ALL MAXIMAL UNIT-REGULAR ELEMENTS OF
$\text{Relhyp}((m),(n))$

PORNPIMOL KUNAMA AND SORASAK LEERATANAVALEE

Abstract. Any relational hypersubstitution for algebraic systems of type $(\tau,\tau') = ((m_i)_{i \in I}, (n_j)_{j \in J})$ is a mapping which maps any $m_i$-ary operation symbol to an $m_i$-ary term and maps any $n_j$-ary relational symbol to an $n_j$-ary relational term preserving arities, where $I, J$ are indexed sets. The set of all relational hypersubstitutions for algebraic systems of type $(\tau,\tau')$ together with a binary operation defined on the set and its identity forms a monoid. The properties of this structure are expressed by terms and formulas. Some algebraic properties of the monoid of a special type, especially the set of all unit-regular elements, were studied. In this paper, we determine all maximal unit-regular submonoids of this monoid of type $((m),(n))$ for arbitrary natural numbers $m, n \geq 2$.

1. Introduction

In universal algebra, identities are used to classify algebras into collections called varieties, and hyperidentities are used to classify varieties into collections called hypervarieties. The tool which is used to study hyperidentities and hypervarieties is the concept of a hypersubstitution introduced by W. Taylor [15]. The notation of a hypersubstitution was developed by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [6]. The authors used this concept for the characterization of solid varieties of type $\tau$. A solid variety is a variety which is closed under the following operation: taking a universal algebra $(A, (f^A_i)_{i \in I})$ of type $\tau = (m_i)_{i \in I}$ with the universe $A$ and family $(f^A_i)_{i \in I}$ of $m_i$-ary operations $f^A_i$ on $A$ for $i \in I$ of a variety, then we replace the operation $f^A_i$ by any $m_i$-ary term operation $\sigma(f^A_i)$, for $i \in I$, and obtain a new universal algebra $(A, (\sigma(f^A_i))_{i \in I})$, which also belongs to the variety. Hence, a hypersubstitution of a given type $\tau = (m_i)_{i \in I}$ is a mapping which maps every $m_i$-ary operation symbol $f^A_i$ to an $m_i$-ary term of the same type, for $i \in I$. Moreover, the set $Hyp(\tau)$ of all hypersubstitutions of type $\tau$ together with an associative binary operation $\circ_h$ forms a monoid; see more details in [6,17].

However, we can consider algebraic systems in the sense of Mal’cev [11]. An algebraic system of type $(\tau,\tau')$ is a triple $(A, (f^A_i)_{i \in I}, (\gamma_j^A)_{j \in J})$ consisting of a non-empty set $A$, a sequence $(f^A_i)_{i \in I}$ of $n_i$-ary operations defined on $A$ and a sequence $(\gamma_j^A)_{j \in J}$ of $n_j$-ary relations on $A$, where $\tau = (n_i)_{i \in I}$ is a sequence of the arity of

MSC (2020): primary 20M07, 08B15, 08B25.
Keywords: hypersubstitutions, algebraic systems, regular elements.

The research was supported by Chiang Mai University, Chiang Mai 50200, Thailand.
each operation $f^A_i$ and $\tau' = (n_j)_{j \in J}$ is a sequence of the arity of each relation $\gamma^A_j$. The pair $(\tau, \tau')$ is called a type of an algebraic system; see more details in [12–14].

In 2008 [5], K. Denecke and D. Phusanga introduced the concept of a hypersubstitution for algebraic systems, which is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a formula which preserves the arity. The set of all hypersubstitutions for algebraic systems of type $(\tau, \tau')$ is denoted by $Hyp(\tau, \tau')$. They defined an associative operation $\circ_r$ on this set and proved that $(Hyp(\tau, \tau'), \circ_r, \sigma_{id})$ forms a monoid, where $\sigma_{id}$ is an identity hypersubstitution. In 2018 [14], D. Phusanga and J. Koppitz introduced the concept of a relational hypersubstitution for algebraic systems of type $(\tau, \tau')$ and proved that the set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau')$ together with an associative binary operation and the identity element forms a monoid.

Firstly, we recall the definition of some special elements in a semigroup. An element $a$ of a semigroup $S$ is called unit-regular if there exists $u \in U(S)$ such that $a = auu$, where $U(S)$ is the set of all unit elements of $S$ and $S$ is called unit-regular if every element of $S$ is unit-regular. Later in 1980, H. D'Alarcao showed that a monoid $S$ is factorisable if and only if it is unit-regular [4]. A number of authors studied different factorisable semigroups. S. Y. Chen and S. C. Hsieh studied factorisable inverse semigroups [3]. Y. Tirasupa studied factorisable transformation semigroups [16]. In 2001 P. Jampachon, M. Saichalee and R. P. Sullivan used the concept of factorisability to study locally factorisable transformation semigroups [7]. In 2016, A. Boonmee and S. Leeratanavalee studied factorisable monoid of generalized hypersubstitutions of type $(n)$ [2]. In this paper, we determine all maximal unit-regular elements of relational hypersubstitutions of type $((m), (n))$ for arbitrary natural numbers $m, n \geq 2$.

Next, we recall the concept of an $n$-ary term of type $\tau$ and an $n$-ary relational term of type $(\tau, \tau')$, respectively. Let $X := \{x_1, \ldots, x_n, \ldots\}$ be a countably infinite set of symbols called variables. For each $n \geq 1$, let $X_n := \{x_1, \ldots, x_n\}$. Let $\{f_i : i \in I\}$ be the set of $m_i$-ary operation symbols indexed by $I$, where $m_i \geq 1$ is a natural number. Let $\tau$ be a function which assigns to every $f_i$ the number $m_i$ as its arity. The function $\tau = (m_i)_{i \in I}$ is called a type. An $n$-ary term of type $\tau$ is defined inductively as follows.

