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ALL MAXIMAL UNIT-REGULAR ELEMENTS OF *Relhyp*$((m),(n))$

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Abstract. Any relational hypersubstitution for algebraic systems of type $(\tau,\tau') = ((m_i)_{i\in I},(n_j)_{j\in J})$ is a mapping which maps any m_i-ary operation symbol to an m_i-ary term and maps any n_j-ary relational symbol to an n_j-ary relational term preserving arities, where I, J are indexed sets. The set of all relational hypersubstitutions for algebraic systems of type (τ,τ') together with a binary operation defined on the set and its identity forms a monoid. The properties of this structure are expressed by terms and formulas. Some algebraic properties of the monoid of a special type, especially the set of all unit-regular elements, were studied. In this paper, we determine all maximal unit-regular submonoids of this monoid of type $((m),(n))$ for arbitrary natural numbers $m, n \geq 2$.

1. Introduction

In universal algebra, identities are used to classify algebras into collections called *varieties*, and hyperidentities are used to classify varieties into collections called *hypervarieties*. The tool which is used to study hyperidentities and hypervarieties is the concept of a hypersubstitution introduced by W. Taylor [15]. The notation of a hypersubstitution was developed by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [6]. The authors used this concept for the characterization of solid varieties of type τ. A solid variety is a variety which is closed under the following operation: taking a universal algebra $(A,(f_i^A)_{i\in I})$ of type $\tau = (m_i)_{i\in I}$ with the universe A and family $(f_i^A)_{i\in I}$ of m_i-ary operations f_i^A on A for $i \in I$ of a variety, then we replace the operation f_i^A by any m_i-ary term operation $\sigma(f_i)^A$, for $i \in I$, and obtain a new universal algebra $(A,(\sigma(f_i)^A)_{i\in I})$, which also belongs to the variety. Hence, a hypersubstitution of a given type $\tau = (m_i)_{i\in I}$ is a mapping which maps every m_i-ary operation symbol f_i to an m_i-ary term of the same type, for $i \in I$. Moreover, the set $Hyp(\tau)$ of all hypersubstitutions of type τ together with an associative binary operation $\circ_h$ forms a monoid; see more details in [6,17].

However, we can consider algebraic systems in the sense of Mal'cev [11]. An algebraic system of type $(\tau,\tau^{'})$ is a triple $(A,(f_i^A)_{i\in I},(\gamma_j^A)_{j\in J})$ consisting of a non-empty set A, a sequence $(f_i^A)_{i\in I}$ of n_i-ary operations defined on A and a sequence $(\gamma_j^A)_{j\in J}$ of n_j-ary relations on A, where $\tau = (n_i)_{i\in I}$ is a sequence of the arity of

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each operation f_i^A and $\tau^{'} = (n_j)_{j\in J}$ is a sequence of the arity of each relation γ_j^A. The pair $(\tau,\tau^{'})$ is called a type of an algebraic system; see more details in [12–14].

In 2008 [5], K. Denecke and D. Phusanga introduced the concept of a hypersubstitution for algebraic systems, which is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a formula which preserves the arity. The set of all hypersubstitutions for algebraic systems of type $(\tau,\tau^{'})$ is denoted by $Hyp(\tau,\tau^{'})$. They defined an assosiative operation $\circ_r$ on this set and proved that $(Hyp(\tau,\tau^{'}),\circ_r,\sigma_{id})$ forms a monoid, where σ_{id} is an identity hypersubstitution. In 2018 [14], D. Phusanga and J. Koppitz introduced the concept of a relational hypersubstitution for algebraic systems of type (τ,τ') and proved that the set of all relational hypersubstitutions for algebraic systems of type (τ,τ') together with an assosiative binary operation and the identity element forms a monoid.

Firstly, we recall the definition of some special elements in a semigroup. An element a of a semigroup S is called *unit-regular* if there exists $u\in U(S)$ such that $a = aua$, where $U(S)$ is the set of all unit elements of S and S is called *unit-regular* if every element of S is unit-regular. Later in 1980, H. D'Alarcao showed that a monoid S is factorisable if and only if it is unit-regular [4]. A number of authors studied different factorisable semigroups. S. Y. Chen and S. C. Hsieh studied factorisable inverse semigroups [3]. Y. Tirasupa studied factorisable transformation semigroups [16]. In 2001 P. Jampachon, M. Saichalee and R. P. Sullivan used the concept of factorisability to study locally factorisable transformation semigroups [7]. In 2016, A. Boonmee and S. Leeratanavalee studied factorisable monoid of generalized hypersubstitutions of type (n) [2]. In this paper, we determine all maximal unit-regular elements of relational hypersubstitutions of type $((m),(n))$ for arbitrary natural numbers $m,n\geq 2$.

Next, we recall the concept of an n-ary term of type τ and an n-ary relational term of type (τ,τ'), respectively. Let $X := \{x_1,\ldots,x_n,\ldots\}$ be a countably infinite set of symbols called variables. For each $n\geq 1$, let $X_n := \{x_1,\ldots,x_n\}$. Let $\{f_i : i\in I\}$ be the set of m_i-ary operation symbols indexed by I, where $m_i\geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number m_i as its arity. The function $\tau = (m_i)_{i\in I}$ is called a type. An *n-ary term of type* τ is defined inductively as follows.

(i) Every variable $x_k\in X_n$ is an n-ary term of type τ.
(ii) If $t_1,\ldots,t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\ldots,t_{n_i})$ is an n-ary term of type τ.

