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TOPOLOGIES INDUCED BY GRAPH METRICS ON THE VERTEX SET OF GRAPHS

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Abstract. This paper presents a method of constructing topologies on the vertex set of a graph G induced by open balls with respect to the graph metric viz. geodesic distance, detour distance, circular distance and circular D-distance on the vertex set of G. Also, this paper explores the topologies induced by eccentric neighbourhoods of vertices of a graph and presents the nature of topologies generated by various graph metrics on the vertex set of some standard graphs.

1. Introduction

Graphs are used for modelling multiple relations and processes in computer, engineering, physical and biological sciences. The application of distance in graphs can be found in image processing, optimization, networking, pattern recognition and navigation. Distance metrics are a key part of several machine learning algorithms. A number of machine learning algorthims, supervised or unsupervised, use distance metrics to know the input data pattern in order to make any data based decision. These distance metrics are used in both supervised and unsupervised learning, generally to calculate the similarity between data points.

A metric space is a non-empty set together with a metric on the set. The metric is a function that defines the concept of distance between any two members of the set, which are usually called points. A metric on a space induces topological properties like open and closed sets which lead to the study of more abstract topological spaces. The metric is used to generate a subbasis for a topology, the metric topology. From this, all the usual objects in a topology are easily defined. The topologies generated by an effective distance metric can be used to improve the performance of a machine learning model. Diesto and Gervacio [5] constructed a topology on a vertex set of graphs using neighbourhoods. Also, it was further studied in [7] and [9]. Nianga and Canoy in [6,10,11] used the hop neighbourhoods to generate a topology on a vertex set of graphs and studied the properties of topologies induced by some unary, binary operations on graphs.

A biological system is a complex network which connects several biologically relevant entities. The myriad components of a biological system and their interactions are best characterized as networks and they are mainly represented as

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graphs. Many topological structures on the vertex set can be generated by different relations on the vertex set of graphs. The dependence scales of subgraphs computed from the topology generated by graphs are used to detect, diagnose and monitor diseases. Graph distance plays a vital role in the study of a graph model of blood circulation in the human heart and urinary system.

The significance of circular distance can be viewed in logistic management. For instance, a salesman who delivers goods from a source to various destinations, takes a long trip so that he can cover as many places as he can on his way. On his return, he selects the shortest way to his source so as to minimize the time and fuel consumption. In his journey from source to destination and from destination to source, he chooses the shortest path and the longest path.

In many real life situations, in addition to the length of the path, calculation of the degree of every vertex of a graph is of much importance. For instance, a van delivering goods to various places has to stop at each point and deliver the goods. Regarding the points as vertices and the number of goods delivered as the degree of vertices, it has to travel and deliver the goods at all delivery points, by taking it as a detour D-distance and return to source, by taking it as a D-distance. It helps in saving time and fuel usage.

In this direction, this paper explores some methods of generating topologies via graph metrics which ensure efficient decision making.

2. Preliminaries

The definitions stated in this section are defined by referring to the references [1–4].

In a graph, the degree of a vertex v denoted by $\deg(v)$ is defined as the number of vertices adjacent to v. A complete graph on n vertices denoted by K_n is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A path graph on n vertices denoted by P_n is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$, where $i = 1, 2, \ldots, n-1$. A cycle graph on n vertices denoted by C_n is a graph that consists of a single cycle. A wheel graph on n vertices denoted by $W_n, n \geqslant 4$ is a graph formed by connecting a single vertex to all vertices of a cycle of length $n-1$ and that single vertex is referred to as the centre vertex of W_n. A star graph $St_{1,n}$ is a special type of graph in which n vertices have degree 1 and a single vertex has degree n and that single vertex is referred to as the central vertex of $St_{1,n}$.

A path in a graph is a sequence of vertices in which each vertex is connected by an edge to the next. The path length corresponds to the number of edges in the path. The length of the shortest path between two vertices x and y is called the distance between x and y, which is denoted by $d(x, y)$. A metric space defined over a set of vertices in terms of distances in a graph defined over the vertex set is called a graph metric. The vertex set and the distance function form a metric space if and only if the graph is connected. In a connected graph G, the geodesic distance $d(a, b)$ between two distinct vertices a and b is defined as the length of the shortest path between a and b and zero, if $a = b$; the detour distance $D(a, b)$ between distinct vertices a and b is the length of the longest path between a and b and zero, if $a = b$. The maximum geodesic distance (detour distance) to any other vertex from the vertex a is defined as the eccentricity $e_d(a)$ ($e_D(a)$) of the vertex a. The geodesic radius $r_d(G)$ (detour radius $r_D(G)$) and geodesic diameter $\mathrm{diam}_d(G)$ (detour diameter $\mathrm{diam}_D(G)$) of a graph G is defined as

$$r_d(G) = \min\{e_d(a) : a \in V(G)\} \quad (r_D(G) = \min\{e_D(a) : a \in V(G)\}),$$
$$\mathrm{diam}_d(G) = \max\{e_d(a) : a \in V(G)\} \quad (\mathrm{diam}_D(G) = \max\{e_D(a) : a \in V(G)\}).$$

P. L. N. Varma *et al.* [12,13] introduced D-length of a path, D-distance between vertices, detour D-distance between vertices in a connected graph.

If P is a path connecting the vertices a and b, then the D-length of P is defined as $Dl(P) = l(P) + \deg(a) + \deg(b) + \sum_x \deg(x)$, where the summation is taken over all the internal vertices x of P; the geodesic D-distance between the two distinct vertices a and b is defined as $d^D(a,b) = \min\{Dl(P)\}$, where the minimum is taken over all paths P connecting a and b and zero, if $a = b$, detour D-distance between the two distinct vertices a and b is defined as $D^D(a,b) = \max\{Dl(P)\}$ where maximum is taken over all paths P connecting a and b and zero, if $a = b$.

Janagam Veeranjaneyulu and Peruri Lakshmi Narayana Varma [8] defined circular distance $cd(a,b)$ between two distinct vertices a and b in a connected graph as the sum of the geodesic distance $d(a,b)$ and the detour distance $D(a,b)$ and the circular D-distance $cd^D(a,b)$ as the sum of the geodesic D-distance between the vertices a and b and the detour D-distance between the vertices a and b. The maximum circular D-distance to any other vertex from the vertex a is defined as the circular D-eccentricity $e_c^D(a)$ of the vertex a. The circular D-radius $r_{cd}^D(G)$ and circular D-diameter $\mathrm{diam}_{cd}^D(G)$ of a graph G is defined as

$$r_{cd}^D(G) = \min\{e_c^D(a) : a \in V(G)\}, \quad \mathrm{diam}_{cd}^D(G) = \max\{e_c^D(a) : a \in V(G)\}.$$

Throughout this paper, graphs under discussion are connected nontrivial simple graphs.

