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ON COMPLEX TRINOMIAL ROOTS DISTRIBUTION

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Abstract. The paper deals with a certain trinomial with two strictly complex coefficients. The root locus technique is utilized to obtain a distribution of roots with respect to a unit circle in the complex plane. A number of roots inside the unit disk is described in a relation with the two parameter values. Several appropriate graphs illustrate the trinomial roots distribution in particular cases.

1. Introduction

We consider a trinomial with two complex coefficients

$$T_{k,m}(\lambda) = \lambda^k + \mathrm{i}a\lambda^{k-m} + \mathrm{i}b, \tag{1.1}$$

where a, b are real parameters and $k > m$, $k, m \in \mathbb{N}$. Our aim is to analyze the dependence of the number of roots of (1.1) with a modulus lower than one, equal to one and greater than one on the pair of parameters (a, b). We call these numbers *root location numbers*, and we denote them as r_{in}, r_{on} and r_{out}, respectively. This notation reflects the trinomial $T_{k,m}$ roots position with respect to a unit disk in the complex plane. We also introduce a pair of non-negative integers $r_{in} - r_{out}$ for the root location description with respect to the pair of parameters (a, b). This efficient notation modifies the one used in [2], where a root location of a trinomial $P(\lambda) = \lambda^k + a\lambda^{k-m} + b$ with two strictly real coefficients was analyzed. We emphasize that the trinomial (1.1) has strictly complex coefficients. The trinomial $T_{k,m}(\lambda)$ can be considered as a characteristic polynomial of the difference equation

$$y(n+k) + \mathrm{i}ay(n+k-m) + \mathrm{i}by(n) = 0, \quad n = 0, 1, 2, \dots .$$

If all the roots of the characteristic polynomial lie inside a unit disk in the complex plane, then asymptotic stable solutions of this difference equation occur. We analyze a root distribution of (1.1) in a more general sense in the paper, not only as an investigation of the asymptotic stability of the corresponding difference equation.

The considered trinomial (1.1), whose unimodal roots we are going to investigate, is a special case of polynomial

$$P(\lambda) = \lambda^k - a\lambda^{k-m} - b \tag{1.2}$$

with general complex coefficients $a, b \in \mathbb{C}$. The conditions laid on coefficients a, b, which guarantee that the polynomial (1.2) has all its roots with the modulus lower

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than one, are equivalent to the asymptotic stability conditions for the difference equation

$$y(n+k) + ay(n+k-m) + by(n) = 0, \quad n = 0, 1, 2, \dots .$$

Many papers dealing with the asymptotic stability of difference equations analyze various special cases of the polynomial (1.2). One of the first papers dealing with this issue was [12], where the asymptotic stability of difference equations was studied. The necessary and sufficient conditions were derived for the polynomial $P(\lambda) = \lambda^k - \lambda^{k-1} + b$, $b \in \mathbb{R}$ to have all its roots inside the unit disk. An analysis of the roots distribution with respect to the unit circle in the complex plane is performed here using the so-called boundary locus technique. Such analysis procedure was also utilized in other later works (e.g., [10]). The result in [12] was later generalized in [11], where the polynomial $P(\lambda) = \lambda^k - a\lambda^{k-1} + b$, $a, b \in \mathbb{R}$ was studied. Next particular generalization came in [8], where the roots of trinomial $P(\lambda) = \lambda^k - \lambda^{k-m} + b$, $b \in \mathbb{R}$ were investigated. Polynomial $P(\lambda) = \lambda^k + a\lambda^{k-m} + b$, $a, b \in \mathbb{R}$ was analyzed in [6] in a similar sense. The necessary and sufficient conditions for polynomial $P(\lambda) = \lambda^k - a\lambda + b$, $a, b \in \mathbb{R}$ to have all the roots inside the unit disk were derived in [7]. Paper [13] deals with the stability of solutions of a difference system. The necessary and sufficient conditions are introduced for the roots of polynomial $P(\lambda) = \lambda^k - a\lambda^{k-1} + \mathrm{i}b$, $a, b \in \mathbb{R}$ to be inside the unit disk in the complex plane.