(i) Every variable $x_k \in X_n$ is an $n$-ary term of type $\tau$.
(ii) If $t_1, \ldots, t_n$, are $n$-ary terms of type $\tau$ and $f_i$ is an $n_i$-ary operation symbol, then $f_i(t_1, \ldots, t_n)$ is an $n$-ary term of type $\tau$.

We denote the set of all $n$-ary terms of type $\tau$ which contains $x_1, \ldots, x_n$ and is closed under finite application of (ii), by $W_\tau(X_n)$ and $W_\tau(X) := \bigcup_{n \in \mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type $\tau$.

2. THE MONOID OF RELATIONAL HYPERSUBSTITUTIONS FOR ALGEBRAIC SYSTEMS

Any relational hypersubstitution for algebraic systems is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a relational term which preserves the arity. Let $(\tau, \tau')$ be a type. An $n$-ary relational term of
type $(\tau, \tau')$ and a relational hypersubstitution for algebraic systems are defined as follows.

**Definition 2.1.** ([12]) Let $I, J$ be indexed sets. If $i \in I$, $j \in J$ and $t_1, t_2, \ldots, t_{n_j}$ are $n$-ary terms of type $\tau = (n_i)_{i \in I}$ and $\gamma_j$ is an $n_j$-ary relation symbol, then $\gamma_j(t_1, t_2, \ldots, t_{n_j})$ is an $n$-ary relational term of type $(\tau, \tau') = ((n_i)_{i \in I}, (n_j)_{j \in J})$.

We denote the set of all $n$-ary relational terms of type $(\tau, \tau')$ by $rF(\tau, \tau')(X_n)$ and $rF(\tau, \tau')(X_n) := \cup_{n \in \mathbb{N}} rF(\tau, \tau')(X_n)$ be the set of all relational terms of type $(\tau, \tau')$.

A relational hypersubstitution for algebraic systems of type $(\tau, \tau') = ((m_i)_{i \in I}, (n_j)_{j \in J})$ is a mapping

$$\sigma : \{f_i \mid i \in I\} \cup \{\gamma_j \mid j \in J\} \rightarrow W_\tau(X) \cup rF(\tau, \tau')(X)$$

with $\sigma(f_i) \in W_\tau(X_{m_i})$ and $\sigma(\gamma_j) \in rF(\tau, \tau')(X_{n_j})$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau, \tau')$ is denoted by $Rhyp(\tau, \tau')$. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S_n^m : W_\tau(X_m) \times (W_\tau(X_n))^m \rightarrow W_\tau(X_n)$ by the following steps. For any $t, t_1, \ldots, t_{m_i} \in W_\tau(X_m)$, $s_1, \ldots, s_m \in W_\tau(X_n)$,

(i) if $t = x_j$ for $1 \leq j \leq n$, then $S_n^m(t, s_1, \ldots, s_m) := s_j$;

(ii) if $t = f_i(t_1, \ldots, t_{m_i})$, then

$$S_n^m(t, s_1, \ldots, s_m) := f_i(S_n^m(t_1, s_1, \ldots, s_m), \ldots, S_n^m(t_{m_i}, s_1, \ldots, s_m)).$$

For any $F = \gamma_j(t_1, \ldots, t_{n_j}) \in rF(\tau, \tau')(X_m)$, we define the superposition of relational terms $R_n^m : (W_\tau(X_m) \cup rF(\tau, \tau')(X_m)) \times (W_\tau(X_n))^m \rightarrow W_\tau(X_n) \cup rF(\tau, \tau')(X_n)$ by

(i) $R_n^m(t, s_1, \ldots, s_m) := S_n^m(t, s_1, \ldots, s_m)$,

(ii) $R_n^m(F, s_1, \ldots, s_m) := \gamma_j(S_n^m(t_1, s_1, \ldots, s_m), \ldots, S_n^m(t_{n_j}, s_1, \ldots, s_m)).$

Every relational hypersubstitution for algebraic systems $\sigma$ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \cup rF(\tau, \tau')(X) \rightarrow W_\tau(X) \cup rF(\tau, \tau')(X)$ as follows:

(i) $\hat{\sigma}(x_i) := x_i$ for $i \in \mathbb{N}$;

(ii) $\hat{\sigma}(f_i(t_1, \ldots, t_{m_i})) := S_n^m(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{m_i}])$, where $i \in I$ and $t_1, \ldots, t_{m_i} \in W_\tau(X_{m_i})$, i.e., any occurrence of the variable $x_k$ in $\sigma(f_i)$ is replaced by the term $\hat{\sigma}[t_k], 1 \leq k \leq m_i$;

(iii) $\hat{\sigma}(\gamma_j(s_1, \ldots, s_{n_j})) := R_n^m(\sigma(\gamma_j), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_{n_j}])$, where $j \in J$, $s_1, \ldots, s_{n_j} \in W_\tau(X_{n_j})$, i.e., any occurrence of the variable $x_k$ in $\sigma(\gamma_j)$ is replaced by the term $\hat{\sigma}[s_k], 1 \leq k \leq n_j$.

We define a binary operation $\circ_r$ on $Rhyp(\tau, \tau')$ by $\sigma \circ_r \alpha := \hat{\sigma} \circ \alpha$, where $\circ$ is the usual composition of mappings and $\sigma, \alpha \in Rhyp(\tau, \tau')$. Let $\sigma_{id}$ be the relational hypersubstitution which maps each $m_i$-ary operation symbol $f_i$ to the term $f_i(x_1, \ldots, x_{m_i})$ and maps each $n_j$-ary relation symbol $\gamma_j$ to the relational term $\gamma_j(x_1, \ldots, x_{n_j})$. D. Phusanga and J. Koppitz [14] proved that $(Rhyp(\tau, \tau'), \circ_r, \sigma_{id})$ is a monoid.