3. Topologies induced by open balls

This section explores the method of generating topologies induced by open balls with respect to graph metrics viz. geodesic distance, detour distance, circular distance and D-circular distance.

From the definitions, it is easy to prove that geodesic distance $d(x,y)$, detour distance $D(x,y)$, circular distance $cd(x,y)$ and circular D-distance $cd^D(x,y)$ are metrics on $V(G)$. With respect to each of these metrics, open balls can be defined, through which the subbasis for topologies on the vertex set of G can be found.

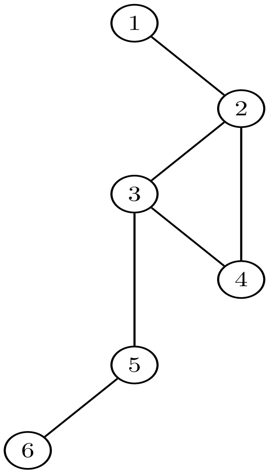
Let $dis(x,y)$ denote any of the above mentioned distance metric. Then, the open ball and the corresponding subbasis for topologies can be defined as follows:

Definition 3.1. In a graph G, the open balls with respect to metric *dis* are defined as

$$B_{dis}(x, r_{dis}(G)) = \{y \in V(G) : dis(x,y) < r_{dis}(G)\}.$$

The definition of $dis(x,y)$ imparts that $x \in B_{dis}(x, r_{dis}(G))$ for every $x \in V(G)$. Hence, the non empty collection of open balls $B_{dis}(x, r_{dis}(G))$ forms a subbasis for a topology τ_{dis} on the vertex set of G.

Example 3.2. Consider the following graph.



$$d(1,2) = d(2,1) = 1, \quad d(1,3) = d(3,1) = 2, \quad d(1,4) = d(4,1) = 2,$$
$$d(1,5) = d(5,1) = 3, \quad d(1,6) = d(6,1) = 4, \quad d(2,3) = d(3,2) = 1,$$
$$d(2,4) = d(4,2) = 1, \quad d(2,5) = d(5,2) = 2, \quad d(2,6) = d(6,2) = 3,$$
$$d(3,4) = d(4,3) = 1, \quad d(3,5) = d(5,3) = 1, \quad d(3,6) = d(6,3) = 2,$$
$$d(4,5) = d(5,4) = 2, \quad d(4,6) = d(6,4) = 3, \quad d(5,6) = d(6,5) = 1,$$
$$e_d(1) = 4,\ e_d(2) = 3,\ e_d(3) = 2,\ e_d(4) = 3,\ e_d(5) = 3,\ e_d(6) = 4,\ r_d(G) = 2,$$
$$B_d(1, r_d(G)) = \{1,2\}; \quad B_d(2, r_d(G)) = \{1,2,3,4\}, \quad B_d(3, r_d(G)) = \{2,3,4,5\},$$
$$B_d(4, r_d(G)) = \{2,3,4\}, \quad B_d(5, r_d(G)) = \{3,5,6\}, \quad B_d(6, r_d(G)) = \{5,6\}.$$

Figure 1 illustrates the subbasis for τ_d in which the vertices joined under differently colored lines form the elements of the subbasis for τ_d.

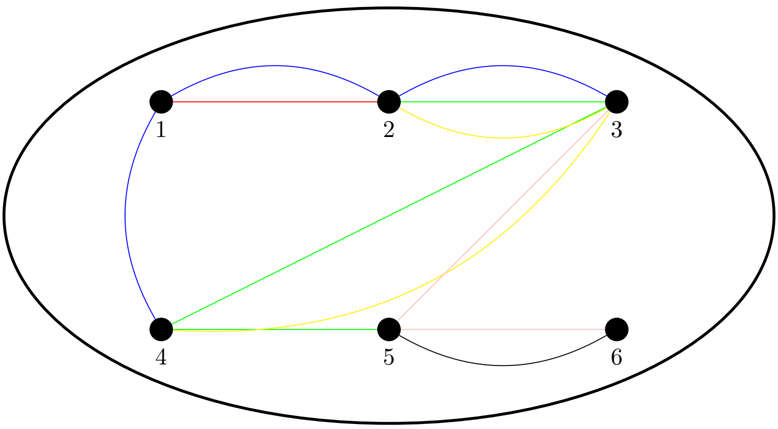


Figure 1

Subbasis for $\tau_d = \{\{1,2\}, \{1,2,3,4\}, \{2,3,4,5\}, \{2,3,4\}, \{3,5,6\}, \{5,6\}\}$. Basis for $\tau_d = \{\emptyset, \{1,2\}, \{1,2,3,4\}, \{2,3,4,5\}, \{2,3,4\}, \{3,5,6\}, \{5,6\}, \{2\}, \{3\}, \{3,5\}, \{5\}\}$, $\tau_d = \{\{\emptyset, \{1,2\}, \{1,2,3,4\}, \{2,3,4,5\}, \{2,3,4\}, \{3,5,6\}, \{5,6\}, \{2\}, \{3\}, \{3,5\}, \{5\},$

$\{1,2,3,4,5\},\{1,2,3,5,6\},\{1,2,5,6\},\{1,2,3,4,5,6\},\{2,3,4,5,6\}\}\}$,

$$\begin{array}{lll} D(1,2)=D(2,1)=1, & D(1,3)=D(3,1)=3, & D(1,4)=D(4,1)=3, \\ D(1,5)=D(5,1)=4, & D(1,6)=D(6,1)=5, & D(2,3)=D(3,2)=2, \\ D(2,4)=D(4,2)=2, & D(2,5)=D(5,2)=3, & D(2,6)=D(6,2)=4, \\ D(3,4)=D(4,3)=2, & D(3,5)=D(5,3)=1, & D(3,6)=D(6,3)=2, \\ D(4,5)=D(5,4)=3, & D(4,6)=D(6,4)=4, & D(5,6)=D(6,5)=1, \end{array}$$

$$e_D(1)=5,\ e_D(2)=4,\ e_D(3)=3,\ e_D(4)=4,\ e_D(5)=4, e^D(6)=5, r_D(G)=3,$$

$$B_D(1,r_D(G))=\{1,2\},\quad B_D(2,r_D(G))=\{1,2,3,4\},$$
$$B_D(3,r_D(G))=\{2,3,4,5,6\},\quad B_D(4,r_D(G))=\{2,3,4\},$$
$$B_D(5,r_D(G))=\{3,5,6\},\quad B_D(6,r_D(G))=\{3,5,6\}.$$

Figure 2 illustrates the subbasis for τ_D in which the vertices joined under differently colored lines form the elements of the subbasis for τ_D.