The necessary and sufficient conditions were formulated in the above-mentioned papers in such a form where there is a need for solving a nonlinear equation to determine the boundary of a region in the (a, b) plane within which, for all pairs (a, b), the roots of the investigated polynomial lie inside the unit disk in the complex plane. An analytical solution can be found only in very simple particular cases. In general, it can only be solved numerically. Other proof procedures were derived later. They lead to other formulations of these conditions. We mention paper [9] dealing with the roots of $P(\lambda) = \lambda^k - a\lambda^{k-1} - b$, $a, b \in \mathbb{C}$, where the conditions were formulated in a different sense. Similar approaches were also considered in [3–5].

Another proof procedure was considered in [10], where the polynomial $P(\lambda) = \lambda^k - a\lambda^{k-m} - b$, $a, b \in \mathbb{R}$ was analyzed. In that paper the conditions were formulated via curves which form the boundary of a region in the (a, b) plane, where all the roots of the polynomial lie inside the unit disk in the complex plane. The curves are given in a parametric form. We emphasize that condition formulations are significantly different in papers [9], [10] and [11]. Moreover, the proof of their equivalence is a very difficult task. We mention a recent paper [2], where a connection of the main results introduced in [10] and [5] was discussed. Finally, we mention a remarkable paper [1], which recalls a result related to the roots distribution of (1.2) with respect to unit modulus from 1908.

2. Roots distribution

In this section, we determine the boundaries of regions in the (a, b) plane, in which the root location numbers r_{in} and r_{out} are preserved ($r_{on} = 0$). The boundary curves are determined by the existence of unimodal roots ($r_{on} > 0$) with respect to a continuous dependence of the roots of the trinomial (1.1) on parameters a, b.

Theorem 2.1. *Let $a, b \in \mathbb{R}$, $k, m \in \mathbb{N}$, $k > m$ be coprime. Then, the trinomial (1.1) has a unimodal root if and only if*

$$a = \frac{\cos(k\omega)}{\sin((k-m)\omega)}, \qquad b = -\frac{\cos(m\omega)}{\sin((k-m)\omega)}, \qquad \omega \in I_s, \tag{2.1}$$

$$I_s := \left(\frac{(s-1)\pi}{k-m}, \frac{s\pi}{k-m}\right), \qquad s = m-k+1, m-k+2, \dots, k-m,$$

and, in the case of both k and m odd, the following holds

$$b = a(-1)^{1+(k-m)/2} - 1 \quad \text{or} \quad b = a(-1)^{1+(k-m)/2} + 1. \tag{2.2}$$

Proof. We consider a root of (1.1) with a unit modulus $\lambda^* = \exp\{\omega \mathrm{i}\}$, where $\omega \in (-\pi, \pi]$. Substitution of such a root to (1.1) gives

$$T_{k,m}(\mathrm{e}^{\omega \mathrm{i}}) = \mathrm{e}^{k\omega \mathrm{i}} + \mathrm{i}a\mathrm{e}^{(k-m)\omega \mathrm{i}} + \mathrm{i}b = 0.$$

We use the Euler formula and we obtain

$$\cos(k\omega) - a\sin((k-m)\omega) = 0, \tag{2.3}$$

$$\sin(k\omega) + a\cos((k-m)\omega) + b = 0 \tag{2.4}$$

considering the real and the imaginary parts separately. The above system of equations defines a set of curves in the (a, b) plane. When a point (a, b) of these curves is considered, a unimodal root of (1.1) occurs. The curves also comprise the boundaries of regions with a various number of roots inside and outside of the unit disk in the complex plane. To get a more convenient formulation of these curves we consider

$$\omega \in \bigcup_{s=m-k+1}^{k-m} I_s \quad \text{and} \quad \omega \in \left\{\frac{s\pi}{k-m}\right\}_{s=m-k+1}^{k-m}.$$