In 2015, W. Wongpinit and S. Leeratanavalee [18] introduced the concept of the $i$-most of terms as follows.
Definition 2.2. ([18]) Let \( \tau = (m) \) be a type with an \( m \)-ary operation symbol \( f, t \in W_{(m)}(X) \) and \( 1 \leq i \leq m \). An \( i \)-most \( t \) is defined inductively as follows

(i) If \( t \) is a variable, then \( i \)-most \( t \) = \( t \),
(ii) If \( t = f(t_1, \ldots, t_n) \), where \( t_1, \ldots, t_n \in W_{(m)}(X) \), then \( i \)-most \( t \) := \( i \)-most \( t_i \).

Example 2.3. Let \( \tau = (3) \) be a type, \( t = f(x_3, f(x_2, x_1), f(x_2, x_1)) \).
Then, \( 1 \)-most \( t \) = \( x_3 \), \( 2 \)-most \( t \) = \( 2 \)-most \( f(x_2, x_1) \) = \( x_2 \) and \( 3 \)-most \( t \) = \( 3 \)-most \( f(x_2, x_1) \) = \( x_1 \).

Lemma 2.4. ([18]) Let \( s, t \in W_{(m)}(X) \). If \( j \)-most \( t \) = \( x_k \in X_m \) and \( k \)-most \( s \) = \( x_i \), then \( j \)-most \( \sigma_t[s] \) = \( x_i \).

The above lemma can be applied to any relational hypersubstitution for algebraic systems of type \((m), (n)\), such as \( s, t \in W_{(m)}(X_m) \) and \( F \in rF_{(m),(n)}(X_n) \) if \( i \)-most \( t \) = \( x_j \), then \( i \)-most \( \sigma_t[F] \) = \( j \)-most \( s \) where \( \sigma_t[F] \) is a relational hypersubstitution of type \((\tau, \tau')\) which maps \( f \) to a term \( t \) and maps \( \gamma \) to a relational term \( F \).

3. Maximal unit-regular elements in \( \text{Relhyp}((m), (n)) \)

Let \((\tau, \tau') = ((m), (n))\) be a type with an \( m \)-ary operation symbol \( f \), \( \gamma \) be an \( n \)-ary relation symbol, \( t \in W_{(m)}(X_m) \) and \( F \in rF_{(m),(n)}(X_n) \). We denote by \( \sigma_{t,F} \) the set of the relational hypersubstitution of type \((\tau, \tau') = ((m), (n))\) which maps \( f \) to a term \( t \) in \( W_{(m)}(X_m) \) and maps \( \gamma \) to a relational term \( F \in rF_{(m),(n)}(X_n) \):

\[
\text{var}(t) \text{ be the set of all variables occurring in the term } t; \\
\text{var}(F) \text{ be the set of all variables occurring in the relational term } F.
\]

\[
R'_{X} := \{ \sigma_{x, \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \{ i \text{-most}(s_k) : 1 \leq k \leq n = \text{var}(\gamma(s_1, \ldots, s_n)) \} \}; \\
R_{T} := \{ \sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \text{var}(f(t_1, \ldots, t_m)) \subseteq \{ t_1, \ldots, t_m \}, \text{var}(\gamma(s_1, \ldots, s_n)) \subseteq \{ s_1, \ldots, s_n \} \}.
\]

In [10], P. Kunama and S. Leeratanavalee showed that \( R'_{X} \cup R_{T} \) is the set of all unit-regular elements in \( \text{Relhyp}((m), (n)) \).

For any \( i \in \{1, \ldots, m\} \), \( j \in \{1, \ldots, n\} \), we denote

\[
R'_{X,i} := \{ \sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \{ i \text{-most}(s_k) : 1 \leq k \leq n = \text{var}(\gamma(s_1, \ldots, s_n)) \} \}; \\
R_{X,i} := \{ \sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \text{var}(s_k) = 1 : 1 \leq k \leq n \}; \\
R_{X} := \{ \sigma_{x, \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \text{var}(\gamma(s_1, \ldots, s_n)) = 1 \}; \\
R_{T,i} := \{ \sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : \text{var}(f(t_1, \ldots, t_m)) \subseteq \{ t_1, \ldots, t_m \}, \text{var}(\gamma(s_1, \ldots, s_n)) \subseteq \{ s_1, \ldots, s_n \} \} \text{ and } t_j = s_j = x_j, \text{var}(f(t_1, \ldots, t_m)) = \text{var}(\gamma(s_1, \ldots, s_n)) = 1 \}; \\
R_{T} := \{ \sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in \text{Relhyp}((m), (n)) : t_i \in X_m : \forall i \in \{1, \ldots, m\}, s_j \in X_n : \forall j \in \{1, \ldots, n\} \}.
\]

Let \( m, n \) be natural numbers, where \( m \geq n \geq 2, i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \).
We denote \((MUR)'_{i,j} := R'_{X,i} \cup R_{X} \cup R_{T,i} \) and \((MUR)'' := R'_{X} \cup R_{X} \cup R_{T} \). Firstly, we recall some definitions used for the proof of our main theorems.
Definition 3.1. ([1]) Let \( t \in W(m)(X_m) \setminus X \), where \( t = (t_1, \ldots, t_m) \) for some \( t_1, \ldots, t_m \in W(m)(X_m) \). Let \( x_j^{(l)} \) be a variable \( x_j \) occurring in the \( l \)th component of \( t \) (from the left). For \( i_1 = 1, \ldots, m \), let \( \pi_i : W(m)(X_m) \setminus X \to W(m)(X_m) \) be defined by \( \pi_i(t) = t_{i_1} \). The sequence of \( x_j^{(l)} \) in \( t \) is denoted by \( \text{seq}^l(x_j^{(l)}) \). If \( x_j^{(l)} = \pi_{i_k} \circ \cdots \circ \pi_{i_1}(t) \) for some \( k \in \mathbb{N} \), then \( \text{seq}^l(x_j^{(l)}) = (i_1, \ldots, i_k) \).