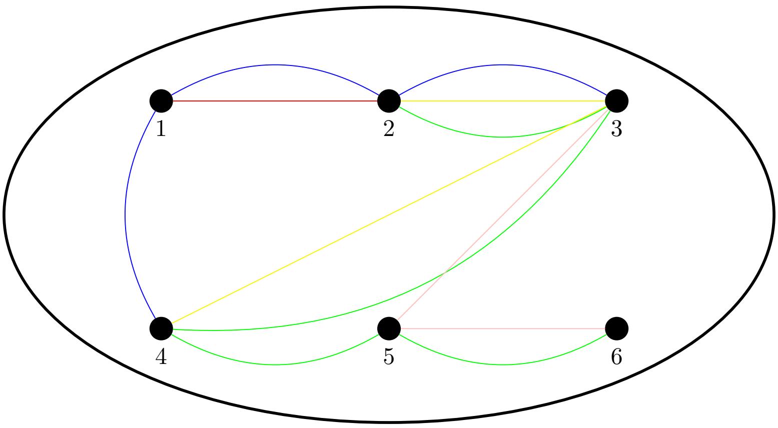


Figure 2

Subbasis for $\tau_D=\{\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{3,5,6\}\}$. Basis for $\tau_D=\{\emptyset,\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{2\},\{3\}\}$, $\tau_D=\{\emptyset,\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{2\},\{3\},\{2,3\},\{1,2,3,4,5,6\},\{2,3,5,6\}\}$,

$$\begin{array}{lll} cd(1,2)=cd(2,1)=2, & cd(1,3)=cd(3,1)=5, & cd(1,4)=cd(4,1)=5, \\ cd(1,5)=cd(5,1)=7, & cd(1,6)=cd(6,1)=9, & cd(2,3)=cd(3,2)=3, \\ cd(2,4)=cd(4,2)=3, & cd(2,5)=cd(5,2)=5, & cd(2,6)=cd(6,2)=7, \\ cd(3,4)=cd(4,3)=3, & cd(3,5)=cd(5,3)=2, & cd(3,6)=cd(6,3)=4, \\ cd(4,5)=cd(5,4)=5, & cd(4,6)=cd(6,4)=7, & cd(5,6)=cd(6,5)=2 \end{array}$$

$$e_{cd}(1)=9, e_{cd}(2)=7,\ e_{cd}(3)=5,\ e_{cd}(4)=7,\ e_{cd}(5)=7,\ e_{cd}(6)=9,\ r_{cd}(G)=5,$$

$$B_{cd}(1,r_{cd}(G))=\{1,2\},\quad B_{cd}(2,r_{cd}(G))=\{1,2,3,4\},$$
$$B_{cd}(3,r_{cd}(G))=\{2,3,4,5,6\},\quad B_{cd}(4,r_{cd}(G))=\{2,3,4\},$$
$$B_{cd}(5,r_{cd}(G))=\{3,5,6\},\quad B_{cd}(6,r_{cd}(G))=\{3,5,6\}.$$

Figure 3 illustrates the subbasis for τ_{cd} in which the vertices joined under differently colored lines form the elements of the subbasis for τ_{cd}.

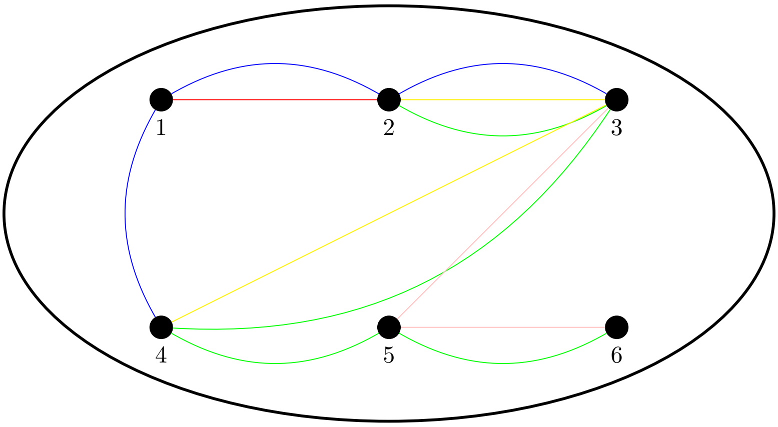


Figure 3

Subbasis for $\tau_{cd} = \{\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{3,5,6\}\}$.
Basis for $\tau_{cd} = \{\emptyset,\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{2\},\{3\}\}$,
$\tau_{cd} = \{\emptyset,\{1,2\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{3,5,6\},\{2\},\{3\},\{2,3\},\{1,2,3,4,5,6\},\{2,3,5,6\}\}$,

$$\begin{aligned}
&cd^D(1,2) = cd^D(2,1) = 10, \quad cd^D(1,3) = cd^D(3,1) = 21,\\
&cd^D(1,4) = cd^D(4,1) = 20, \quad cd^D(1,5) = cd^D(5,1) = 27,\\
&cd^D(1,6) = cd^D(6,1) = 31, \quad cd^D(2,3) = cd^D(3,2) = 17,\\
&cd^D(2,4) = cd^D(4,2) = 16, \quad cd^D(2,5) = cd^D(5,2) = 23,\\
&cd^D(2,6) = cd^D(6,2) = 27, \quad cd^D(3,4) = cd^D(4,3) = 16,\\
&cd^D(3,5) = cd^D(5,3) = 12, \quad cd^D(3,6) = cd^D(6,3) = 16,\\
&cd^D(4,5) = cd^D(5,4) = 22, \quad cd^D(4,6) = cd^D(6,4) = 26,\\
&cd^D(5,6) = cd^D(6,5) = 8,
\end{aligned}$$

$$\begin{aligned}
&e^D_{cd}(1) = 31,\ e^D_{cd}(2) = 27,\ e^D_{cd}(3) = 21,\ e^D_{cd}(4) = 26,\\
&e^D_{cd}(5) = 27,\ e^D_{cd}(6) = 31,\ r^D_{cd}(G) = 21,\\
&B^D_{cd}(1, r^D_{cd}(G)) = \{1,2,4\}, \quad B^D_{cd}(2, r^D_{cd}(G)) = \{1,2,3,4\},\\
&B^D_{cd}(3, r^D_{cd}(G)) = \{2,3,4,5,6\}, \quad B^D_{cd}(4, r^D_{cd}(G)) = \{1,2,3,4\},\\
&B^D_{cd}(5, r^D_{cd}(G)) = \{3,6,5\}, \quad B^D_{cd}(6, r^D_{cd}(G)) = \{3,6,5\}.
\end{aligned}$$

Figure 4 illustrates the subbasis for τ^D_{cd} in which the vertices joined under differently colored lines form the elements of the subbasis for τ^D_{cd}.