In the first case, we divide (2.3) by $\sin((k-m)\omega)$. Together with (2.4) it gives (2.1). However, in the case of $\omega = s\pi/(k-m)$, it is impossible. For these values of ω we must take into account the parity of exponents k and m. We omit the case of both k and m even, which is in contradiction to the assumption of their coprimality. It is obvious that we have $\sin((k-m)\omega) = 0$ and (2.3), (2.4) turn to $\cos(k\omega) = 0$ and $\cos(m\omega)$=0. Thus, $k\omega = \pi/2 + p\pi$, $m\omega = \pi/2 + q\pi$, $p, q \in \mathbb{Z}$. Since $\omega = (\pi/2+p\pi)/k = (\pi/2+q\pi)/m$, we get equation $(k-m)\pi/2 = (mp-kq)\pi$. This can be satisfied only for $(k-m)$ even. Thus, both $kandm$ must be odd. The only two possibilities for (2.3) and (2.4) to be simultaneously satisfied is in the case of $\omega = \pm\pi/2$. Then, (2.3) is fulfilled trivially and (2.4) defines the straight parallel lines (2.2). □

Notation. We denote by C_s the curves given by (2.1) for $\omega \in I_s$, $s = m-k+1$, $m-k+2, \dots, k-m$. Furthermore, we denote by C_- and C_+ the straight lines given by (2.2) for $\omega = -\pi/2$ and $\omega = \pi/2$, respectively.

Remark 2.2. The straight lines given by (2.2) can occur only in the case of k odd and $(k-m)$ even. Then, there exist curves given by (2.1), whose endpoints

$$[a,b] = \left[\frac{k}{k-m}(-1)^{(k-m)/2}\sin(k\pi/2), \frac{-m}{k-m}(-1)^{(k-m)/2}\sin(m\pi/2)\right],$$
$$[a,b] = \left[\frac{-k}{k-m}(-1)^{(k-m)/2}\sin(k\pi/2), \frac{m}{k-m}(-1)^{(k-m)/2}\sin(m\pi/2)\right]$$

are located on the straight lines (2.2). Both the points arise from finite limits for the pair (a,b) given by (2.1) as $\omega \to -\pi/2$ and as $\omega \to \pi/2$, respectively. Moreover, in this case, two unimodal roots are located on curves (2.1) since they are passed twice along the considered $\omega \in (-\pi, \pi]$.

Remark 2.3. The system of curves C_s, $s = m-k+1, m-k+2, \dots, k-m$ (together with C_- and C_+ in the case of k, m odd) is point symmetric with the center of symmetry in the point $[a,b] = [0,0]$. This follows immediately from (2.1), since expressions a, b are odd functions of parameter ω. It is obvious that parallel straight lines (2.2) have the same property, too.

The same line of arguments as in the proof of Theorem 2.1 gives the next assertion about the existence of roots with a general modulus $r > 0$.

Theorem 2.4. *Let $a, b \in \mathbb{R}$, $k, m \in \mathbb{N}$, $k > m$ be coprime. Then, the trinomial* (1.1) *has a root with modulus $r > 0$ if and only if*

$$a = r^m \frac{\cos(k\omega)}{\sin((k-m)\omega)}, \qquad b = -r^k \frac{\cos(m\omega)}{\sin((k-m)\omega)}, \qquad \omega \in I_s, \tag{2.5}$$
$$I_s := \left(\frac{(s-1)\pi}{k-m}, \frac{s\pi}{k-m}\right), \qquad s = m-k+1, m-k+2, \dots, k-m,$$

and, in the case of both k and m odd, the following holds

$$b = ar^{k-m}(-1)^{1+(k-m)/2} - r^k \quad \text{or} \quad b = ar^{k-m}(-1)^{1+(k-m)/2} + r^k.$$

Next, we present another assertion which formulates the location of unimodal roots of $T_{k,m}(\lambda)$ in the (a,b) plane.