We denote the set of all n-ary terms of type τ which contains $x_1,\ldots,x_n$ and is closed under finite application of (ii), by $W_\tau(X_n)$ and $W_\tau(X) := \bigcup_{n\in\mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

2. The monoid of relational hypersubstitutions for algebraic systems

Any relational hypersubstitution for algebraic systems is a mapping that assigns any operation symbol to a term and assigns any relation symbol to a relational term which preserves the arity. Let (τ,τ') be a type. An n-ary relational term of type (τ,τ') and a relational hypersubstitution for algebraic systems are defined as follows.

Definition 2.1. ([12]) Let I,J be indexed sets. If $i\in I, j\in J$ and $t_1,t_2,\ldots,t_{n_j}$ are $n-$ary terms of type $\tau=(n_i)_{i\in I}$ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1,t_2,\ldots,t_{n_j})$ is an $n-$ary relational term of type $(\tau,\tau^{'})=((n_i)_{i\in I},(n_j)_{j\in J})$.

We denote the set of all $n-$ary relational terms of type $(\tau,\tau^{'})$ by $rF_{(\tau,\tau')}(X_n)$ and $rF_{(\tau,\tau')}(X):=\cup_{n\in\mathbb{N}}rF_{(\tau,\tau')}(X_n)$ be the set of all relational terms of type $(\tau,\tau^{'})$.

A relational hypersubstitution for algebraic systems of type $(\tau,\tau^{'})=((m_i)_{i\in I},(n_j)_{j\in J})$ is a mapping

$$\sigma:\{f_i|\ i\in I\}\cup\{\gamma_j|\ j\in J\}\to W_\tau(X)\cup rF_{(\tau,\tau')}(X)$$

with $\sigma(f_i)\in W_\tau(X_{m_i})$ and $\sigma(\gamma_j)\in rF_{(\tau,\tau')}(X_{n_j})$. The set of all relational hypersubstitutions for algebraic systems of type $(\tau,\tau^{'})$ is denoted by *Relhyp*$(\tau,\tau^{'})$. To define a binary operation on this set, we define inductively the concept of a superposition of terms $S^m_n:W_\tau(X_m)\times(W_\tau(X_n))^m\to W_\tau(X_n)$ by the following steps. For any $t,t_1,\ldots,t_{m_i}\in W_\tau(X_m)$, $s_1,\ldots,s_m\in W_\tau(X_n)$,

(i) if $t=x_j$ for $1\le j\le n$, then $S^m_n(t,s_1,\ldots,s_m):=s_j$;
(ii) if $t=f_i(t_1,\ldots,t_{m_i})$, then

$$S^m_n(t,s_1,\ldots,s_m):=f_i(S^m_n(t_1,s_1,\ldots,s_m),\ldots,S^m_n(t_{m_i},s_1,\ldots,s_m)).$$

For any $F=\gamma_j(t_1,\ldots,t_{n_j})\in rF_{(\tau,\tau')}(X_m)$, we define the superposition of relational terms $R^m_n:(W_\tau(X_m)\cup rF_{(\tau,\tau')}(X_m))\times(W_\tau(X_n))^m\to W_\tau(X_n)\cup rF_{(\tau,\tau')}(X_n)$ by

(i) $R^m_n(t,s_1,\ldots,s_m):=S^m_n(t,s_1,\ldots,s_m)$,
(ii) $R^m_n(F,s_1,\ldots,s_m):=\gamma_j(S^m_n(t_1,s_1,\ldots,s_m),\ldots,S^m_n(t_{n_j},s_1,\ldots,s_m))$.

Every relational hypersubstitution for algebraic systems σ can be extended to a mapping $\widehat{\sigma}:W_\tau(X)\cup rF_{(\tau,\tau')}(X)\to W_\tau(X)\cup rF_{(\tau,\tau')}(X)$ as follows:

(i) $\widehat{\sigma}[x_i]:=x_i$ for $i\in\mathbb{N}$;
(ii) $\widehat{\sigma}[f_i(t_1\ldots,t_{m_i})]:=S^{m_i}_m(\sigma(f_i),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{m_i}])$, where $i\in I$ and $t_1,\ldots,t_{m_i}\in W_\tau(X_m)$, i.e., any occurrence of the variable x_k in $\sigma(f_i)$ is replaced by the term $\widehat{\sigma}[t_k], 1\le k\le m_i$;
(iii) $\widehat{\sigma}[\gamma_j(s_1\ldots,s_{n_j})]:=R^{n_j}_n(\sigma(\gamma_j),\widehat{\sigma}[s_1],\ldots,\widehat{\sigma}[s_{n_j}])$, where $j\in J$, $s_1,\ldots,s_{n_j}\in W_\tau(X_n)$, i.e., any occurrence of the variable x_k in $\sigma(\gamma_j)$ is replaced by the term $\widehat{\sigma}[s_k], 1\le k\le n_j$.

We define a binary operation $\circ_r$ on *Relhyp*$(\tau,\tau^{'})$ by $\sigma\circ_r\alpha:=\hat{\sigma}\circ\alpha$, where $\circ$ is the usual composition of mappings and $\sigma,\alpha\in$ *Relhyp*$(\tau,\tau^{'})$. Let σ_{id} be the relational hypersubstitution which maps each m_i-ary operation symbol f_i to the term $f_i(x_1,\ldots,x_{m_i})$ and maps each n_j-ary relation symbol γ_j to the relational term $\gamma_j(x_1,\ldots,x_{n_j})$. D. Phusanga and J. Koppitz [14] proved that (*Relhyp*$(\tau,\tau^{'}),\circ_r,\sigma_{id})$ is a monoid.