Definition 3.2. ([8]) Let \( F \in rF_{(m), (n)}(X_n) \), where \( F = \gamma(s_1, \ldots, s_n) \) for some \( s_1, \ldots, s_n \in W(n)(X_n) \). Let \( x_j^{(l)} \) be a variable \( x_j \) occurring in the \( l \)th component of \( F \) (from the left). For \( i_1 = 1, \ldots, n \), let \( \varphi_i : rF_{(m), (n)}(X_n) \to W(n)(X_n) \) be defined by \( \varphi_i(F) = \varphi_i(\gamma(s_1, \ldots, s_n)) = s_{i_1} \). For \( i_k \in \{1, \ldots, m\} \), let \( \phi_{i_k} : W(n)(X_n) \setminus X \to W(n)(X_n) \) be defined by \( \phi_{i_k}(t) = t_{i_k} \). The sequence of \( x_j^{(l)} \) in \( F \) is denoted by \( \text{seq}^F(x_j^{(l)}) \). If \( x_j^{(l)} = \phi_{i_k} \circ \cdots \circ \phi_{i_2} \circ \varphi_{i_1}(F) \) for some \( k \in \mathbb{N} \), then \( \text{seq}^F(x_j^{(l)}) = (i_1, \ldots, i_k) \).

We denote the set of all sequences of \( x_j \) in term \( t \) and \( x_j \) in relational term \( F \) by \( \text{Seq}^l(x_j) \) and \( \text{Seq}^F(x_j) \), respectively, i.e.

\[
\text{Seq}^l(x_j) = \{\text{seq}^l(x_j^l) \mid l \in \mathbb{N}\};
\]

\[
\text{Seq}^F(x_j) = \{\text{seq}^F(x_j^l) \mid l \in \mathbb{N}\}.
\]

Example 3.3. Let \( (\tau, \tau') = ((3), (2)) \) and \( t \in W(3)(X_3) \), \( F \in rF_{(3), (2)}(X_2) \), where \( t = f(x_2, f(x_3, f(x_1, f(x_2, x_3, x_1), x_2), x_1), f(f(x_3, x_2, f(x_2, x_3, x_1)), x_1, x_3)) \) and \( F = \gamma(f(x_2, f(x_1, f(x_2, x_3, x_1), x_2), x_1), f(f(x_3, x_2, f(x_2, x_3, x_1)), x_1, x_3)) \). Then, \( \text{seq}^l(x_1^{(1)}) = (2, 2, 1) \), \( \text{seq}^l(x_1^{(2)}) = (2, 1, 2) \), \( \text{seq}^F(x_1^{(1)}) = (2, 3) \), \( \text{seq}^F(x_1^{(2)}) = (3, 1, 3, 3) \), \( \text{seq}^F(x_1^{(3)}) = (3, 2) \), \( \text{seq}^F(x_1^{(4)}) = (1, 2, 1) \), \( \text{seq}^F(x_2^{(1)}) = (1, 2, 3) \), \( \text{seq}^F(x_2^{(2)}) = (1, 3) \), \( \text{seq}^F(x_2^{(3)}) = (2) \). So \( \text{Seq}^l(x_1) = \{(2, 2, 1), (2, 2, 2, 3), (2, 3), (3, 1, 3, 3), (3, 2)\} \) and \( \text{Seq}^F(x_1) = \{(1, 2, 1), (1, 2, 3), (1, 3), (2)\} \).

Lemma 3.4. Let \( t = f(t_1, \ldots, t_m), F = \gamma(s_1, \ldots, s_n) \) with \( \text{var}(t) = \{x_{a_1}, \ldots, x_{a_k}\} \), \( \text{var}(F) = \{x_{b_1}, \ldots, x_{b_l}\} \), and there exists \( a_i' \in \{1, \ldots, m\}, b_j' \in \{1, \ldots, n\} \) such that \( t_{a_i'} = x_{a_i} \) and \( s_{b_j'} = x_{b_j} \). If \( x_{a_c} \in \text{var}(t_p) \) for some \( c \in \{1, \ldots, k\} \), \( p \in \{1, \ldots, m\} \setminus \{a_1', \ldots, a_k'\} \) and \( x_{b_d} \in \text{var}(s_q) \) for some \( d \in \{1, \ldots, l\} \), \( q \in \{1, \ldots, n\} \setminus \{b_1', \ldots, b_l'\} \), where \( (p_1', \ldots, p_r') \in \text{Seq}^r(x_{a_c}) \) for some \( p_1', \ldots, p_r' \in \{1, \ldots, m\} \setminus \{a_1', \ldots, a_k'\} \) and \( (q_1', \ldots, q_g') \in \text{Seq}^s(x_{b_d}) \), then there exists \( \sigma_{a,H} \in \text{Relhyp}(m), (n) \) such that \( \sigma_{a,H} \circ \tau \circ \tau_F \) is not a unit-regular element in \( \text{Relhyp}(m), (n) \).