Theorem 2.5. *Let k, m be positive integers such that $k > m$. If λ^* is a root of* (1.1) *with $|\lambda^*| = 1$, then either* (2.2) *holds or*

$$|a| + |b| > 1, \quad |a| - 1 < |b| < 1 + |a|,$$

$k \arcsin \frac{b^2-a^2-1}{2|a|} + m \arcsin \frac{a^2-b^2-1}{2|b|} = s\pi$ *for suitable $s \in \mathbb{Z}$.*

Proof. Let polynomial $T_{k,m}$ have a unimodal root $\lambda = \exp(\omega \mathrm{i})$, $\omega \in (-\pi, \pi]$. We consider parameters a, b of (1.1) as $a = |a|\exp(\mathrm{i}\theta_a)$ and $b = |b|\exp(\mathrm{i}\theta_b)$, where $\theta_a, \theta_b \in \{0, \pi\}$. Then, we obtain the trinomial (1.1) in the form

$$\exp(\mathrm{i}k\omega) + \mathrm{i}|a|\exp(\mathrm{i}(k-m)\omega + \mathrm{i}\theta_a) + \mathrm{i}|b|\exp(\mathrm{i}\theta_b) = 0.$$

Considering the real and the imaginary parts separately, we get

$$\cos(k\omega) - |a|\sin((k-m)\omega + \theta_a) - |b|\sin(\theta_b) = 0, \tag{2.6}$$
$$\sin(k\omega) + |a|\cos((k-m)\omega + \theta_a) + |b|\cos(\theta_b) = 0. \tag{2.7}$$

Although the term $\sin(\theta_b)$ in (2.6) has zero value, it will be useful to be considered in this form also in the following steps. We rewrite (2.6), (2.7) in an equivalent sense to

$$|a|\sin((k-m)\omega+\theta_a-\theta_b)=\cos(k\omega-\theta_b),$$
$$|b|\sin((k-m)\omega+\theta_a-\theta_b)=-\cos(m\omega-\theta_a).$$

The sign of the term $\sin((k-m)\omega+\theta_a-\theta_b)$ splits the proof into the next three possibilities.

I) Let $\sin((k-m)\omega+\theta_a-\theta_b)>0$. Then, $\cos(k\omega-\theta_b)>0$ and $\cos(m\omega-\theta_a)<0$. But this is satisfied if and only if

$$k\omega-\theta_b\in\left(-\frac{\pi}{2}+p\pi,\frac{\pi}{2}+p\pi\right),\qquad m\omega-\theta_a\in\left(-\frac{\pi}{2}+q\pi,\frac{\pi}{2}+q\pi\right)\tag{2.8}$$

for a suitable p even and q odd. Now, we rearrange the system (2.6), (2.7) in two different ways:

a) We isolate the term $|a|\sin((k-m)\omega+\theta_a)$ from the first equation and the term $|a|\cos((k-m)\omega+\theta_a)$ from the second one. Then, we square both the equations and sum them to obtain

$$\sin(k\omega-\theta_b)=\frac{a^2-b^2-1}{2|b|}.\tag{2.9}$$

b) We isolate the term $|b|\sin(\theta_b)$ from the first equation and the term $|b|\cos(\theta_b)$ from the second one. Then, we square both the equations and sum them to obtain

$$\sin(m\omega-\theta_a)=\frac{b^2-a^2-1}{2|a|}.\tag{2.10}$$

Equations (2.9) and (2.10), using (2.8), give

$$\sin(k\omega-\theta_b-p\pi)=\frac{a^2-b^2-1}{2|b|},\qquad \sin(q\pi-(m\omega-\theta_a))=\frac{b^2-a^2-1}{2|a|}.$$