In 2015, W. Wongpinit and S. Leeratanavalee [18] introduced the concept of the $i-$ *most* of terms as follows.

Definition 2.2. ([18]) Let $\tau = (m)$ be a type with an m-ary operation symbol f, $t \in W_{(m)}(X)$ and $1 \leq i \leq m$. An $i - most(t)$ is defined inductively as follows

(i) If t is a variable, then $i - most(t) = t$,

(ii) If $t = f(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n \in W_{(m)}(X)$, then $i - most(t) := i - most(t_i)$.

Example 2.3. Let $\tau = (3)$ be a type, $t = f(x_3, f(x_3, x_2, x_1), f(x_2, x_1, x_1))$. Then, $1 - most(t) = x_3$, $2 - most(t) = 2 - most(f(x_3, x_2, x_1)) = x_2$ and $3 - most(t) = 3 - most(f(x_2, x_1, x_1)) = x_1$.

Lemma 2.4. ([18]) *Let $s, t \in W_{(m)}(X)$. If $j - most(t) = x_k \in X_m$ and $k - most(s) = x_i$, then $j - most(\hat{\sigma_t}[s]) = x_i$.*

The above lemma can be applied to any relational hypersubstitution for algebraic systems of type $((m), (n))$, such as $s, t \in W_{(m)}(X_m)$ and $F \in rF_{((m),(n))}(X_n)$ if $i - most(t) = x_j$, then $i - most(\widehat{\sigma}_{t,F}[s]) = j - most(s)$ where $\sigma_{t,F}$ is a relational hypersubstitution of type (τ, τ') which maps f to a term t and maps γ to a relational term F.

3. Maximal unit-regular elements in *Relhyp*$((m), (n))$

Let $(\tau, \tau^{'}) = ((m), (n))$ be a type with an $m-$ary operation symbol f, γ be an $n-$ary relation symbol, $t \in W_{(m)}(X_m)$ and $F \in rF_{((m),(n))}(X_n)$. We denote by $\sigma_{t,F}$ the set of the relational hypersubstitution of type $(\tau, \tau') = ((m), (n))$ which maps f to a term $t \in W_{(m)}(X_m)$ and maps γ to a relational term $F \in rF_{((m),(n))}(X_n)$;

$var(t)$ be the set of all variables occurring in the term t;

$var(F)$ be the set of all variables occurring in the relational term F.

$R'_X := \{\sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : \{i - most(s_k) : 1 \leq k \leq n = var(\gamma(s_1, \ldots, s_n))\}\}$;

$R_T := \{\sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : var(f(t_1, \ldots, t_m)) \subseteq \{t_1, \ldots, t_m\}, var(\gamma(s_1, \ldots, s_n)) \subseteq \{s_1, \ldots, s_n\}\}$.

In [10], P. Kunama and S. Leeratanavalee showed that $R'_X \cup R_T$ is the set of all unit-regular elements in *Relhyp*$((m), (n))$.

For any $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$, we denote

$R^*_{x_i} := \{\sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : \{i - most(s_k) : 1 \leq k \leq n = var(\gamma(s_1, \ldots, s_n))\}\}$;

$R^*_X := \{\sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : |var(s_k)| = 1 : 1 \leq k \leq n\}$;

$R^{**}_X := \{\sigma_{x_i, \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : |var(\gamma(s_1, \ldots, s_n))| = 1\}$;

$R^*_{t_j} := \{\sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : var(f(t_1, \ldots, t_m)) \subseteq \{t_1, \ldots, t_m\}, var(\gamma(s_1, \ldots, s_n)) \subseteq \{s_1, \ldots, s_n\}$ and $t_j = s_j = x_j, |var(f(t_1, \ldots, t_m)| = |var(\gamma(s_1, \ldots, s_m))| = 1\}$;

$R^*_T := \{\sigma_{f(t_1, \ldots, t_m), \gamma(s_1, \ldots, s_n)} \in Relhyp((m), (n)) : t_i \in X_m : \forall i \in \{1, \ldots, m\}, s_j \in X_n : \forall j \in \{1, \ldots, n\}\}$.

Let m, n be natural numbers, where $m \geq n \geq 2$, $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$. We denote $(MUR)'_{(i,j)} := R^*_{x_i} \cup R^{**}_X \cup R^*_{t_j}$ and $(MUR)'' := R^*_X \cup R^{**}_X \cup R^*_T$. Firstly, we recall some definitions used for the proof of our main theorems.

Definition 3.1. ([1]) Let $t \in W_{(m)}(X_m)\backslash X$, where $t = f(t_1, \ldots, t_m)$ for some $t_1, \ldots, t_m \in W_{(m)}(X_m)$. Let $x_j^{(l)}$ be a variable x_j occurring in the l^{th} component of t (from the left). For $i_l = 1, \ldots, m$, let $\pi_{i_l} : W_{(m)}(X_m)\backslash X \to W_{(m)}(X_m)$ be defined by $\pi_{i_l}(f(t_1, \ldots, t_m)) = t_{i_l}$. The sequence of $x_j^{(l)}$ in t is denoted by $seq^t(x_j^{(l)})$. If $x_j^{(l)} = \pi_{i_k} \circ \cdots \circ \pi_{i_1}(t)$ for some $k \in \mathbb{N}$, then $seq^t(x_j^{(l)}) = (i_1, \ldots, i_k)$.

