Math. Appl. **12** (2023), 1–25
DOI: 10.13164/ma.2023.01

CONNECTIONS BETWEEN MATRIX PRODUCTS FOR 3-VECTORS AND GEOMETRIC ALGEBRA

BERTOLD BONGARDT AND HARALD LÖWE

Abstract. The *geometric product* represents a core concept for establishing geometric algebras and in case of vectors matches the formal sum of their *inner product* and their *wedge product*. The geometric product is reconsidered for the case of 3-vectors by means of usual matrix algebra in this article. Therefore, a symmetric matrix product and an antisymmetric matrix product are introduced, whose matrix sum yields a third product that renders the information of a vector pair's geometric product in terms of a matrix associated to the vectors. The three matrices – that correspond to inner product, wedge product, and geometric product, respectively – are named *wheel product*, *curl product*, and *full product*. The observation about the structural correspondence of the geometric product with matrix theory may be used for future practical computations and unveils connections of geometric algebra with related disciplines.

1. Introduction

Line geometry is "concerned with the set of lines of three-dimensional space" and has "relations to mechanics and spatial kinematics" and "therefore applications [..] in mechanism design and robotics" [26]. In this context, the present article has been motivated by an analysis [4] in which the kinematics of spatial chains with three cylindrical joints are solved analytically by employing the algebra and geometry of oriented lines in space. In the scope of that analysis, the symmetric matrix $\boldsymbol{a} \circledast \boldsymbol{b} \in \mathsf{sym}(3)$ has been defined for two linear independent vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ as

$$\boldsymbol{a} \circledast \boldsymbol{b} := \boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} + \boldsymbol{a}^{\otimes} * \boldsymbol{b}^{\otimes} = 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - (\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) \tag{1.1}$$

to express (3×3)-blocks of (6×6)-matrices for orthogonal decomposition of oriented lines in terms of 6-vectors in a compact form. The obvious symmetry of the matrix $\boldsymbol{a} \circledast \boldsymbol{b}$ raises the interest in the significance of that matrix both in geometric and algebraic terms. In view of this interest, we consider the "unified system" [12] of *geometric algebras*, also known as *Clifford algebras*, to explore the context of the symmetric matrix product. A geometric algebra can roughly be described as an extension of a linear '*space of vectors*', equipped with an inner product, to a linear '*space of multivectors*'. Thereby, a multivector is defined as the weighted sum of *wedge products* of basis vectors.[1] The framework of geometric algebras has been developed for "efficiently expressing and exploiting the full range of geometric

MSC (2020): primary 15A66; secondary 97H60, 15A69, 26B12, 51M30.
Keywords: geometric algebra, linear algebra, line geometry, Lie–Jordan products.

[1]Similar to a vector, that is defined as the weighted sum of basis vectors.

concepts in mathematics" [12]. Due to its generality, such diverse concepts as classic vector algebra, line geometry [18], screw theory [11], quaternions, dual quaternions and domains as computer graphics, kinematics [14], theoretical physics have been approached by using geometric algebras.

For analyzing the role of the symmetric matrix product $\boldsymbol{a} \circledast \boldsymbol{b}$ of Equation (1.1), only the three-dimensional vector space $\mathbb{R}^3$ is considered within this article. With $\langle \cdot | \cdot \rangle$, a symmetric bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is indicated[2] that induces a norm[3] with $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a} | \boldsymbol{a} \rangle}$ for $\langle \cdot | \cdot \rangle$ positive definite (see also Appendix A). In particular, an orthonormal basis $(\hat{\boldsymbol{e}}_\mathsf{x}, \hat{\boldsymbol{e}}_\mathsf{y}, \hat{\boldsymbol{e}}_\mathsf{z})$ for $\mathbb{R}^3$ is characterized by vectors of unit length

$$\langle \hat{\boldsymbol{e}}_\mathsf{x} | \hat{\boldsymbol{e}}_\mathsf{x} \rangle = 1 \qquad \langle \hat{\boldsymbol{e}}_\mathsf{y} | \hat{\boldsymbol{e}}_\mathsf{y} \rangle = 1 \qquad \langle \hat{\boldsymbol{e}}_\mathsf{z} | \hat{\boldsymbol{e}}_\mathsf{z} \rangle = 1 \tag{1.2}$$

and of perpendicularity

$$\langle \hat{\boldsymbol{e}}_\mathsf{x} | \hat{\boldsymbol{e}}_\mathsf{y} \rangle = 0 \qquad \langle \hat{\boldsymbol{e}}_\mathsf{y} | \hat{\boldsymbol{e}}_\mathsf{z} \rangle = 0 \qquad \langle \hat{\boldsymbol{e}}_\mathsf{z} | \hat{\boldsymbol{e}}_\mathsf{x} \rangle = 0 \,. \tag{1.3}$$