Applying the appropriate inverse functions, we get

$$k\omega-\theta_b-p\pi=\arcsin\frac{a^2-b^2-1}{2|b|},\qquad q\pi-m\omega+\theta_a=\arcsin\frac{b^2-a^2-1}{2|a|}.$$

Elimination of the parameter ω gives

$$k\arcsin\frac{b^2-a^2-1}{2|a|}+m\arcsin\frac{a^2-b^2-1}{2|b|}=L,$$

where $L=(kq-mp)\pi+k\theta_a-m\theta_b$. Since there exists $\ell\in\mathbb{Z}$ such that $k\theta_a-m\theta_b=\ell\pi$, we can write $L=(kq-mp+\ell)\pi$. Thus, $L=s\pi$ for a suitable $s=kq-mp+\ell$, $s\in\mathbb{Z}$.

II) Let $\sin((k-m)\omega+\theta_a-\theta_b)<0$. Then, an analogous procedure as in the previous case gives the theorem assertion.

III) Let $\sin((k-m)\omega+\theta_a-\theta_b)=0$. Then, an analogous procedure as in the proof of Theorem 2.1 gives (2.2). $\square$

So far, we have made some considerations about the location of unimodal roots. Now, we turn to an analysis to determine the number of roots with a modulus lower and greater than one. I.e., we try to analyze how the numbers of roots r_{in} and

r_{out} are set for various regions bounded by curves C_s, and also by C_- and C_+ in the case of k, m odd. We start with the obvious fact that the area in the (a, b) plane, which contains the open vertical line segment $\overline{AB}$, $A = [0, -1]$, $B = [0, 1]$, determines that all the roots of $T_{k,m}$ are within the unit disk in the complex plane ($r_{in} = k$, $r_{out} = 0$). This follows immediately from $T_{k,m} = \lambda^k + \mathrm{i}b$ considering $a = 0$ and $b \in (-1, 1)$. On the other hand, the regions in the (a, b) plane, which contain vertical half-lines $a = 0$, $b \in (-\infty, -1)$ and $a = 0$, $b \in (1, \infty)$, determine the case when all the roots of $T_{k,m}$ have the modulus greater than one ($r_{in} = 0$, $r_{out} = k$) for an analogous reason. Similarly, in the horizontal direction we have the following roots distribution: the region which contains the open line segment $\overline{DE}$, $D = [-1, 0]$, $E = [1, 0]$, determines that all the roots of $T_{k,m}$ are within the unit disk in the complex plane ($r_{in} = k$, $r_{out} = 0$). This follows immediately from $T_{k,m} = (\lambda^m + ai)\lambda^{k-m}$ considering $a \in (-1, 1)$ and $b = 0$. On the other hand, the regions in the (a, b) plane which contain half-lines $a \in (-\infty, -1)$, $b = 0$ and $a \in (1, \infty)$, $b = 0$ determine the case of $T_{k,m}$ with root location numbers $r_{in} = k - m$, $r_{out} = m$ for an analogous reason. Thus, we have information about these four important particular cases. To determine the settings of the pair r_{in}, r_{out} in other regions, we can utilize Theorem 2.4. If we consider a small perturbation δ of the unit modulus $r = 1 + \delta$, $\delta > 0$, we find a direction where, originally, the unit modulus increases its value (we can consider similar procedure also for $\delta < 0$). In coincidence with (2.5), one can say that for a perturbation $\delta > 0$ the considered direction points away from the plane origin in the case of curves C_s. An appropriate change of the position of the straight lines C_- and C_+ is also illustrated in the case of $T_{3,1}$ in Example 3.2. The greater the values of coprimes k, m, the more difficult the analysis is and an analytical description of each region remains to be a great challenge. Nevertheless, plotting the curves C_s, C_- and C^+ for a particular case of k, m, we can determine the regions via the theorems introduced above. Much more about this issue becomes clear after its demonstration on particular examples in the following section.