Proof. Assume the condition holds. Since \( (p_1', \ldots, p_r') \in \text{Seq}^r(x_{a_c}) \) and \( (q_1', \ldots, q_g') \in \text{Seq}^s(x_{b_d}) \), then \( (p, p_1', \ldots, p_r') \in \text{Seq}^l(x_{a_c}) \) and \( (q, q_1', \ldots, q_g') \in \text{Seq}^F(x_{b_d}) \). Let \( p_1^* \), \ldots, \( p_r^* \) be distinct for \( p, p_1', \ldots, p_r' \) and \( q_1^* \), \ldots, \( q_g^* \) be distinct for \( q, q_1', \ldots, q_g' \). Choose \( \sigma_{a,H} \in \text{Relhyp}(m), (n) \), where \( u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n) \) such that \( u_1 = x_{p_1^*}, \ldots, u_r = x_{p_r^*}, u_{r+1}, \ldots, u_m \in W(m)(X_m) \setminus \{x_{a_c'}\} \) and \( h_1 = x_{q_1^*}, \ldots, q_g = x_{q_g^*}, h_{g+1}, \ldots, h_n \in W(n)(X_n) \setminus \{x_{b_d'}\} \). Then, \( u_i \neq x_{a_c} \) for all \( i \in \ldots \).
\{1, \ldots, m\} \text{ and } h_j \neq x_{b_d} \forall j \in \{1, \ldots, n\}. \text{ Consider }
\begin{align*}
(\sigma_{u,H} \circ \sigma_{t,F})(f) &= \tilde{\sigma}_{u,H} [f(t_1, \ldots, t_m)] \\
&= S^m(t(u_1, \ldots, u_m), \tilde{\sigma}_{u,H}[t_1], \ldots, \tilde{\sigma}_{u,H}[t_m]) \\
&= f(w_1, \ldots, w_m)
\end{align*}
where \(w_i = S^m(u_i, \tilde{\sigma}_{u,H}[t_1], \ldots, \tilde{\sigma}_{u,H}[t_m]),\)
and
\begin{align*}
(\sigma_{u,H} \circ \sigma_{t,F})(\gamma) &= \tilde{\sigma}_{u,H} [\gamma(s_1, \ldots, s_n)] \\
&= R^m(\gamma(h_1, \ldots, h_n), \tilde{\sigma}_{u,H}[s_1], \ldots, \tilde{\sigma}_{u,H}[s_n]) \\
&= \gamma(z_1, \ldots, z_n)
\end{align*}
where \(z_i = S^m(h_i, \tilde{\sigma}_{u,H}[s_1], \ldots, \tilde{\sigma}_{u,H}[s_n]).\)

Since \(u_i \neq x_{a'}, s_j \neq x_{a'}, \) so \(w_i \neq x_{a'}, z_j \neq x_{b} \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, n\}. \)
By Theorem 3.1., 3.2. of [8], we get \(x_{a'} \in \var{\tilde{\sigma}_{u,H}[l]}, x_{b} \in \var{\tilde{\sigma}_{u,H}[F]}\) such that \(x_{a'} \in \var{w_v}, \) where \(w_v \in W_{(m)}(X_m) \setminus X_m \text{ for some } v \in \{1, \ldots, m\} \) and \(x_{b} \in \var{z_y}, \) where \(z_y \in W_{(n)}(X_n) \setminus X_n \text{ for some } y \in \{1, \ldots, n\}. \) Hence, \(\sigma_{u,H} \circ \sigma_{t,F} \notin R^*_X \cup R_T. \) So, \(\sigma_{u,H} \circ \sigma_{t,F} \) is not a unit-regular element in \(\text{Relhyp}((m), (n)). \)

**Theorem 3.5.** \((\text{MUR})_{(i,j)}\) is a unit-regular submonoid of \(\text{Relhyp}((m), (n)). \)

**Proof.** We get that every element in \((\text{MUR})'\) is unit-regular. Next, we show that \((\text{MUR})'_{(i,j)} := R^*_x \cup R^*_X \cup R^*_t\) is closed under \(\circ. \) Let \(\sigma_{t,F}, \sigma_{u,H} \in (\text{MUR})'_{(i,j)}. \)

**Case 1:** \(\sigma_{t,F} \in R^*_x. \) Then, \(t = x_i \in X_m \) and \(F = \gamma(s_1, \ldots, s_n) \) with \(\var{F} = \{x_{b_1}, \ldots, x_{b_l}\} \) such that \(i - \text{most}(s_{a'}_{k}) = x_{b_k} \forall k = 1, \ldots, l. \)

**Case 1.1:** \(\sigma_{u,H} \in R^*_x. \) Then, \(u = x_i \in X_m \) and \(H = \gamma(h_1, \ldots, h_n) \) with \(\var{H} = \{x_{d_1}, \ldots, x_{d_q}\} \) such that \(i - \text{most}(h_{a'}_{k}) = x_{d_k} \forall k = 1, \ldots, q. \)

Consider
\begin{align*}
(\sigma_{t,F} \circ \sigma_{u,H})(f) &= \tilde{\sigma}_{x_i,F}[x_i] = x_i \\
(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) &= R^m(F, \tilde{\sigma}_{x_i,F}[h_1], \ldots, \tilde{\sigma}_{x_i,F}[h_n]) \\
&= \gamma(S^m(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, \\
&= \gamma(s_{a'}_{k}) \text{ where } \var(\gamma(s_{a'}_{1}, \ldots, s_{a'}_{n})) \subseteq \{x_{d_1}, \ldots, x_{d_q}\}
\end{align*}

such that \(i - \text{most}(s_{a'}_{k}) = x_{d_k}; k = 1, \ldots, q. \)

**Case 1.2:** \(\sigma_{u,H} \in R^*_X. \) Then, \(u = x_j \in X_m, H = \gamma(h_1, \ldots, h_n) \) with \(|\var{H}| = 1. \)

Consider
\begin{align*}
(\sigma_{t,F} \circ \sigma_{u,H})(f) &= \tilde{\sigma}_{x_j,F}[x_j] = x_j, \text{ and }
\end{align*}
Consider $\forall k = j$ such that $\exists h \in (H \cup \sigma) \cap \mathbb{R}^n_{\text{max}} \text{ relhyp}(m, n)$:

$$\gamma(S_n^m(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, S_n^m(s_n, i - \text{most}(h_1), \ldots, i - \text{most}(h_n))) = \gamma(s_1', \ldots, s_n') \text{ where } |\text{var}(\gamma(s_1', \ldots, s_n'))| = 1.$$

**Case 1.3:** $\sigma_{u,H} \in R^*_t$. Then, $u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W(m)(X_m) \setminus X_m; \forall k \in \{1, \ldots, m\} \setminus \{j\}, h_k \in W(n)(X_n) \setminus X_n; \forall k \in \{1, \ldots, n\} \setminus \{j\}$ with $\text{val}(u) = \text{var}(H) = \{x_j\}$. Consider

$$(\sigma_{t,F} \circ \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = x_j,$$ and

$$(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R^n_m(F, \hat{\sigma}_{x_i,F}[h_1], \ldots, \hat{\sigma}_{x_i,F}[h_n]) = \gamma(S_n^m(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, S_n^m(s_n, i - \text{most}(h_1), \ldots, i - \text{most}(h_n))) = \gamma(s_1', \ldots, s_n') \text{ where } |\text{var}(\gamma(s_1', \ldots, s_n'))| = 1.$$

**Case 2:** $\sigma_{t,F} \in R^*_X$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1, \ldots, s_n)$ with $|\text{var}(F)| = 1$.

