Y}\!/\!\sim, \widetilde{\tau_2})$ such that the following diagram commutes:

$$\begin{tikzcd}
(X, \tau_1) \arrow[r, "f"] \arrow[d, "\rho_X"'] \arrow[dr, phantom, "\circlearrowleft"] & (Y, \tau_2) \arrow[d, "\rho_Y"] \\
({}^{X}\!/\!\sim, \widetilde{\tau_1}) \arrow[r, "\tilde{f}"] & ({}^{Y}\!/\!\sim, \widetilde{\tau_2})
\end{tikzcd}$$

The following definitions are introduced in order to give the main result of this section.

Definition 4.4. Let $f : X \to Y$ be a neutrosophic continuous map between two neutrosophic saturated topological spaces.

(1) f is said to be N-onto (N-surjective) if, for each $y \in Y$, there exists $x \in X$ such that $\overline{y_{1,1,0}} = \overline{f(x)_{1,1,0}}$;

(2) f is said to be N-one-to-one (N-injective) if, for each $x, y \in X$ such that $\overline{f(x)_{1,1,0}} = \overline{f(y)_{1,1,0}} \Rightarrow \overline{x_{1,1,0}} = \overline{y_{1,1,0}}$;
(3) f is said to be N-bijective if it is both N-one-to-one and N-onto.

Lemma 4.5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a neutrosophic continuous map between neutrosophic saturated topological spaces and $\tilde{f}$ is as in the commuting diagram above. Then, the following properties hold:*

(1) *f is N-injective if and only if $\tilde{f}$ is injective.*
(2) *f is N-surjective if and only if $\tilde{f}$ is surjective.*
(3) *f is N-bijective if and only if $\tilde{f}$ is bijective.*

Proof. (1) Suppose that $\tilde{f}$ is injective. Let $x, y \in X$ such that $\overline{f(x)_{1,1,0}} = \overline{f(y)_{1,1,0}}$. Then, $\rho_Y(f(x)) = \rho_Y(f(y))$, and thus $\tilde{f}(\rho_X(x)) = \tilde{f}(\rho_X(y))$. As $\tilde{f}$ is injective, then $\rho_X(x) = \rho_X(y)$, which implies that $\overline{x_{1,1,0}} = \overline{y_{1,1,0}}$. Therefore, f is N-one-to-one.

Conversely, suppose f is N-injective. Let $x, y \in X$ such that $\tilde{f}(\rho_X(x)) = \tilde{f}(\rho_X(y))$. Then, $\rho_Y(f(x)) = \rho_Y(f(y))$, and thus $\overline{f(x)_{1,1,0}} = \overline{f(y)_{1,1,0}}$. Since f is N-injective, $\overline{x_{1,1,0}} = \overline{y_{1,1,0}}$. So, $\rho_X(x) = \rho_X(y)$, which proves that $\tilde{f}$ is injective.

(2) Suppose that $\tilde{f}$ is surjective. If $y \in Y$, then there exists $x \in X$ such that $\tilde{f}(\rho_X(x)) = \rho_Y(y)$, then $\rho_Y(f(x)) = \rho_Y(y)$. Hence, $\overline{f(x)_{1,1,0}} = \overline{y_{1,1,0}}$.

Conversely, suppose that f is N-surjective. Let $y \in Y$; there exists $x \in X$ such that $\overline{y_{1,1,0}} = \overline{f(x)_{1,1,0}}$. Then, $\rho_Y(y) = \rho_Y(f(x))$, which implies that $\rho_Y(y) = \tilde{f}(\rho_X(x))$. Therefore, $\tilde{f}$ is a surjective map.

(3) An immediate consequence of (1) and (2). □

Theorem 4.6. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a neutrosophic continuous map between two neutrosophic saturated topological spaces. Then, the following statements are equivalent:*

(i) *f is a N-onto, N-quasihomeomorphism;*
(ii) *$\tilde{f}$ is a neutrosophic homeomorphism.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 4.5, f is a N-onto map, then $\tilde{f}$ is an onto map.

Let us prove that $\tilde{f}$ is one-to-one. Using Lemma 4.5, it is sufficient to show that f is N-one-to-one. For this, let $x, y \in X$ such that $\overline{f(x)_{1,1,0}} = \overline{f(y)_{1,1,0}}$. If $\overline{x_{1,1,0}} \neq \overline{y_{1,1,0}}$, then there exists a neutrosophic saturated open set A of X such that, for example, $x_{1,1,0} \subseteq A$ and $y_{1,1,0} \subseteq A^c$. Since f is a N-quasihomeomorphism, there exists a neutrosophic saturated open set B of Y such that $f^{-1}(B) = A$. Then, $f(x)_{1,1,0} \subseteq B$ and $f(y)_{1,1,0} \subseteq B^c$, which is a contradiction. We conclude that f is N-one-to-one so that $\tilde{f}$ is one-to-one.

Since $\tilde{f} \circ \rho_X = \rho_Y \circ f$, $\rho_Y \circ f$ and ρ_X are neutrosophic continuous maps, so $\tilde{f}$ is a neutrosophic continuous map.

$\tilde{f}^{-1}$ is a neutrosophic continuous map. Let $A \in \widetilde{\tau_1}$. Since ρ_X is a neutrosophic continuous map, then $\rho_X^{-1}(A) \in \tau_1$. Moreover, f is a N-quasihomeomorphism, so there exists B a neutrosophic saturated open set of Y such that $f^{-1}(B) = \rho_X^{-1}(A)$, which implies that $f^{-1}(\rho_Y^{-1}(\rho_Y(B)) = \rho_X^{-1}(A)$, and thus

$$\rho_X^{-1}(A) = (\rho_Y \circ f)^{-1}(\rho_Y(B)) = (\tilde{f} \circ \rho_X)^{-1}(\rho_Y(B)) = \rho_X^{-1}(\tilde{f}^{-1}(\rho_Y(B)));$$

hence, $\widetilde{f}(A) = \rho_Y(B)$. Furthermore, we have $\rho_Y(B) \in \widetilde{\tau_2}$. Therefore, $\tilde{f}(A)$ is a neutrosophic saturated open set of $(Y/\sim, \widetilde{\tau_2})$. We conclude that $\tilde{f}^{-1}$ is a neutrosophic continuous map.

Finally, $\tilde{f}$ is a bijective neutrosophic continuous map with its inverse $\tilde{f}^{-1}$ also being neutrosophic continuous, so $\tilde{f}$ is a neutrosophic homeomorphism.

(ii) $\Rightarrow$ (i) By Lemma 4.5, f is N-onto.

Clearly, f is a N-quasihomeomorphism by Proposition 4.3 and also ρ_X is an N-quasihomeomorphism. □

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