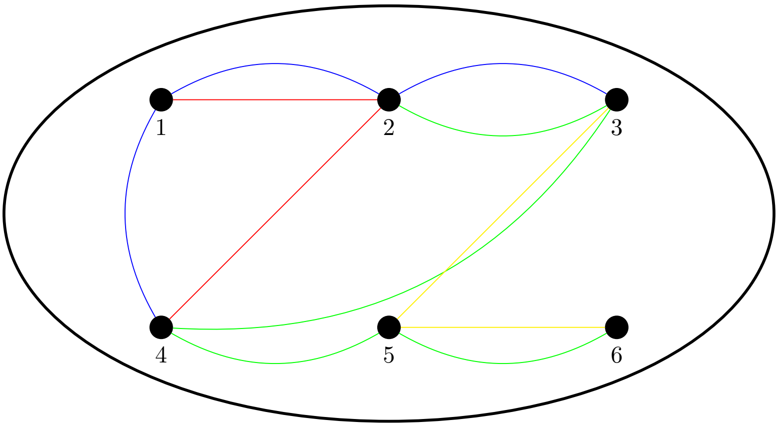


Figure 4

Subbasis for $\tau_{cd}^{D} = \{\{1,2,4\},\{1,2,3,4\},\{2,3,4,5,6\},\{1,2,3,4\},\{3,6,5\},\{3,6,5\}\}$.
Basis for $\tau_{cd}^{D} = \{\emptyset,\{1,2,4\},\{1,2,3,4\},\{2,3,4,5,6\},\{3,6,5\},\{2,4\},\{2,3,4\},\{3\}\}$,
$\tau_{cd}^{D} = \{\emptyset,\{1,2,4\},\{1,2,3,4\},\{2,3,4,5,6\},\{3,6,5\},\{2,4\},\{2,3,4\},\{3\},\{1,2,3,4,5,6\}\}$.

Theorem 3.3. *For any graph G, $B_D(x, r_D(G)) = B_{cd}(x, r_{cd}(G))$.*

Proof. In any graph G, $cd(u,v) = d(u,v) + D(u,v)$ for all $u, v \in V(G)$ and $r_{cd}(G) = r_d(G) + r_D(G)$. Now, $y \in B_D(x, r_D(G)) \Rightarrow D(x,y) < r_D(G) < r_D(G) + r_d(G) = r_{cd}(G)$. Also $d(x,y) < D(x,y)$. Hence, $d(x,y) + D(x,y) < r_{cd}(G)$ and so $cd(x,y) < r_{cd}(G)$. Thus, $y \in B_{cd}(x, r_{cd}(G))$.

Now, $y \in B_{cd}(x, r_{cd}(G)) \Rightarrow cd(x,y) < r_{cd}(G) \Rightarrow d(x,y) + D(x,y) < r_d(G) + r_D(G) \Rightarrow d(x,y) < r_d(G)$ and $D(x,y) < r_D(G)$. Hence, $y \in B_D(x, r_D(G))$. Thus, for any graph G, $B_D(x, r_D(G)) = B_{cd}(x, r_{cd}(G))$. □

Next, the structures of balls that are described by different graph metrics on vertex set of some standard graphs are dealt.

Theorem 3.4. *For a cycle graph C_n with $n \geq 4$, the following holds:*

(i) *$B_d(v, r_d(C_n)) \cup B_D(v, r_D(C_n)) = V(C_n)$ for all $v \in V(C_n)$,*
(ii) *$B_{cd}(v, r_{cd}(C_n)) = \{v\}$ for all $v \in V(C_n)$,*
(iii) *$B_{cd}^{D}(v, r_d^{D}(C_n)) = \{v\}$ for all $v \in V(C_n)$.*

Proof. (i) For any two vertices in C_n, there are exactly two paths connecting them, one of which is a geodesic path P_1 and the other is a detour path P_2.

Note that the length of P_1 + the length of $P_2 = n$.

Also, $e_d(v) = [\frac{n}{2}]$ and $e^D(v) = n - 1$ for all $v \in V(C_n)$. So, $r_d(C_n) = [\frac{n}{2}]$ and $r_D(C_n) = n - 1$. Clearly, $v \in B_d(v, r_d(C_n))$ and $v \in B_D(v, r_D(C_n))$ for all $v \in V(C_n)$. Let $w \in V(C_n)$ be arbitrary. If $d(v,w) < [\frac{n}{2}]$, then $w \in B_d(v, r_d(C_n))$; otherwise $D(v,w) = n - [\frac{n}{2}] < n - 1$, so that $w \in B_D(v, r_D(C_n))$. Hence, $B_d(v, r_d(C_n)) \cup B_D(v, r_D(C_n)) = V(C_n)$ for all $v \in V(C_n)$.

(ii) Let $v \in V(C_n)$ be arbitrary. Since there are exactly two paths between any two vertices in C_n, $cd(v,u) = n$ for all $u \in V(C_n) - \{v\}$. Hence, $e_{cd}(v) = n$ for all $v \in V(C_n)$ and $r_{cd}(C_n) = n$. So, $B_{cd}(v, r_{cd}(C_n)) = \{v\}$.

(iii) Let $v \in V(C_n)$ be arbitrary. Since there are exactly two paths between any two vertices in C_n and the degree of every vertex in C_n is 2, the $cd^D(v,u)$ values are equal for each $u \in V(C_n) - \{v\}$. Let $cd^D(v,u) = b$ for every $u \in V(C_n) - \{v\}$.

Hence, $e^D_{cd}(v) = b$ for all $v \in V(C_n)$ and $r^D_{cd}(C_n) = b$. So, $B^D_{cd}(v, r^D_{cd}(C_n)) = \{v\}$. □

Theorem 3.5. *For a path graph P_n with $n \geq 2$,*

$$B_d(v, r_d(P_n)) = B_D(v, r_D(P_n)) = B_{cd}(v, r_{cd}(P_n)) = B^D_{cd}(v, r^D_{cd}(P_n))$$

for all $v \in V(P_n)$.

Proof. In a path graph P_n, there exists exactly one path between any two vertices. Hence, for any $v \in V(P_n), d(v,u) = D(v,u)$ for all $u \in V(P_n)$. So, $r_d(P_n) = r_D(P_n)$, which implies $B_d(v, r_d(P_n)) = B_D(v, r_D(P_n))$ for all $v \in V(P_n)$. Also, for any two vertices $u, v \in V(P_n), cd(v,u) = 2d(v,u)$ and $cd^D(v,u) = 2\{d(v,u) + \deg(v) + \deg(u) + \sum_x \deg(x)$, where the summation is taken over all the internal vertices x of the path connecting v and $u\}$. Hence, $r_{cd}(P_n) = 2r_d(P_n)$ and $r^D_{cd}(P_n) = 2\{r_d(P_n)\deg(v) + \deg(u) + \sum_x \deg(x)$, where the summation is taken over all the internal vertices x of the path connecting v and $u\}$. Now, $u \in B_{cd}(v, r_{cd}(P_n)) \Leftrightarrow cd(v,u) < r_{cd}(P_n) \Leftrightarrow 2d(u,v) < 2r_d(P_n) \Leftrightarrow d(u,v) < r_d(P_n) \Leftrightarrow u \in B_d(v, r_d(P_n))$. Hence, $B_{cd}(v, r_{cd}(P_n)) = B_d(v, r_d(P_n))$.