Definition 3.2. ([8]) Let $F \in rF_{((m),(n))}(X_n)$, where $F = \gamma(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n \in W_{(n)}(X_n)$. Let $x_j^{(l)}$ be a variable x_j occurring in the l^{th} component of F (from the left). For $i_l = 1, \ldots, n$, let $\varphi_{i_l} : rF_{((m),(n))}(X_n) \to W_{(n)}(X_n)$ be defined by $\varphi_{i_l}(F) = \varphi_{i_l}(\gamma(s_1, \ldots, s_n)) = s_{i_l}$. For $i_k \in \{1, \ldots, m\}$, let $\phi_{i_k} : W_{(n)}(X_n)\backslash X \to W_{(n)}(X_n)$ be defined by $\phi_{i_k}(f(t_1, \ldots, t_m)) = t_{i_k}$. The sequence of $x_j^{(l)}$ in F is denoted by $seq^F(x_j^{(l)})$. If $x_j^{(l)} = \phi_{i_k} \circ \cdots \circ \phi_{i_2} \circ \varphi_{i_1}(F)$ for some $k \in \mathbb{N}$, then $seq^F(x_j^{(l)}) = (i_1, \ldots, i_k)$.

We denote the set of all sequences of x_j in term t and x_j in relational term F by $Seq^t(x_j)$ and $Seq^F(x_j)$, respectively, i.e.

$$Seq^t(x_j) = \{seq^t(x_j^l) \mid l \in \mathbb{N}\};$$
$$Seq^F(x_j) = \{seq^F(x_j^l) \mid l \in \mathbb{N}\}.$$

Example 3.3. Let $(\tau, \tau') = ((3),(2))$ and $t \in W_{(3)}(X_3)$, $F \in rF_{((3),(2))}(X_2)$, where $t = f(x_2, f(x_3, f(x_1, f(x_2, x_3, x_1), x_2), x_1), f(f(x_3, x_2, f(x_2, x_3, x_1)), x_1, x_3))$ and $F = \gamma(f(x_2, f(x_1, x_2, x_1), x_1), x_1)$. Then, $seq^t(x_1^{(1)}) = (2, 2, 1)$, $seq^t(x_1^{(2)}) = (2, 2, 2, 3)$, $seq^t(x_1^{(3)}) = (2, 3)$, $seq^t(x_1^{(4)}) = (3, 1, 3, 3)$, $seq^t(x_1^{(5)}) = (3, 2)$, $seq^F(x_1^{(1)}) = (1, 2, 1)$, $seq^F(x_1^{(2)}) = (1, 2, 3)$, $seq^F(x_1^{(3)}) = (1, 3)$, $seq^F(x_1^{(4)}) = (2)$. So $Seq^t(x_1) = \{(2, 2, 1), (2, 2, 2, 3), (2, 3), (3, 1, 3, 3), (3, 2)\}$ and $Seq^F(x_1) = \{(1, 2, 1), (1, 2, 3), (1, 3), (2)\}$.

Lemma 3.4. *Let* $t = f(t_1, \ldots, t_m), F = \gamma(s_1, \ldots, s_n)$ *with* $var(t) = \{x_{a_1}, \ldots, x_{a_k}\}$, $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$, *and there exists* $a_i' \in \{1, \ldots, m\}$, $b_j' \in \{1, \ldots, n\}$ *such that* $t_{a_i'} = x_{a_i}$ *and* $s_{b_j'} = x_{b_j}$. *If* $x_{a_c} \in var(t_p)$ *for some* $c \in \{1, \ldots, k\}$, $p \in \{1, \ldots, m\}\backslash\{a_1', \ldots, a_k'\}$ *and* $x_{b_d} \in var(s_q)$ *for some* $d \in \{1, \ldots, l\}$, $q \in \{1, \ldots, n\}\backslash\{b_1', \ldots, b_l'\}$, *where* $(p_1', \ldots, p_r') \in Seq^{t_p}(x_{a_c})$ *for some* $p_1', \ldots, p_r' \in \{1, \ldots, m\}\backslash\{a_c'\}$ *and* $(q_1', \ldots, q_g') \in Seq^{s_q}(x_{b_d})$ $\exists$ $q_1', \ldots, q_g' \in \{1, \ldots, n\}\backslash\{b_d'\}$, *then there exists* $\sigma_{u,H} \in$ *Relhyp*$((m),(n))$ *such that* $\sigma_{u,H} \circ_r \sigma_{t,F}$ *is not a unit-regular element in Relhyp*$((m),(n))$.

Proof. Assume the condition holds. Since $(p_1', \ldots, p_r') \in Seq^{t_p}(x_{a_c})$ and $(q_1', \ldots, q_g') \in Seq^{s_q}(x_{b_d})$, then $(p, p_1', \ldots, p_r') \in Seq^t(x_{a_c})$ and $(q, q_1', \ldots, q_g') \in Seq^F(x_{b_d})$. Let $p_1^*, \ldots, p_{r^*}^*$ be distinct for $p, p_1', \ldots, p_r'$ and $q_1^*, \ldots, q_{g^*}^*$ be distinct for $q, q_1', \ldots, q_g'$. Choose $\sigma_{u,H} \in$ *Relhyp*$((m),(n))$, where $u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)$ such that $u_1 = x_{p_1^*}, \ldots, u_r = x_{p_{r^*}^*}, u_{r+1}, \ldots, u_m \in W_{(m)}(X_m)\backslash\{x_{a_c'}\}$ and $h_1 = x_{q_1^*}, \ldots, q_g = x_{q_{g^*}^*}, h_{g+1}, \ldots, h_n \in W_{(n)}(X_n)\backslash\{x_{b_d'}\}$. Then, $u_i \neq x_{a_c'}$ for all $i \in$