Choosing coordinates, one may achieve that $\hat{\boldsymbol{e}}_\mathsf{x} = (1,0,0)^\top$, $\hat{\boldsymbol{e}}_\mathsf{y} = (0,1,0)^\top$, and $\hat{\boldsymbol{e}}_\mathsf{z} = (0,0,1)^\top$. Together with $\langle \boldsymbol{a} | \boldsymbol{b} \rangle \equiv \boldsymbol{a} * \boldsymbol{b} := \sum_i a_i \cdot b_i$, the positive definite Euclidean space is given, abbreviated as $(\mathbb{R}^3, *)$. The associated geometric algebra[4] $\mathsf{C}_{3,0,0} \cong \mathsf{G}(\mathbb{R}^3, *)$ consists of multivectors of the form

$$A = \{A\}_0 + \{A\}_1 + \{A\}_2 + \{A\}_3 \in \mathsf{C}_{3,0,0}$$

as the sum of wedge products of four different grades k. Here, the term $\{A\}_k$ denotes a k-blade. The 0-blade $\{A\}_0$ is a *scalar*, the 1-blade $\{A\}_1$ is a *vector*, the 2-blade $\{A\}_2$ is a *bivector*, and the 3-blade $\{A\}_3$ is a *trivector*. Further, a d-blade in dimension d is a *pseudoscalar* and a $(d-1)$-blade is a *pseudovector*. A bivector $\{A\}_2 = \bigwedge_{i=1}^2 \boldsymbol{a}_i = \boldsymbol{a}_1 \wedge \boldsymbol{a}_2$ with wedge product is introduced below in Equation (1.4). The geometric algebra $\mathsf{G}(\mathbb{R}^3, *)$ can be written as a graded structure

$$\mathsf{G}(\mathbb{R}^3, *) = \mathsf{G}_0 \oplus \mathsf{G}_1 \oplus \mathsf{G}_2 \oplus \mathsf{G}_3$$

with $\mathsf{G}_0 \cong \mathbb{R}$ and $\mathsf{G}_1 \cong \mathbb{R}^3$ and $\mathsf{G}_2 \cong \mathbb{R}^3 \wedge \mathbb{R}^3$ and $\mathsf{G}_3 \cong I_3\mathbb{R}$. The term I_3 is the pseudoscalar introduced in Equation (1.8) below. In Table 1, an overview of the basis elements of the geometric algebra $\mathsf{G}(\mathbb{R}^3, *)$ of four grades is given.

The core operation of geometric algebras is the geometric product: the geometric product of two multivectors A and B is usually denoted by juxtaposition as AB. It is typically not symmetric, $AB \neq BA$. A geometric product is defined by the axioms of (i) the existence of $1 \in \mathsf{G}$, the multiplicative *identity element*, (ii) the *associativity* of the product with $(AB)C = A(BC)$, (iii) the *distributivity* of the product with respect to addition,[5] and (iv) the *compatibility* of the geometric product with the bilinear form $\langle \cdot | \cdot \rangle$ by requiring that $\boldsymbol{v}\boldsymbol{v} = 1 \cdot \langle \boldsymbol{v} | \boldsymbol{v} \rangle$ holds for any vector $\boldsymbol{v} \in \mathbb{R}^d$. Letting the power $\boldsymbol{v}^2 := \boldsymbol{v}\boldsymbol{v}$, the requirement of 'geometric-inner compatibility' is expressed as $\boldsymbol{v}^2 = 1 \cdot \langle \boldsymbol{v} | \boldsymbol{v} \rangle = 1 \cdot Q(\boldsymbol{v})$, where $Q(\boldsymbol{v}) = \langle \boldsymbol{v} | \boldsymbol{v} \rangle$

---

[2]See Equation (A.1) of the appendix.

[3]The norm for a general multivector [20] can be established as $\|A\| := \{A\underline{A}\}_0$ by means of the reversion $\underline{A}$ of Equation (C.2) in the appendix.

[4]A geometric algebra $\mathsf{C}_{p,q,r} \cong \mathsf{G}(\mathbb{R}^d, \langle \cdot | \cdot \rangle)$ of signature (p,q,r) is characterized by a quadratic form $Q(\boldsymbol{a}) = \langle \boldsymbol{a} | \boldsymbol{a} \rangle$ (Equation (A.2) and Equation (A.3)) which is *positive definite* in p, *negative definite* in q, and *degenerated* in r dimensions, with $p + q + r = d$.

[5]The requirement of *distributivity* reads $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$.

Table 1. Overview of the eight basis elements of the geometric algebra $\mathsf{C}_{3,0,0} \cong \mathsf{G}(\mathbb{R}^3, *)$ associated to the three-dimensional Euclidean space.