Similarly, $B_D(v, r_D(P_n)) = B^D_{cd}(v, r^D_{cd}(P_n)) = B_d(v, r_d(P_n))$. □

Theorem 3.6. *For a wheel graph W_n with $n \geq 4$,*

$$B_d(v, r_d(W_n)) = B_D(v, r_D(W_n)) = B_{cd}(v, r_{cd}(W_n)) = \{v\}$$

for all $v \in V(W_n)$.

Proof. For any two vertices $u \neq v$ in W_n, $d(u,v) = 1$ or 2, $D(u,v) = n-1$, $cd(u,v) = n$ or $n+1$. In particular, if w is the centre vertex of W_n, then $d(w,x) = 1$ and $cd(w,x) = n$ for all $x \in V(W_n) - \{w\}$. Hence, $e_d(w) = 1$ and $e_{cd}(w) = n$. So, $r_d(W_n) = 1, r_D(W_n) = n-1$ and $r_{cd}(W_n) = n$. Thus, $B_d(v, r_d(W_n)) = B_D(v, r_D(W_n)) = B_{cd}(v, r_{cd}(W_n)) = \{v\}$ for all $v \in V(W_n)$. □

Theorem 3.7. *For a star graph $St_{1,n}$ with $n \geq 3, B_d(v, r_d(St_{1,n})) = B_D(v, r_D(St_{1,n})) = B_{cd}(v, r_{cd}(St_{1,n})) = B^D_{cd}(v, r^D_{cd}(St_{1,n})) = \{v\}$ for all $v \in V(St_{1,n})$.*

Proof. In a star graph, there is exactly one path between any two vertices and so $d(u,v) = D(u,v)$ for all $u, v \in V(St_{1,n})$. If w is a central vertex of $St_{1,n}$, then $d(w,v) = 1$ for all $v \in V(St_{1,n})$ and for all $u \in V(St_{1,n}), d(u,v) = 2$, where $v \neq w$. Hence, $e_d(v) = 2$ for all $v \neq w$ and $e_d(w) = 1$. So, $r_d(St_{1,n}) = 1$ and $B_d(v, r_d(St_{1,n})) = \{v\}$. Consequently, $B_D(v, r_D(St_{1,n})) = \{v\}$. Also, $cd(w,v) = 2$ for all $v \in V(St_{1,n})$ and for all $u \in V(St_{1,n}), cd(u,v) = 4$, where $v \neq w$. Hence, $e_{cd}(v) = 4$ for all $v \neq w$ and $e_{cd}(w) = 2$. So, $r_{cd}(St_{1,n}) = 2$ and $B_{cd}(v, r_{cd}(St_{1,n})) = \{v\}$. Since there exists exactly one path between any two vertices in $St_{1,n}$ and the degree of every vertex other than w in $St_{1,n}$ is 1, for each $u \in V(St_{1,n})$, the $cd^D(v,u)$ values are equal for every $v \in V(St_{1,n}) - \{w\}$. Let $cd^D(v,u) = b$ for every $v \in V(St_{1,n}) - \{w\}$ and $cd^D(w,u) < b$ for all $u \in V(St_{1,n})$. Let $cd^D(w,u) = a$ for all $u \in V(St_{1,n})$, where $a < b$. Hence,

$e^D_{cd}(v) = b$ for all $v \neq w \in V(St_{1,n})$ and $e^D_{cd}(w) = a$. So $r^D_{cd}(St_{1,n}) = a$. So, $B^D_{cd}(v, r^D_{cd}(St_{1,n})) = \{v\}$. □

Theorem 3.8. *For a complete graph K_n with $n \geq 4$,*

$$B_d(v, r_d(K_n)) = B_D(v, r_D(K_n)) = B_{cd}(v, r_{cd}(K_n)) = B^D_{cd}(v, r^D_{cd}(K_n)) = \{v\}$$

for all $v \in V(K_n)$.

Proof. In a complete graph K_n, for any two vertices $u \neq v, d(u,v) = 1, D(u,v) = n-1, cd(u,v) = n$. Hence, $r_d(K_n) = 1, r_D(K_n) = n-1, r_{cd}(K_n) = n$. Consequently, $B_d(v, r_d(K_n)) = B_D(v, r_D(K_n)) = B_{cd}(v, r_{cd}(K_n)) = \{v\}$ for all $v \in V(K_n)$. Since every two vertices in K_n are adjacent and the degree of every vertex is $n-1$, the $cd^D(u,v)$ values are equal for each $u \in V(K_n)$. Let $cd^D(u,v) = b$ for all $u \in V(K_n)$. So, $e^D_{cd}(v) = b$ for all $v \in V(K_n)$. Hence, $r^D_{cd}(K_n) = b$ and $B^D_{cd}(v, r^D_{cd}(K_n)) = \{v\}$ for all $v \in V(K_n)$. □

From the above theorems, we can infer:

(i) For any graph G, $\tau_D = \tau_{cd}$;
(ii) τ_{cd} and τ^D_{cd} on the vertex set of cycle graphs are discrete topologies;
(iii) $\tau_d, \tau_D, \tau_{cd}$ and τ^D_{cd} on the vertex set of path graphs are the same;
(iv) τ_d, τ_D and τ_{cd} on the vertex set of wheel graphs are discrete topologies;
(v) $\tau_d, \tau_D, \tau_{cd}$ and τ^D_{cd} on the vertex set of star graphs and complete graphs are discrete topologies.

4. Topologies induced by eccentric neighbourhoods

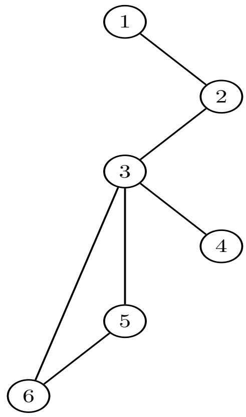
This section presents the method of generating topologies induced by eccentric neighbourhoods of vertices viz. geodesic eccentric neighbourhoods, detour eccentric neighbourhoods, circular eccentric neighbourhoods and circular D-eccentric neighbourhoods.