**Case 2.1:** $\sigma_{u,H} \in R^*_X$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $\text{var}(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ such that $i - \text{most}(h_{d_k}) = x_{d_k}$ for all $k = 1, \ldots, q$. Consider

$$(\sigma_{t,F} \circ \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[x_j] = x_j,$$ and

$$(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R^n_m(F, \hat{\sigma}_{x_i,F}[h_1], \ldots, \hat{\sigma}_{x_i,F}[h_n]) = \gamma(S_n^m(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, S_n^m(s_n, i - \text{most}(h_1), \ldots, i - \text{most}(h_n))) = \gamma(s_1', \ldots, s_n') \text{ where } |\text{var}(\gamma(s_1', \ldots, s_n'))| = 1.$$

**Case 2.2:** $\sigma_{u,H} \in R^*_X$. Then, $u = x_j \in X_m, H = \gamma(h_1, \ldots, h_n)$ with $|\text{var}(H)| = 1$.

Consider

$$(\sigma_{t,F} \circ \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[x_j] = x_j,$$ and

$$(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R^n_m(F, \hat{\sigma}_{x_i,F}[h_1], \ldots, \hat{\sigma}_{x_i,F}[h_n]) = \gamma(S_n^m(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, S_n^m(s_n, i - \text{most}(h_1), \ldots, i - \text{most}(h_n))) = \gamma(s_1', \ldots, s_n') \text{ where } |\text{var}(\gamma(s_1', \ldots, s_n'))| = 1.$$

**Case 2.3:** $\sigma_{u,H} \in R^*_t$. Then, $u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W(m)(X_m) \setminus X_m; \forall k \in \{1, \ldots, m\} \setminus \{j\}, h_k \in W(n)(X_n) \setminus X_n; \forall k \in \{1, \ldots, n\} \setminus \{j\}$ with $\text{val}(u) = \text{var}(H) = \{x_j\}$. Consider

$$(\sigma_{t,F} \circ \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = x_j,$$ and
\[(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R_n^u(F, \tilde{\sigma}_{x_i,F}[h_1], \ldots, \tilde{\sigma}_{x_i,F}[h_n]) = \gamma(S_m^n(s_1, i - \text{most}(h_1), \ldots, i - \text{most}(h_n)), \ldots, S_m^n(s_i, i - \text{most}(h_1), \ldots, i - \text{most}(h_n))) = \gamma(s_1', \ldots, s_n') \text{ where } \var{\gamma(s_1', \ldots, s_n')} = \{x_j\} \]

Case 3: \(\sigma_{t,F} \in R^*_t\). Then, \(t = (t_1, \ldots, t_m), F = \gamma(s_1, \ldots, s_n)\) where \(t_j = x_j = s_j\) such that \(t_k \in W_m(X_m) \setminus X_m; \forall k \in \{1, \ldots, m\} \setminus \{j\}, s_k \in W_n(X_n) \setminus X_n; \forall k \in \{1, \ldots, n\} \setminus \{j\}\) with \(\var{t} = \var(F) = \{x_j\}\).

Case 3.1: \(\sigma_{u,H} \in R^*_X\). Then, \(u = x_i \in X_m\) and \(H = \gamma(h_1, \ldots, h_n)\) with \(\var(H) = \{x_{d_1}, \ldots, x_{d_q}\}\) such that \(i - \text{most}(h_{d_k'}) = x_{d_k}\) for all \(k = 1, \ldots, q\).

Consider
\[
(\sigma_{t,F} \circ \sigma_{u,H})(f) = \tilde{\sigma}_{t,F}[x_i] = x_i, \text{ and }
\]
\[
(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R_n^u(F, \tilde{\sigma}_{t,F}[h_1], \ldots, \tilde{\sigma}_{t,F}[h_n]) = \gamma(s_1', \ldots, s_n') \text{ where } |\var{\gamma(s_1', \ldots, s_n')}| = 1.
\]

Case 3.2: \(\sigma_{u,H} \in R^*_X\). Then, \(u = x_j \in X_m, H = \gamma(h_1, \ldots, h_n)\) with \(|\var(H)| = 1\).

Consider
\[
(\sigma_{t,F} \circ \sigma_{u,H})(f) = \tilde{\sigma}_{t,F}[x_j] = x_j, \text{ and }
\]
\[
(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R_n^u(F, \tilde{\sigma}_{t,F}[h_1], \ldots, \tilde{\sigma}_{t,F}[h_n]) = \gamma(s_1', \ldots, s_n') \text{ where } |\var{\gamma(s_1', \ldots, s_n')}| = 1.
\]

Case 3.3: \(\sigma_{u,H} \in R^*_t\). Then, \(u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)\) where \(u_j = x_j = h_j\) such that \(u_k \in W_m(X_m) \setminus X_m; \forall k \in \{1, \ldots, m\} \setminus \{j\}, h_k \in W_n(X_n) \setminus X_n; \forall k \in \{1, \ldots, n\} \setminus \{j\}\) with \(\var{u} = \var(H) = \{x_j\}\). Consider
\[
(\sigma_{t,F} \circ \sigma_{u,H})(f) = \tilde{\sigma}_{t,F}[f(u_1, \ldots, u_m)] = S_m^n(t, \tilde{\sigma}_{t,F}[u_1], \ldots, \tilde{\sigma}_{t,F}[x_j], \ldots, \tilde{\sigma}_{t,F}[u_m]) = f(t_1, \ldots, x_j, \ldots, t_m) \text{ where } \var{f(t_1, \ldots, x_j, \ldots, t_m)} = \{x_j\}, \text{ and }
\]
\[
(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = \tilde{\sigma}_{t,F}[\gamma(h_1, \ldots, h_n)] = R_n^u(F, \tilde{\sigma}_{t,F}[h_1], \ldots, \tilde{\sigma}_{t,F}[x_j], \ldots, \tilde{\sigma}_{t,F}[h_n]) = \gamma(s_1, \ldots, x_j, \ldots, s_n) \text{ where } \var{\gamma(s_1, \ldots, x_j, \ldots, s_n)} = \{x_j\}.
\]