3. Examples

In this section, we present particular cases of the trinomial $T_{k,m}$ and we describe its unimodal roots dislocation in the parameter plane (a, b). The curves C_s and occasionally parallel straight lines form the boundaries of regions with constant numbers of roots inside or outside of the unit disk. These curves are shown as solid bold lines in all the figures presented below. In addition, there are also shown dotted curves, which represent pairs (a, b) when the perturbed root λ_p^*, $|\lambda_p^*| = 1.05$ of $T_{k,m}$ exists (unit modulus perturbation $\delta = 0.05$). It shows in which direction from the solid curves r_{out} increases and r_{in} decreases in accordance with the continuous dependency of roots on the parameter pairs (a, b). All the figures also document the symmetry property mentioned in Remark 2.3.

Example 3.1. We consider the trinomial

$$T_{2,1}(\lambda) = \lambda^2 + \mathrm{i}a\lambda + \mathrm{i}b.$$

Distribution of the roots of $T_{2,1}$ with respect to modulus one in the parameter plane (a, b) is shown in Figure 1. Only two boundary curves C_0, C_1 defined by

(2.1) are present, since we have only $I_0 = (-\pi, 0)$ and $I_1 = (0, \pi)$ according to Theorem 2.1. They only split the whole plane into five distinct regions, which were the subjects of consideration at the end of the previous section. The pairs $r_{in} - r_{out}$ are specified for these regions. As we can see, the dotted lines for the perturbed unimodal roots with modulus $r = 1.05$ illustrate the direction from the solid line in which r_{out} is increased and r_{in} is decreased.

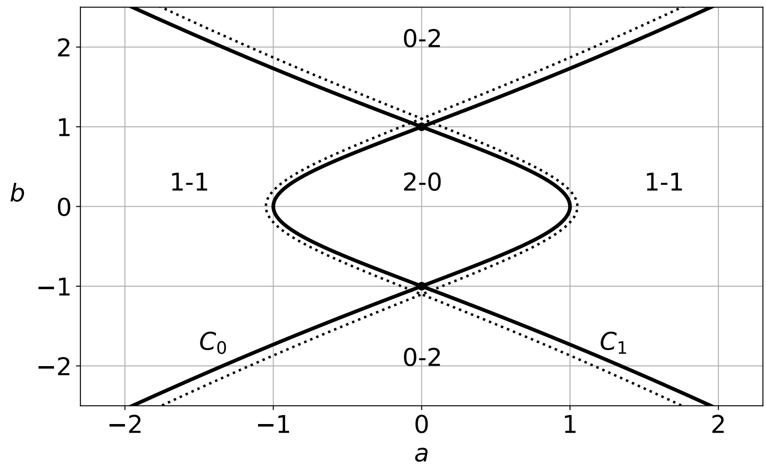

Figure 1. Roots distribution for $T_{2,1}(\lambda)$, $r_{in} - r_{out}$

Example 3.2. We consider the trinomial

$$T_{3,1}(\lambda) = \lambda^3 + \mathrm{i}a\lambda^2 + \mathrm{i}b.$$

Roots distribution in the parameter plane (a, b) is shown in Figure 2. The area where when all the roots are inside the unit disk is located around the origin of the plane. There are four curves C_{-1}, C_0, C_1, C_2 by (2.1) with parameter ω in $I_{-1} = (-\pi, -\pi/2)$, $I_0 = (-\pi/2, \pi)$, $I_1 = (\pi, 3\pi/2)$, $I_2 = (3\pi/2, \pi)$, respectively. As we can see, the curves C_{-1}, C_0 are the same and C_1, C_2 also comprise one curve. According to Remark 2.2 they have endpoints $[3/2, 1/2]$ and $[-3/2, -1/2]$ on appropriate straight lines C_- and C_+, respectively. Again, the dotted lines mean the analogous curves for roots with modulus $r = 1.05$.