$\{1,\ldots,m\}$ and $h_j \neq x_{b'_d}$ $\forall j \in \{1,\ldots,n\}$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(f) &= \widehat{\sigma}_{u,H}[f(t_1,\ldots,t_m)]] \\ &= S^m_m(f(u_1,\ldots,u_m),\widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_m]) \\ &= f(w_1,\ldots,w_m) \\ &\text{where } w_i = S^m_m(u_i,\widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_m]),\end{aligned}$$

and

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{u,H}[\gamma(s_1,\ldots,s_n)]] \\ &= R^n_n(\gamma(h_1,\ldots,h_n),\widehat{\sigma}_{u,H}[s_1],\ldots,\widehat{\sigma}_{u,H}[s_n]) \\ &= \gamma(z_1,\ldots,z_n) \\ &\text{where } z_i = S^n_n(h_i,\widehat{\sigma}_{u,H}[s_1],\ldots,\widehat{\sigma}_{u,H}[s_n]).\end{aligned}$$

Since $u_i \neq x_{a'_c}$, $s_j \neq x_{b'_d}$, so $w_i \neq x_{a_c}$, $z_j \neq x_{b_d}$ $\forall i \in \{1,\ldots,m\}$, $\forall j \in \{1,\ldots,n\}$. By Theorem 3.1., 3.2. of [8], we get $x_{a_c} \in var(\widehat{\sigma}_{u,H}[t])$, $x_{b_d} \in var(\widehat{\sigma}_{u,H}[F])$ such that $x_{a_c} \in var(w_v)$, where $w_v \in W_{(m)}(X_m)\backslash X_m$ for some $v \in \{1,\ldots,m\}$ and $x_{b_d} \in var(z_y)$, where $z_y \in W_{(n)}(X_n)\backslash X_n$ for some $y \in \{1,\ldots,n\}$. Hence, $\sigma_{u,H} \circ_r \sigma_{t,F} \notin R'_X \cup R_T$. So, $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not a unit-regular element in *Relhyp*$((m),(n))$. □

Theorem 3.5. $(MUR)'_{(i,j)}$ *is a unit-regular submonoid of Relhyp*$((m),(n))$.

Proof. We get that every element in $(MUR)'$ is unit-regular. Next, we show that $(MUR)'_{(i,j)} := R^*_{x_i} \cup R^{**}_X \cup R^*_{t_j}$ is closed under $\circ_r$. Let $\sigma_{t,F}, \sigma_{u,H} \in (MUR)'_{(i,j)}$.

Case 1: $\sigma_{t,F} \in R^*_{x_i}$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1,\ldots,s_n)$ with $var(F) = \{x_{b_1},\ldots,x_{b_l}\}$ such that $i - most(s_{b'_k}) = x_{b_k}$ $\forall$ $k = 1,\ldots,l$.

Case 1.1: $\sigma_{u,H} \in R^*_{x_i}$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1,\ldots,h_n)$ with $var(H) = \{x_{d_1},\ldots,x_{d_q}\}$ such that $i - most(h_{d'_k}) = x_{d_k}$ for all $k = 1,\ldots,q$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^n_n(F,\widehat{\sigma}_{x_i,F}[h_1],\ldots,\widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S^n_n(s_1, i - most(h_1),\ldots, i - most(h_n)),\ldots, \\ &S^n_n(s_n, i - most(h_1),\ldots, i - most(h_n))) \\ &= \gamma(s'_1,\ldots,s'_n) \text{ where } var(\gamma(s'_1,\ldots,s'_n)) \subseteq \{x_{d_1},\ldots,x_{d_q}\} \\ &\text{such that } i - most(s'_{d'_k}) = x_{d_k};, k = 1,\ldots,q.\end{aligned}$$

Case 1.2: $\sigma_{u,H} \in R^{**}_X$. Then, $u = x_j \in X_m, H = \gamma(h_1,\ldots,h_n)$ with

$$|var(H)| = 1.$$

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_i] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } |var(\gamma(s_1', \ldots, s_n'))| = 1.\end{aligned}$$

Case 1.3: $\sigma_{u,H} \in R^*_{t_j}$. Then, $u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W_{(m)}(X_m)\backslash X_m; \forall k \in \{1, \ldots, m\}\backslash\{j\}$, $h_k \in W_{(n)}(X_n)\backslash X_n$; $\forall k \in \{1, \ldots, n\}\backslash\{j\}$ with $vat(u) = var(H) = \{x_j\}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } var(\gamma(s_1', \ldots, s_n')) = \{x_j\}.\end{aligned}$$

Case 2: $\sigma_{t,F} \in R^{**}_X$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1, \ldots, s_n)$ with

$$|var(F)| = 1.$$

Case 2.1: $\sigma_{u,H} \in R^*_{x_i}$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ such that $i - most(h_{d_k'}) = x_{d_k}$ for all $k = 1, \ldots, q$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } |var(\gamma(s_1', \ldots, s_n'))| = 1.\end{aligned}$$

Case 2.2: $\sigma_{u,H} \in R^{**}_X$. Then, $u = x_j \in X_m$, $H = \gamma(h_1, \ldots, h_n)$ with

$$|var(H)| = 1.$$

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } |var(\gamma(s_1', \ldots, s_n'))| = 1.\end{aligned}$$