Therefore, \(\text{(MUR)}_{(i,j)}^\prime\) is a unit-regular submonoid of \(\text{Relhyp}((m), (n))\).

**Theorem 3.6.** \(\text{(MUR)}''\) is a unit-regular submonoid of \(\text{Relhyp}((m), (n))\).

**Proof.** We get that every element in \(\text{(MUR)}''\) is unit-regular. Next, we show that \(\text{(MUR)}'' := R^*_X \cup R^*_X \cup R^*_T\) is closed under \(\circ_r\). Let \(\sigma_{t,F}, \sigma_{u,H} \in \text{(MUR)}''\).

Case 1: \(\sigma_{t,F} \in R^*_X\). Then, \(t = x_i \in X_m\) and \(F = \gamma(s_1, \ldots, s_n)\) with \(\var(F) = \{x_{b_1}, \ldots, x_{b_l}\}\) and \(|\var(s_j)| = 1 \forall j = 1, \ldots, n\).
Consider Case 2. Then, $u = x_j \in X_m$ and $H = \gamma(h_1,\ldots,h_n)$ with $\var(H) = \{x_{d_1},\ldots,x_{d_q}\}$ and $|\var(h_j)| = 1 \ \forall j = 1,\ldots,n$. Consider
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[x_j] = x_j
\]
and
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R^n_{\sigma}(F,\hat{\sigma}_{x_i,F}[h_1],\ldots,\hat{\sigma}_{x_i,F}[h_n])
= \gamma(S^m_n(s_1,i-\text{most}(h_1),\ldots,i-\text{most}(h_n)),\ldots,
S^m_n(s_n,i-\text{most}(h_1),\ldots,i-\text{most}(h_n)))
= \gamma(s'_1,\ldots,s'_n) \text{ where } |\var(s'_j)| = 1 \ \forall j = 1,\ldots,n.
\]

Consider Case 1. If $u = x_j \in X_m$ then $H = \gamma(h_1,\ldots,h_n)$ with $|\var(H)| = 1$.

Consider Case 1.2. Then, $u = x_j \in X_m$ and $H = \gamma(h_1,\ldots,h_n)$ with $|\var(H)| = 1$.

Consider Case 1.3. Then, $u = f(u_1,\ldots,u_m)$; $u_i \in X_m \ \forall i = 1,\ldots,m$ and $H = \gamma(h_1,\ldots,h_n)$; $h_i \in X_n \ \forall i = 1,\ldots,n$. Consider
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \hat{\sigma}_{x_i,F}[f(u_1,\ldots,u_m)] = \hat{\sigma}_{x_i,F}[u_i] = x_j \in X_m,
\]
and
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R^n_{\sigma}(F,\hat{\sigma}_{x_i,F}[h_1],\ldots,\hat{\sigma}_{x_i,F}[h_n])
= \gamma(S^m_n(s_1,i-\text{most}(h_1),\ldots,i-\text{most}(h_n)),\ldots,
S^m_n(s_n,i-\text{most}(h_1),\ldots,i-\text{most}(h_n)))
= \gamma(s'_1,\ldots,s'_n) \text{ where } |\var(s'_j)| = 1 \ \forall j = 1,\ldots,n.
\]

Consider Case 2. Then, $t = x_i \in X_m$ and $F = \gamma(s_1,\ldots,s_n)$ with $|\var(F)| = 1$.

Consider Case 2.1. Then, $u = x_j \in X_m$ and $H = \gamma(h_1,\ldots,h_n)$ with $|\var(H)| = 1 \ \forall j = 1,\ldots,n$. Consider
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \hat{\sigma}_{t,F}[x_i] = x_i,
\]
and
\[
(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) = R^n_{\sigma}(F,\hat{\sigma}_{t,F}[h_1],\ldots,\hat{\sigma}_{t,F}[h_n])
= \gamma(s'_1,\ldots,s'_n) \text{ where } |\var(s'_j)| = 1.
\]

Consider Case 2.2. The proof is similar to case 2.2 of Theorem 3.5.
Therefore, $(MUR)''$ is a unit-regular submonoid of $Relhyp((m), (n))$. □
**Theorem 3.7.** $(MUR)_{(i,j)}^\prime$ is a maximal unit-regular submonoid of $Relhyp((m),(n))$.

*Proof.* Let $K$ be a proper unit-regular submonoid of $Relhyp((m),(n))$ such that $(MUR)_{(i,j)} \subseteq K \subseteq Relhyp((m),(n))$. Let $\sigma_{t,F} \in K$. Then, $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_X \setminus R^*_X \cup R^{**}_X$. Then, $t = x_j \in X_m$ for $F = \gamma(s_1, \ldots, s_n)$ with $var(F) = \{x_{b_1}, \ldots, x_{b_k}\}$ such that $j - most(s'_{k_i}) = x_{b_k} \forall k = 1, \ldots, l$. Choose $\sigma_{u,H} \in R^*_X$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_r}\}$ such that $i - most(h'_{d_p}) = x_{d_p} \forall p = 1, \ldots, r$. Consider

$$\sigma_{u,H} \circ_r \sigma_{t,F}(f) = \tilde{\sigma}_{x_i,H}[x_j] = x_j,$$

and

$$(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) = R^n_u(H, \tilde{\sigma}_{x_i,H}[s_1], \ldots, \tilde{\sigma}_{x_i,H}[s_n]) = \gamma(S^n_u(h_1, i - most(s_1), \ldots, i - most(s_n)), \ldots, \nonumber \quad S^n_u(h_n, i - most(s_1), \ldots, i - most(s_n))).$$