Definition 4.1. In a graph G, the eccentric neighbourhoods with respect to metrics geodesic distance $d(x,y)$, detour distance $D(x,y)$, circular distance $cd(x,y)$ and circular D-distance $cd^D(x,y)$ are defined as follows:

$$N_d(v) = \{u \in V(G) : d(u,v) = e_d(v)\};$$
$$N_D(v) = \{u \in V(G) : D(u,v) = e_D(v)\};$$
$$N_{cd}(v) = \{u \in V(G) : cd(u,v) = e_c(v)\};$$
$$N^D_{cd}(v) = \{u \in V(G) : cd^D(u,v) = e^D_c(v)\}.$$

Let $M_d(v), M_D(v), M_{cd}(v)$ and $M^D_{cd}(v)$ be the complement of $N_d(v), N_D(v), N_{cd}(v)$ and $N^D_{cd}(v)$, respectively. By definition of $d(x,y), D(x,y), cd(x,y), cd^D(x,y)$, for any $v \in V(G)$, $v \notin$ any of $N_d(v), N_D(v), N_{cd}(v)$ and $N^D_{cd}(v)$. So, $v \in M_d(v)$, $M_D(v), M_{cd}(v)$ and $M^D_{cd}(v)$ for every $v \in V(G)$. Hence, the family of $M_d(v)$ forms a subbasis for the topology τ_{M-d}; the family of $M_D(v)$ forms a subbasis for the topology τ_{M-D}; the family of $M_{cd}(v)$ forms a subbasis for the topology τ_{M-cd}; the family of $M^D_{cd}(v)$ forms a subbasis for the topology τ^D_{M-cd} on a vertex set of G.

Example 4.2. Consider the following graph.



$$d(1,2) = d(2,1) = 1, \quad d(1,3) = d(3,1) = 2, \quad d(1,4) = d(4,1) = 3,$$
$$d(1,5) = d(5,1) = 3, \quad d(1,6) = d(6,1) = 3, \quad d(2,3) = d(3,2) = 1,$$
$$d(2,4) = d(4,2) = 2, \quad d(2,5) = d(5,2) = 2, \quad d(2,6) = d(6,2) = 2,$$
$$d(3,4) = d(4,3) = 1, \quad d(3,5) = d(5,3) = 1, \quad d(3,6) = d(6,3) = 1,$$
$$d(4,5) = d(5,4) = 2, \quad d(4,6) = d(6,4) = 2, \quad d(5,6) = d(6,5) = 1,$$
$$e_d(1) = 3,\ e_d(2) = 2,\ e_d(3) = 2,\ e_d(4) = 3,\ e_d(5) = 3,\ e_d(6) = 3,$$
$$N_d(1) = \{4,5,6\}, \quad N_d(2) = \{4,5,6\}, \quad N_d(3) = \{1\},$$
$$N_d(4) = \{1\}, \quad N_d(5) = \{1\}, \quad N_d(6) = \{1\},$$
$$M_d(1) = \{1,2,3\}, \quad M_d(2) = \{1,2,3\}, \quad M_d(3) = \{2,3,4,5,6\},$$
$$M_d(4) = \{2,3,4,5,6\}, \quad M_d(5) = \{2,3,4,5,6\}, \quad M_d(6) = \{2,3,4,5,6\}.$$

Figure 5 illustrates the subbasis for τ_{M-d} in which the vertices joined under differently colored lines form the elements of the subbasis for τ_{M-d}.

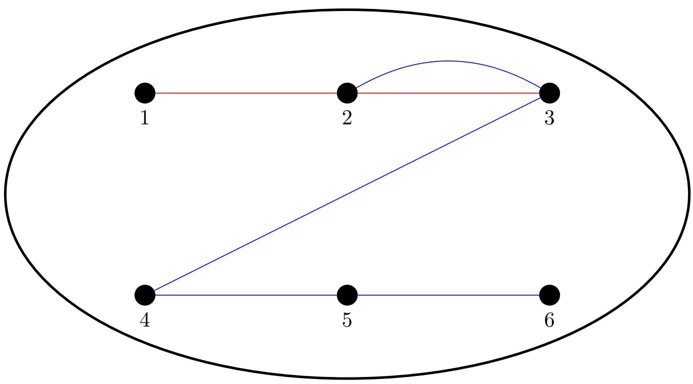


Figure 5

Subbasis for $\tau_{M-d} = \{\{1,2,3\},\{1,2,3\},\{2,3,4,5,6\},\{2,3,4,5,6\},\{2,3,4,5,6\},$ $\{2,3,4,5,6\}\}$. Basis for $\tau_{M-d} = \{\emptyset,\{1,2,3\},\{2,3,4,5,6\},\{2,3\}\}$, $\tau_{M-d} = \{\emptyset,$

$\{1,2,3\},\{2,3,4,5,6\},\{2,3\},\{1,2,3,4,5,6\}\}$,

$$D(1,2)=D(2,1)=1,\quad D(1,3)=D(3,1)=2,\quad D(1,4)=D(4,1)=3,$$
$$D(1,5)=D(5,1)=4,\quad D(1,6)=D(6,1)=4,\quad D(2,3)=D(3,2)=1,$$
$$D(2,4)=D(4,2)=2,\quad D(2,5)=D(5,2)=3,\quad D(2,6)=D(6,2)=3,$$
$$D(3,4)=D(4,3)=1,\quad D(3,5)=D(5,3)=2,\quad D(3,6)=D(6,3)=2,$$
$$D(4,5)=D(5,4)=3,\quad D(4,6)=D(6,4)=3,\quad D(5,6)=D(6,5)=2,$$
$$e_D(1)=4,\ e_D(2)=3,\ e_D(3)=2,\ e_D(4)=3,\ e_D(5)=4,\ e^D(6)=4,$$
$$N_D(1)=\{5,6\},\quad N_D(2)=\{5,6\},\quad N_D(3)=\{1,5,6\},$$
$$N_D(4)=\{1,5,6\},\quad N_D(5)=\{1\},\quad N_D(6)=\{1\},$$
$$M_D(1)=\{1,2,3,4\},\quad M_D(2)=\{1,2,3,4\},\quad M_D(3)=\{2,3,4\},$$
$$M_D(4)=\{2,3,4\},\quad M_D(5)=\{2,3,4,5,6\},\quad M_D(6)=\{2,3,4,5,6\}.$$

Figure 6 illustrates the subbasis for τ_{M-D} in which the vertices joined under differently colored lines form the elements of the subbasis for τ_{M-D}.

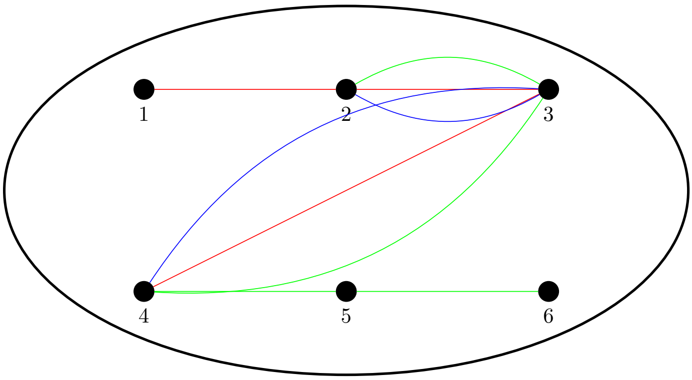


Figure 6

Subbasis for $\tau_{M-D}=\{\{1,2,3,4\},\{1,2,3,4\},\{2,3,4\},\{2,3,4\},\{2,3,4,5,6\},\{2,3,4,5,6\}\}$. Basis for $\tau_{M-D}=\{\emptyset,\{1,2,3,4\},\{2,3,4\},\{2,3,4,5,6\}\}, \tau_{M-D}=\{\emptyset,\{1,2,3,4\},\{2,3,4\},\{2,3,4,5,6\},\{1,2,3,4,5,6\}\}$,