Case 2.3: $\sigma_{u,H} \in R^*_{t_j}$. Then, $u = f(u_1, \ldots, u_m)$, $H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W_{(m)}(X_m)\backslash X_m; \forall k \in \{1, \ldots, m\}\backslash\{j\}$, $h_k \in W_{(n)}(X_n)\backslash X_n$; $\forall k \in \{1, \ldots, n\}\backslash\{j\}$ with $vat(u) = var(H) = \{x_j\}$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } var(\gamma(s'_1, \ldots, s'_n)) = \{x_j\}.\end{aligned}$$

Case 3: $\sigma_{t,F} \in R^*_{t_j}$. Then, $t = f(t_1, \ldots, t_m), F = \gamma(s_1, \ldots, s_n)$ where $t_j = x_j = s_j$ such that $t_k \in W_{(m)}(X_m)\backslash X_m$; $\forall k \in \{1, \ldots, m\}\backslash\{j\}$, $s_k \in W_{(n)}(X_n)\backslash X_n$; $\forall k \in \{1, \ldots, n\}\backslash\{j\}$ with $vat(t) = var(F) = \{x_j\}$.

Case 3.1: $\sigma_{u,H} \in R^*_{x_i}$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ such that $i - most(h_{d'_k}) = x_{d_k}$ for all $k = 1, \ldots, q$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } |var(\gamma(s'_1, \ldots, s'_n))| = 1.\end{aligned}$$

Case 3.2: $\sigma_{u,H} \in R^{**}_X$. Then, $u = x_j \in X_m, H = \gamma(h_1, \ldots, h_n)$ with

$$|var(H)| = 1.$$

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } |var(\gamma(s'_1, \ldots, s'_n))| = 1.\end{aligned}$$

Case 3.3: $\sigma_{u,H} \in R^*_{t_j}$. Then, $u = f(u_1, \ldots, u_m), H = \gamma(h_1, \ldots, h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W_{(m)}(X_m)\backslash X_m$; $\forall k \in \{1, \ldots, m\}\backslash\{j\}$, $h_k \in W_{(n)}(X_n)\backslash X_n$; $\forall k \in \{1, \ldots, n\}\backslash\{j\}$ with $vat(u) = var(H) = \{x_j\}$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(u_1, \ldots, u_m)] \\ &= S_m^m(t, \widehat{\sigma}_{t,F}[u_1], \ldots, \widehat{\sigma}_{t,F}[x_j], \ldots, \widehat{\sigma}_{t,F}[u_m]) \\ &= f(t_1, \ldots, x_j, \ldots, t_m) \\ &\quad \text{where } var(f(t_1, \ldots, x_j, \ldots, t_m)) = \{x_j\}, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{t,F}[\gamma(h_1, \ldots, h_n)] \\ &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[x_j], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s_1, \ldots, x_j, \ldots, s_n) \text{ where } var(\gamma(s_1, \ldots, x_j, \ldots, s_n)) = \{x_j\}.\end{aligned}$$

Therefore, $(MUR)'_{(i,j)}$ is a unit-regular submonoid of *Relhyp*$((m), (n))$. □

Theorem 3.6. $(MUR)''$ *is a unit-regular submonoid of Relhyp*$((m), (n))$.

Proof. We get that every element in $(MUR)''$ is unit-regular. Next, we show that $(MUR)'' := R^*_X \cup R^{**}_X \cup R^*_T$ is closed under $\circ_r$. Let $\sigma_{t,F}, \sigma_{u,H} \in (MUR)''$.

Case 1: $\sigma_{t,F} \in R^*_X$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1, \ldots, s_n)$ with $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$ and $|var(s_j)| = 1$ $\forall j = 1, \ldots, n$.

Case 1.1: $\sigma_{u,H} \in R_X^*$. Then, $u = x_j \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ and $|var(h_j)| = 1 \ \forall j = 1, \ldots, n$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_j] = x_j \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } var(\gamma(s_1', \ldots, s_n')) \subseteq \{x_{d_1}, \ldots, x_{d_q}\} \\ &\quad \text{such that } |var(s_j')| = 1 \ \forall \ j = 1, \ldots, n.\end{aligned}$$

Case 1.2: $\sigma_{u,H} \in R_X^{**}$. Then, $u = x_j \in X_m, H = \gamma(h_1, \ldots, h_n)$ with

$$|var(H)| = 1.$$

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } |var(\gamma(s_1', \ldots, s_n'))| = 1.\end{aligned}$$

Case 1.3: $\sigma_{u,H} \in R_T^*$. Then, $u = f(u_1, \ldots, u_m)$; $u_i \in X_m \ \forall i = 1, \ldots, m$ and $H = \gamma(h_1, \ldots, h_n)$; $h_i \in X_n \ \forall i = 1, \ldots, n$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = \widehat{\sigma}_{x_i,F}[u_i] = x_j \in X_m, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } var(\gamma(s_1', \ldots, s_n')) \subseteq var(H) \\ &\quad \text{such that } |var(s_j')| = 1 \ \forall \ j = 1, \ldots, n.\end{aligned}$$

Case 2: $\sigma_{t,F} \in R_X^{**}$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1, \ldots, s_n)$ with

$$|var(F)| = 1.$$

Case 2.1: $\sigma_{u,H} \in R_X^*$. Then, $u = x_j \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ and $|var(h_j)| = 1 \ \forall j = 1, \ldots, n$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s_1', \ldots, s_n') \text{ where } |var(\gamma(s_1', \ldots, s_n'))| = 1.\end{aligned}$$

Case 2.2: $\sigma_{u,H} \in R_X^{**}$. The proof is similar to case 2.2 of Theorem 3.5.