Since $i - most(h'_{d_p}) = x_{d_p}$ for all $p = 1, \ldots, r$, we have

$$x_{d_p} = i - most(h'_{d_p})$$

$$= S^n_u(i - most(h'_{d_p}), i - most(s_1), \ldots, i - most(s_n))$$

$$= S^n_u(x_{d_p}, i - most(s_1), \ldots, i - most(s_n))$$

$$= i - most(s_{d_p}).$$

Since $u \neq x_j$, by Lemma 2.4., we have $j - most(\tilde{\sigma}_{u,H}[s_{d_p}]) \neq i - most(s_{d_p}) = x_{d_p}$. Thus, $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Case 2: $\sigma_{t,F} \in R_T \setminus R^*_T$. Then, $t = f(t_1, \ldots, t_m)$ and $F = \gamma(s_1, \ldots, s_n)$ such that $var(f(t_1, \ldots, t_m)) \subseteq \{t_1, \ldots, t_m\}$ and $var(\gamma(s_1, \ldots, s_n)) \subseteq \{s_1, \ldots, s_n\}$. Choose $\sigma_{u,H} \in R^*_T$. Then, $u = f(u_1, \ldots, u_m)$, $H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W_{(m)}(X_m) \setminus X_m$; $\forall k \in \{1, \ldots, m\}\{j\}$, $h_k \in W_{(n)}(X_n) \setminus X_n$; $\forall k \in \{1, \ldots, n\}\{j\}$ with $\text{val}(u) = var(H) = \{x_j\}$. Consider

$$(\sigma_{u,H} \circ_r \sigma_{t,F})(f) = S^m_u(u, \tilde{\sigma}_{u,H}[t_1], \ldots, \tilde{\sigma}_{u,H}[t_m])$$

$$= f(u_1, \ldots, u_m) \text{ where } |\text{val}(\gamma(u'_1, \ldots, u'_m))| = 1$$

If $t_j \in W_{(m)}(X_m) \setminus X_m$, then $u'_k \in W_{(m)}(X_m) \setminus X_m$ for all $k = 1, \ldots, m$.

If $t_j \in X_m \\setminus \{x_j\}$, then $u'_j \neq x_j$. By Example 4 of [9], we have that it is not closed into itself. Thus, $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Thus, $\sigma_{t,F} \in (MUR)_{(i,j)}$. Therefore, $K \subseteq (MUR)_{(i,j)}$, and thus

$$K = (MUR)_{(i,j)}^\prime.$$

□

**Theorem 3.8.** $(MUR)^{\prime\prime}$ is a maximal unit-regular submonoid of $Relhyp((m),(n))$.

*Proof.* Let $K$ be a proper unit-regular submonoid of $Relhyp((m),(n))$ such that $(MUR)^{\prime\prime} \subseteq K \subseteq Relhyp((m),(n))$. Let $\sigma_{t,F} \in K$. Then, $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_X \setminus R^*_X \cup R^{**}_X$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1, \ldots, s_n)$
with \( \text{var}(F) = \{x_{b_1}, \ldots, x_{b_l}\} \) such that \( i - \text{most}(s_{b_i}') = x_{b_k} \ \forall k = 1, \ldots, l. \) If \(|\text{var}(F)| = 2\), choose \( \sigma_{u,H} \in R^+_T \), such that \( i - \text{most}(u) = x_j \) with \( j - \text{most}(s_l) = x_{b_k} \ \forall l \in \{1, \ldots, n\} \) and \( H = \gamma(x_1, \ldots, x_n) \). If \(|\text{var}(F)| > 2\), choose \( \sigma_{u,H} \in R^+_T \), such that \( i - \text{most}(u) = x_j \) with \( j - \text{most}(s_l) = x_{b_k} \ \exists l \in \{1, \ldots, n\} \) and \( H = \gamma(x_1, \ldots, x_n) \).

Consider

\[
(\sigma_{t,F} \circ \sigma_{u,H})(f) = \tilde{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = x_j,
\]

and

\[
(\sigma_{t,F} \circ \sigma_{u,H})(\gamma) = R^m_n(F, \tilde{\sigma}_{t,F}[x_1], \ldots, \tilde{\sigma}_{t,F}[x_n]) = \gamma(s_1', \ldots, s_n').
\]

Since \( i - \text{most}(u) = x_j \), so \(|\text{var}(j - \text{most}(s_l))| = 1 < |\text{var}(\gamma(s_1', \ldots, s_n'))|\). Thus, \( \sigma_{t,F} \circ \sigma_{u,H} \) is not unit-regular.

**Case 2:** \( \sigma_{t,F} \in R_T^+ \setminus R^*_T \). By Lemma 3.4., we can choose \( \sigma_{u,H} \in R^+_T \) such that \( \sigma_{u,H} \circ \sigma_{t,F} \) is not unit-regular.

Thus, \( \sigma_{t,F} \in (MUR)' \). Therefore, \( K \subseteq (MUR)'' \), and thus \( K = (MUR)'' \).

**Corollary 3.9.** \((MUR)'_{(i,j)} \), \((MUR)'' \) are maximal factorisable submonoids of the monoid of the relational hypersubstitutions for algebraic systems of type \(((m), (n))\).

**References**

ALL MAXIMAL UNIT-REGULAR ELEMENTS OF $\text{Relhyp}(m, (n))$


Pornpimol Kunama, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
\textit{e-mail:} pornpimol5331@gmail.com

Sorasak Leeratanavalee, Research Group in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand
\textit{e-mail:} sorasak.l@cmu.ac.th