$$cd(1,2)=cd(2,1)=2,\quad cd(1,3)=cd(3,1)=4,\quad cd(1,4)=cd(4,1)=6,$$
$$cd(1,5)=cd(5,1)=7,\quad cd(1,6)=cd(6,1)=7,\quad cd(2,3)=cd(3,2)=2,$$
$$cd(2,4)=cd(4,2)=4,\quad cd(2,5)=cd(5,2)=5,\quad cd(2,6)=cd(6,2)=5,$$
$$cd(3,4)=cd(4,3)=2,\quad cd(3,5)=cd(5,3)=3,\quad cd(3,6)=cd(6,3)=3,$$
$$cd(4,5)=cd(5,4)=5,\quad cd(4,6)=cd(6,4)=5,\quad cd(5,6)=cd(6,5)=3,$$
$$e_{cd}(1)=7,\quad e_{cd}(2)=5,\quad e_{cd}(3)=4,\quad e_{cd}(4)=6,\quad e_{cd}(5)=7,\quad e_{cd}(6)=7,$$
$$N_{cd}(1)=\{5,6\},\qquad N_{cd}(2)=\{5,6\},\quad N_{cd}(3)=\{1\},\quad N_{cd}(4)=\{1\},$$
$$N_{cd}(5)=\{1\},\quad N_{cd}(6)=\{1\},$$
$$M_{cd}(1)=\{1,2,3,4\},\quad M_{cd}(2)=\{1,2,3,4\},$$
$$M_{cd}(3)=\{2,3,4,5,6\},\quad M_{cd}(4)=\{2,3,4,5,6\},$$

$M_{cd}(5) = \{2,3,4,5,6\}, \quad M_{cd}(6) = \{2,3,4,5,6\}.$

Figure 7 illustrates the subbasis for τ_{M-cd} in which the vertices joined under differently colored lines form the elements of the subbasis for τ_{M-cd}.

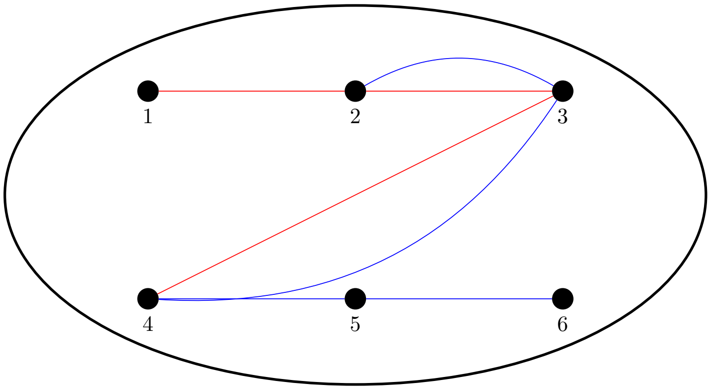


Figure 7

Subbasis for $\tau_{M-cd} = \{\{1,2,3,4\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4,5,6\},\{2,3,4,5,6\},\{2,3,4,5,6\}\}$. Basis for $\tau_{M-cd} = \{\emptyset,\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\}\}$, $\tau_{M-cd} = \{\emptyset,\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{1,2,3,4,5,6\}\}$,

$$cd^D(1,2) = cd^D(2,1) = 8, \quad cd^D(1,3) = cd^D(3,1) = 18,$$
$$cd^D(1,4) = cd^D(4,1) = 22, \quad cd^D(1,5) = cd^D(5,1) = 27,$$
$$cd^D(1,6) = cd^D(6,1) = 27, \quad cd^D(2,3) = cd^D(3,2) = 14,$$
$$cd^D(2,4) = cd^D(4,2) = 18, \quad cd^D(2,5) = cd^D(5,2) = 23,$$
$$cd^D(2,6) = cd^D(6,2) = 23, \quad cd^D(3,4) = cd^D(4,3) = 12,$$
$$cd^D(3,5) = cd^D(5,3) = 17, \quad cd^D(3,6) = cd^D(6,3) = 17,$$
$$cd^D(4,5) = cd^D(5,4) = 21, \quad cd^D(4,6) = cd^D(6,4) = 21,$$
$$cd^D(5,6) = cd^D(6,5) = 15,$$
$$e^D_{cd}(1) = 27,\ e^D_{cd}(2) = 23,\ e^D_{cd}(3) = 18,\ e^D_{cd}(4) = 22,\ e^D_{cd}(5) = 27,\ e^D_{cd}(6) = 27,$$
$$N^D_{cd}(1) = \{6,5\}, \quad N^D_{cd}(2) = \{5,6\}, \quad N^D_{cd}(3) = \{1\},$$
$$N^D_{cd}(4) = \{1\}, \quad N^D_{cd}(5) = \{1\}, \quad N^D_{cd}(6) = \{1\}$$
$$M^D_{cd}(1) = \{1,2,3,4\}, \quad M^D_{cd}(2) = \{1,2,3,4\}, \quad M^D_{cd}(3) = \{2,3,4,5,6\},$$
$$M^D_{cd}(4) = \{2,3,4,5,6\}, \quad M^D_{cd}(5) = \{2,3,4,5,6\}, \quad M^D_{cd}(6) = \{2,3,4,5,6\}.$$

Figure 8 illustrates the subbasis for τ^D_{M-cd} in which the vertices joined under differently colored lines form the elements of the Subbasis for τ^D_{M-cd}.

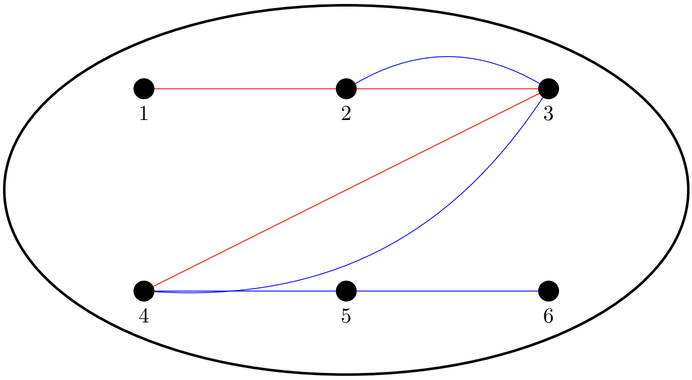


Figure 8

Subbasis for $\tau^D_{M-cd} = \{\{1,2,3,4\},\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4,5,6\},\{2,3,4,6,5\},\{2,3,4,6,5\}\}$. Basis for $\tau^D_{M-cd} = \{\emptyset,\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\}\}$, $\tau^D_{M-cd} = \{\emptyset,\{1,2,3,4\},\{2,3,4,5,6\},\{2,3,4\},\{1,2,3,4,5,6\}\}$.