Case 2.3: $\sigma_{u,H} \in R_T^*$. Then, $u = f(u_1, \ldots, u_m)$; $u_i \in X_m$ $\forall i = 1, \ldots, m$ and $H = \gamma(h_1, \ldots, h_n)$; $h_i \in X_n$ $\forall i = 1, \ldots, n$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] = \widehat{\sigma}_{x_i,F}[u_i] = x_j \in X_m, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } |var(\gamma(s'_1, \ldots, s'_n))| = 1.\end{aligned}$$

Case 3: $\sigma_{t,F} \in R_T^*$. Then, $t = f(t_1, \ldots, t_m)$; $t_i \in X_m$ $\forall i = 1, \ldots, m$ and $F = \gamma(s_1, \ldots, s_n)$; $s_i \in X_n$ $\forall i = 1, \ldots, n$ with $var(t) = \{x_{a_1}, \ldots, x_{a_k}\}$, $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$.
Case 3.1: $\sigma_{u,H} \in R_X^*$. Then, $u = x_j \in X_m$ and $H = \gamma(h_1, \ldots, h_n)$ with $var(H) = \{x_{d_1}, \ldots, x_{d_q}\}$ and $|var(h_j)| = 1$ $\forall j = 1, \ldots, n$. Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_i] = x_i, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{x_i,F}[h_1], \ldots, \widehat{\sigma}_{x_i,F}[h_n]) \\ &= \gamma(S_n^n(s_1, i-most(h_1), \ldots, i-most(h_n)), \ldots, \\ &\quad S_n^n(s_n, i-most(h_1), \ldots, i-most(h_n))) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } var(\gamma(s'_1, \ldots, s'_n)) \subseteq var(H) \\ &\quad \text{such that } |var(s'_j)| = 1 \ \forall\ j = 1, \ldots, n.\end{aligned}$$

Case 3.2: $\sigma_{u,H} \in R_X^{**}$. Then, $u = x_j \in X_m$, $H = \gamma(h_1, \ldots, h_n)$ with

$$|var(H)| = 1.$$

Consider

$$(\sigma_{t,F} \circ_r \sigma_{u,H})(f) = \widehat{\sigma}_{t,F}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } |var(\gamma(s'_1, \ldots, s'_n))| = 1.\end{aligned}$$

Case 3.3: $\sigma_{u,H} \in R_T^*$. Then, $u = f(u_1, \ldots, u_m)$; $u_i \in X_m$ $\forall i = 1, \ldots, m$ and $H = \gamma(h_1, \ldots, h_n)$; $h_i \in X_n$ $\forall i = 1, \ldots, n$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{t,F}[f(u_1, \ldots, u_m)] \\ &= S_m^m(t, \widehat{\sigma}_{t,F}[u_1], \ldots, \widehat{\sigma}_{t,F}[u_m]) \\ &= f(t'_1, \ldots, t'_m) \\ &\quad \text{where } t'_i \in X_m,\ var(f(t'_1, \ldots, t'_m)) \subseteq var(u), \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= \widehat{\sigma}_{t,F}[\gamma(h_1, \ldots, h_n)] \\ &= R_n^n(F, \widehat{\sigma}_{t,F}[h_1], \ldots, \widehat{\sigma}_{t,F}[h_n]) \\ &= \gamma(s'_1, \ldots, s'_n) \text{ where } var(\gamma(s'_1, \ldots, s'_n)) \subseteq var(H).\end{aligned}$$

Therefore, $(MUR)''$ is a unit-regular submonoid of *Relhyp*$((m), (n))$. $\square$

Theorem 3.7. $(MUR)'_{(i,j)}$ *is a maximal unit-regular submonoid of* *Relhyp*$((m),(n))$.

Proof. Let K be a proper unit-regular submonoid of *Relhyp*$((m),(n))$ such that $(MUR)'_{(i,j)} \subseteq K \subset$ *Relhyp*$((m),(n))$. Let $\sigma_{t,F} \in K$. Then, $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_X \backslash R^*_{x_i} \cup R^{**}_X$. Then, $t = x_j \in X_m$ and $F = \gamma(s_1,\ldots,s_n)$ with $var(F) = \{x_{b_1},\ldots,x_{b_l}\}$ such that $j - most(s_{b'_k}) = x_{b_k}\ \forall\ k = 1,\ldots,l$. Choose $\sigma_{u,H} \in R^*_{x_i}$. Then, $u = x_i \in X_m$ and $H = \gamma(h_1,\ldots,h_n)$ with $var(H) = \{x_{d_1},\ldots,x_{d_r}\}$
such that $i - most(h_{d'_p}) = x_{d_p}\ \forall\ p = 1,\ldots,r$. Consider

$$(\sigma_{u,H} \circ_r \sigma_{t,F})(f) = \widehat{\sigma}_{x_i,H}[x_j] = x_j, \text{ and}$$

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(\gamma) &= R^n_n(H, \widehat{\sigma}_{x_i,H}[s_1],\ldots,\widehat{\sigma}_{x_i,H}[s_n]) \\ &= \gamma(S^n_n(h_1, i - most(s_1),\ldots,i - most(s_n)),\ldots, \\ &\quad S^n_n(h_n, i - most(s_1),\ldots,i - most(s_n))).\end{aligned}$$