Example 3.3. We consider the trinomial

$$T_{3,2}(\lambda) = \lambda^3 + \mathrm{i}a\lambda + \mathrm{i}b.$$

Roots distribution in the parameter plane (a, b) is shown in Figure 3. The area where all the roots are inside the unit disk is located around the origin of the plane. Only two boundary curves, C_0, C_1, defined by (2.1) are presented, since we have only $I_0 = (-\pi, 0)$ and $I_1 = (0, \pi)$ for ω. They split the whole plane into nine distinct regions in this case. We again introduce the dotted perturbation lines for the roots of $T_{3,2}$ with modulus $r = 1.05$ to document the appropriate changes in the pairs $r_{in} - r_{out}$.

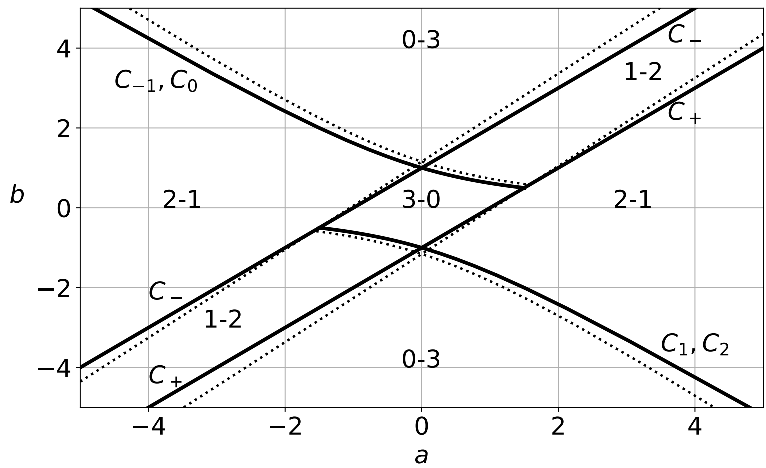

Figure 2. Roots distribution for $T_{3,1}(\lambda)$, $r_{in} - r_{out}$

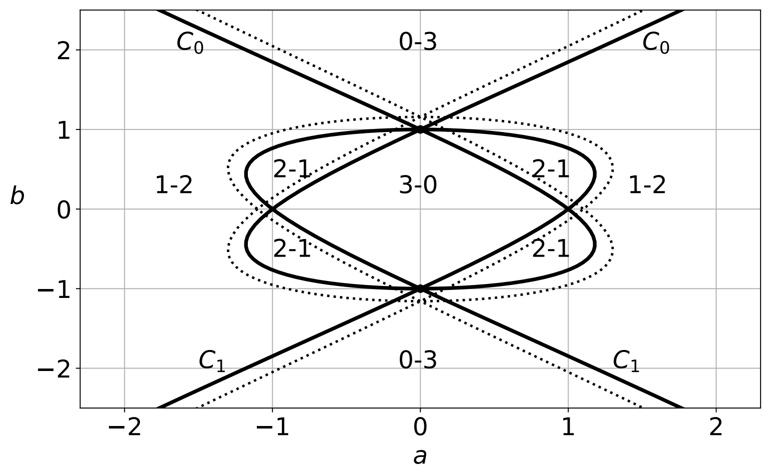

Figure 3. Roots distribution for $T_{3,2}(\lambda)$, $r_{in} - r_{out}$

Example 3.4. We consider the trinomial

$$T_{4,1}(\lambda) = \lambda^4 + \mathrm{i}a\lambda^3 + \mathrm{i}b.$$

Roots distribution in the parameter plane (a, b) is shown in Figure 4. Six boundary curves C_s, $s = -2, -1, \ldots, 3$, defined by (2.1) are presented, which split the plane into thirteen distinct regions. We introduce the dotted perturbation lines for roots of $T_{4,1}$ with modulus $r = 1.015$ to document the appropriate changes in the pairs $r_{in} - r_{out}$.