Theorem 4.3. *In any graph G, $M_{cd}(v) = M_d(v) \cup M_D(v)$ for all $v \in V(G)$.*

Proof. In any graph G, $cd(u,v) = d(u,v) + D(u,v)$ for all $u,v \in V(G)$ and $e_{cd}(v) = e_d(v) + e_D(v)$ for all $v \in V(G)$. Now, $u \in N_{cd}(v) \Leftrightarrow cd(u,v) = e_{cd}(v) \Leftrightarrow d(u,v) + D(u,v) = e_d(v) + e_D(v) \Leftrightarrow d(u,v) = e_d(v)$ and $D(u,v) = e_D(v) \Leftrightarrow u \in N_d(v)$ and $u \in N_D(v) \Leftrightarrow u \in N_d(v) \cap N_D(v)$. Hence, $N_{cd}(v) = N_d(v) \cap N_D(v)$ for all $v \in V(G)$ and $M_{cd}(v) = M_d(v) \cup M_D(v)$ for all $v \in V(G)$. □

Theorem 4.4. *In a cycle graph C_n with $n \geq 4, M_d(v) = B_d(v, r_d(C_n))$ for all $v \in V(C_n)$.*

Proof. In $C_n, e_d(v) = [\frac{n}{2}]$ for all $v \in V(C_n)$ and $r_d(C_n) = [\frac{n}{2}]$.
Now, $u \in B_d(v, r_d(C_n)) \Rightarrow d(u,v) < r_d(C_n) = [\frac{n}{2}] = e_d(v) \Rightarrow u \notin N_d(v) \Rightarrow u \in M_d(v)$. Now, $u \in M_d(v) \Rightarrow d(u,v) < e_d(v) \Rightarrow d(u,v) < r_d(C_n) \Rightarrow u \in B_d(v, r_d(C_n))$. So, $M_d(v) = B_d(v, r_d(C_n))$ for all $v \in V(C_n)$. □

The structure of the standard graphs viz. path graph, cycle graph, complete graph, wheel graph and star graph and the definition of eccentric neighbourhoods, infer the following observations.

Observation 4.5. 1. In a path graph P_n with $n \geq 3$, since there exists a unique path between any two vertices, we have $M_d(v) = M_D(v) = M_{cd}(v)$ for all $v \in V(P_n)$ and so $\tau_{M-d} = \tau_{M-D} = \tau_{M-cd}$. Also, if $v_1, v_2, \ldots, v_n$ are the vertices of P_n, when n is odd, we can see

$$N^D_{cd}(v_1) = N^D_{cd}(v_2) = \cdots = N^D_{cd}(v_{[\frac{n}{2}]}) = \{v_n\}, N^D_{cd}(v_{[\frac{n}{2}]+1}) = \{v_1, v_n\},$$
$$N^D_{cd}(v_{[\frac{n}{2}]+2}) = N^D_{cd}(v_{[\frac{n}{2}]+3}) = \cdots = N^D_{cd}(v_n) = \{v_1\}$$

so that

$$\tau^D_{M-cd} = \{\emptyset, \{v_1, v_2, \ldots, v_{n-1}\}, \{v_2, \ldots, v_n\}, \{v_2, \ldots, v_{n-1}\}, \{v_1, v_2, \ldots, v_n\}\}$$

and when n is even, we can see

$$N^D_{cd}(v_1) = N^D_{cd}(v_2) = \cdots = N^D_{cd}(v_{\frac{n}{2}}) = \{v_n\}, N^D_{cd}(v_{\frac{n}{2}+1}) = \cdots = N^D_{cd}(v_n) = \{v_1\}$$

so that

$$\tau^D_{M-cd} = \{\emptyset, \{v_1, v_2, \dots, v_{n-1}\}, \{v_2, \dots v_n\}, \{v_2, \dots, v_{n-1}\}, \{v_1, v_2, \dots v_n\}\}.$$

2. In a complete graph K_n with $n \geq 4$, we see that

$$N_d(v) = N_D(v) = N_{cd}(v) = N^D_{cd}(v) = V(K_n) - \{v\}$$

for every $v \in V(K_n)$ and so $\tau_{M-d}, \tau_{M-D}, \tau_{M-cd}$ and τ^D_{M-cd} are all discrete topologies on a vertex set of K_n.

3. If $v_1, v_2, \dots, v_{n-1}$ are the vertices of the cycle of a wheel graph W_n with $n \geq 3$ and v_n is the centre vertex of W_n, then for $i = 1, 2, \dots, n-1$,

$$N_d(v_i) = N_{cd}(v_i) = \{v_1, v_2, \dots, v_{i-2}, v_{i+2}, \dots, v_{n-1}\}$$

and

$$N_d(v_n) = N_{cd}(v_n) = \{v_1, v_2, \dots, v_{n-1}\}; \quad N_D(v) = V(W_n) - \{v\}$$

for all $v \in V(W_n)$.

4. If $v_1, v_2, \dots, v_n$ are the leaves of a star graph $St_{1,n}$ with $n \geq 3$, and w is the central vertex of $St_{1,n}$, then for $i = 1, 2, \dots, n$,

$$N_d(v_i) = N_D(v_i) = N_{cd}(v_i) = N^D_{cd}(v_i) = \{v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$$

and

$$N_d(w) = N_D(w) = N_{cd}(w) = N^D_{cd}(w) = \{v_1, v_2, \dots, v_n\}.$$

5. Conclusion

Based on different distances in graphs, the methods of generating topologies by different graph metrics viz. open balls and eccentric neighbourhoods of vertices are presented. The nature of topologies generated by graph metrics on the vertex set of some standard graphs is studied. It is proved that for any graph G, $B_D(x, r_D(G)) = B_{cd}(x, r_{cd}(G))$. It is observed that the open balls obtained through other distances are independent of each other. Also, the relations between the topologies generated by different graph metrics on the vertex set of complete graph, path graph, cycle graph, wheel graph and star graph are explored. For a cycle graph $C_n, n \geq 4$, the relationships between the topologies generated in sections 2 and 3 are studied and it is proved that $M_d(v) = B_d(v, r_d(C_n))$ for all $v \in V(C_n)$. It is observed that, in general, the topologies generated in sections 2 and 3 are independent. Also, the results in this paper can be studied further using the graph metrics viz. closed balls, closed eccentric neighbourhoods, etc. In a further study, one can explore the relationship between the topologies generated by different graph metrics. The topologies generated using the graph metric can be used to solve network problems which focus on distances in graphs. This paper can be regarded as the initial stage of studying a topological structure on a vertex set of graphs using graph metrics, which could lead to significant applications in real life.

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