Since $i - most(h_{d'_p}) = x_{d_p}$ for all $p = 1,\ldots,r$, we have

$$\begin{aligned}x_{d_p} &= i - most(h_{d'_p}) \\ &= S^n_n(i - most(h_{d'_p}), i - most(s_1),\ldots,i - most(s_n)) \\ &= S^n_n(x_{d_p}, i - most(s_1),\ldots,i - most(s_n)) \\ &= i - most(s_{d_p}).\end{aligned}$$

Since $u \neq x_j$, by Lemma 2.4., we have $j - most(\widehat{\sigma}_{u,H}[s_{d_p}]) \neq i - most(s_{d_p}) = x_{d_p}$. Thus, $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Case 2: $\sigma_{t,F} \in R_T \backslash R^*_{t_j}$. Then, $t = f(t_1,\ldots,t_m)$ and $F = \gamma(s_1,\ldots,s_n)$ such that $var(f(t_1,\ldots,t_m)) \subseteq \{t_1,\ldots,t_m\}$ and $var(\gamma(s_1,\ldots,s_n)) \subseteq \{s_1,\ldots,s_n\}$. Choose $\sigma_{u,H} \in R^*_{t_j}$. Then, $u = f(u_1,\ldots,u_m), H = \gamma(h_1,\ldots,h_n)$ where $u_j = x_j = h_j$ such that $u_k \in W_{(m)}(X_m)\backslash X_m;\ \forall k \in \{1,\ldots,m\}\backslash\{j\},\ h_k \in W_{(n)}(X_n)\backslash X_n;$ $\forall k \in \{1,\ldots,n\}\backslash\{j\}$ with $vat(u) = var(H) = \{x_j\}$. Consider

$$\begin{aligned}(\sigma_{u,H} \circ_r \sigma_{t,F})(f) &= S^m_m(u, \widehat{\sigma}_{u,H}[t_1],\ldots,\widehat{\sigma}_{u,H}[t_m]) \\ &= f(u'_1,\ldots,u'_m) \text{ where } |var(\gamma(u'_1,\ldots,u'_m))| = 1\end{aligned}$$

If $t_j \in W_{(m)}(X_m)\backslash X_m$, then $u'_k \in W_{(m)}(X_m)\backslash X_m$ for all $k = \{1,\ldots,m\}$.
If $t_j \in X_m\backslash\{x_j\}$, then $u'_j \neq x_j$. By Example 4 of [9], we have that it is not closed into itself. Thus, $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Thus, $\sigma_{t,F} \in (MUR)'_{(i,j)}$. Therefore, $K \subseteq (MUR)'_{(i,j)}$, and thus

$$K = (MUR)'_{(i,j)}.$$

□

Theorem 3.8. $(MUR)''$ *is a maximal unit-regular submonoid of*

Relhyp$((m),(n))$.

Proof. Let K be a proper unit-regular submonoid of *Relhyp*$((m),(n))$ such that $(MUR)'' \subseteq K \subset$ *Relhyp*$((m),(n))$. Let $\sigma_{t,F} \in K$. Then, $\sigma_{t,F}$ is unit-regular.

Case 1: $\sigma_{t,F} \in R'_X \backslash R^*_X \cup R^{**}_X$. Then, $t = x_i \in X_m$ and $F = \gamma(s_1,\ldots,s_n)$

with $var(F) = \{x_{b_1}, \ldots, x_{b_l}\}$ such that $i - most(s_{b'_k}) = x_{b_k}$ $\forall k = 1, \ldots, l$. If $|var(F)| = 2$, choose $\sigma_{u,H} \in R^*_T$, such that $i - most(u) = x_j$ with $j - most(s_l) = x_{b_k}$ $\forall\, l \in \{1, \ldots, n\}$ and $H = \gamma(x_1, \ldots, x_n)\}$. If $|var(F)| > 2$, choose $\sigma_{u,H} \in R^*_T$, such that $i - most(u) = x_j$ with $j - most(s_l) = x_{b_k}$ $\exists\, l \in \{1, \ldots, n\}$ and $H = \gamma(x_1, \ldots, x_n)$. Consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(f) &= \widehat{\sigma}_{x_i,F}[f(u_1, \ldots, u_m)] \\ &= x_j, \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{u,H})(\gamma) &= R^n_n(F, \widehat{\sigma}_{t,F}[x_1], \ldots, \widehat{\sigma}_{t,F}[x_n]) \\ &= \gamma(s'_1, \ldots, s'_n).\end{aligned}$$

Since $i - most(u) = x_j$, so $|var(j - most(s'_l)); \forall\, l = 1, \ldots, n| = 1 < |var(\gamma(s'_1, \ldots, s'_n))|$. Thus, $\sigma_{t,F} \circ_r \sigma_{u,H}$ is not unit-regular.

Case 2: $\sigma_{t,F} \in R_T \backslash R^*_T$. By Lemma 3.4., we can choose $\sigma_{u,H} \in R^*_T$ such that $\sigma_{u,H} \circ_r \sigma_{t,F}$ is not unit-regular.

Thus, $\sigma_{t,F} \in (MUR)''$. Therefore, $K \subseteq (MUR)''$, and thus $K = (MUR)''$. □

Corollary 3.9. *$(MUR)'_{(i,j)}, (MUR)''$ are maximal factorisable submonoids of the monoid of the relational hypersubstitutions for algebraic systems of type $((m), (n))$.*

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