Example 3.5. We consider the trinomial

$$T_{4,3}(\lambda) = \lambda^4 + \mathrm{i}a\lambda + \mathrm{i}b.$$

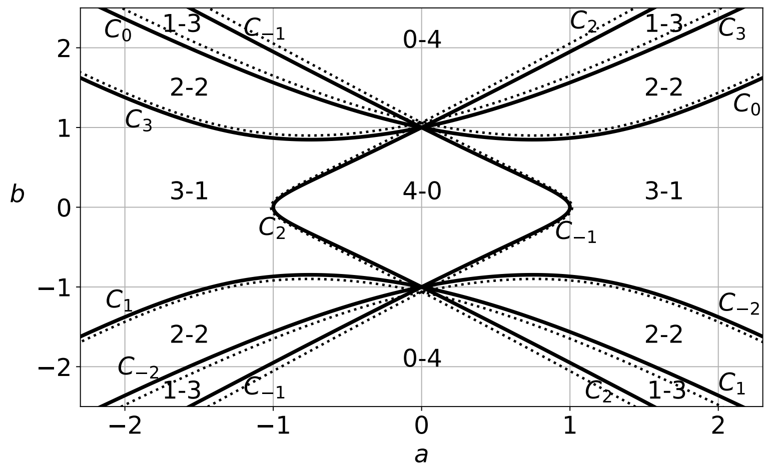

Figure 4. Roots distribution for $T_{4,1}(\lambda)$, $r_{in} - r_{out}$

Roots distribution in the plane (a, b) is shown in Figure 5. Two boundary curves, C_0, C_1, defined by (2.1) are presented. They split the plane into thirteen distinct regions. We again introduce the dotted perturbation lines for the roots of $T_{4,3}$ with modulus $r = 1.015$ to document the appropriate changes in the pairs $r_{in} - r_{out}$.

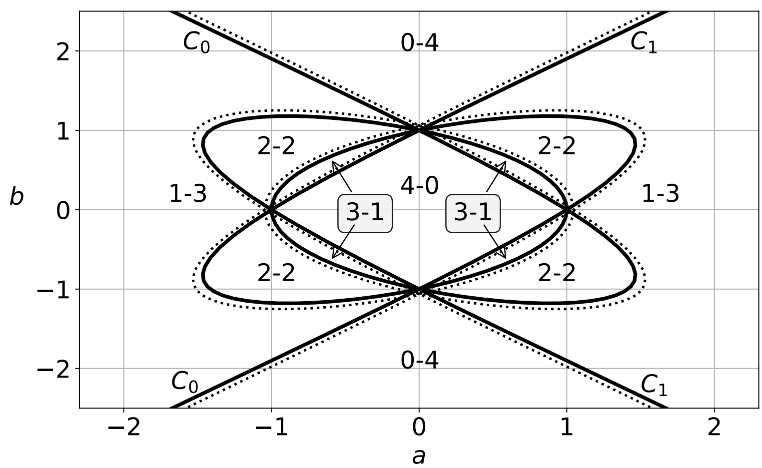

Figure 5. Roots distribution for $T_{4,3}(\lambda)$, $r_{in} - r_{out}$

4. Final remarks

The paper gives a tool for a location analysis of the roots of trinomial (1.1) with respect to the unit modulus. We derived a complete description of curves C_s, $s = m - k + 1, m - k + 2, \dots, k - m$ and C_-, C_+ in Theorem 2.1, which serve as the boundaries of regions where the root location numbers r_{in} and r_{out} for the trinomial $T_{k,m}$ are constant. We introduced an original concept of perturbed curves which correspond to roots with modulus $r = 1 \pm \delta$, $\delta > 0$ sufficiently small. They enable us to determine the change of the root location numbers if we cross the boundary curve from one region to a neighboring one. Moreover, Theorem 2.4 gives a generalized result for a possible analysis considering any positive modulus r of the trinomial (1.1) roots. The approach presented in the paper can be utilized for an analysis of other polynomials.

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