Blade	*Algebraic element*	*Sub-basis*	*Sub-dimension*
3-blade	pseudoscalar	$\mathsf{G}_3 = \{\mathsf{e}_{\mathsf{xyz}}\}$	1
2-blade	bivector	$\mathsf{G}_2 = \{\mathsf{E}_{\mathsf{xy}}, \mathsf{E}_{\mathsf{yz}}, \mathsf{E}_{\mathsf{zx}}\}$	3
1-blade	vector	$\mathsf{G}_1 = \{\hat{\boldsymbol{e}}_{\mathsf{x}}, \hat{\boldsymbol{e}}_{\mathsf{y}}, \hat{\boldsymbol{e}}_{\mathsf{z}}\}$	3
0-blade	scalar	$\mathsf{G}_0 = \{1\}$	1
sum of blades	multivector	$\mathsf{G}_0 \cup \mathsf{G}_1 \cup \mathsf{G}_2 \cup \mathsf{G}_3$	8

(Equation (A.3)) denotes the univariate quadratic form associated to the bilinear form $\langle\cdot|\cdot\rangle$.[6] For an orthonormal basis, the geometric product $\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j$ of two basis vectors $\hat{\boldsymbol{e}}_i$ and $\hat{\boldsymbol{e}}_j$ equals

$$\hat{\boldsymbol{e}}_i\hat{\boldsymbol{e}}_j = \langle\hat{\boldsymbol{e}}_i|\hat{\boldsymbol{e}}_j\rangle + \hat{\boldsymbol{e}}_i \wedge \hat{\boldsymbol{e}}_j = \begin{cases} 1 & i = j \\ \hat{\boldsymbol{e}}_i \wedge \hat{\boldsymbol{e}}_j & i \neq j \end{cases} .$$

For the three dimensional case, the algebra $\mathsf{G}(\mathbb{R}^3, *)$ has a basis of $2^3 = 8$ elements and the geometric product for all pairs of the basis elements is reported in Table 2. For the sake of convenience, the abridged notations

$$\mathsf{E}_{\mathsf{xy}} := \hat{\boldsymbol{e}}_{\mathsf{x}}\hat{\boldsymbol{e}}_{\mathsf{y}} \qquad \mathsf{E}_{\mathsf{yz}} := \hat{\boldsymbol{e}}_{\mathsf{y}}\hat{\boldsymbol{e}}_{\mathsf{z}} \qquad \mathsf{E}_{\mathsf{zx}} := \hat{\boldsymbol{e}}_{\mathsf{z}}\hat{\boldsymbol{e}}_{\mathsf{x}}$$

for three basis bivectors are introduced with the properties

$$\mathsf{E}_{\mathsf{xy}} = \hat{\boldsymbol{e}}_{\mathsf{x}} \wedge \hat{\boldsymbol{e}}_{\mathsf{y}} = -\hat{\boldsymbol{e}}_{\mathsf{y}} \wedge \hat{\boldsymbol{e}}_{\mathsf{x}}, \quad \mathsf{E}_{\mathsf{yz}} = \hat{\boldsymbol{e}}_{\mathsf{y}} \wedge \hat{\boldsymbol{e}}_{\mathsf{z}} = -\hat{\boldsymbol{e}}_{\mathsf{z}} \wedge \hat{\boldsymbol{e}}_{\mathsf{y}}, \quad \mathsf{E}_{\mathsf{zx}} = \hat{\boldsymbol{e}}_{\mathsf{z}} \wedge \hat{\boldsymbol{e}}_{\mathsf{x}} = -\hat{\boldsymbol{e}}_{\mathsf{x}} \wedge \hat{\boldsymbol{e}}_{\mathsf{z}} .$$

For two arbitrary 3-vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, the wedge product $\boldsymbol{a} \wedge \boldsymbol{b} \in \mathsf{G}(\mathbb{R}^3)$ is a bivector that has the components

$$\boldsymbol{a} \wedge \boldsymbol{b} = (a_1b_2 - a_2b_1) \cdot \mathsf{E}_{\mathsf{xy}} + (a_2b_3 - a_3b_2) \cdot \mathsf{E}_{\mathsf{yz}} + (a_3b_1 - a_1b_3) \cdot \mathsf{E}_{\mathsf{zx}} . \tag{1.4}$$

For general multivectors $A, B \in \mathsf{G}(\mathbb{R}^3, *)$, the wedge product is defined [7] by

$$A \wedge B := \sum_r \sum_s \{\{A\}_r \{B\}_s\}_{r+s} ,$$

a sum of maximal-grade blades of geometric products. For a scalar $c \in \mathbb{R}$ and a vector $\boldsymbol{a} \in \mathbb{R}^3$, the wedge product matches the geometric product

$$c \wedge \boldsymbol{a} = c\boldsymbol{a} = c \cdot \boldsymbol{a} = \boldsymbol{a} \cdot c = \boldsymbol{a}c = \boldsymbol{a} \wedge c \in \mathbb{R}^3$$

and the inner product $\langle c|\boldsymbol{a}\rangle = -\langle\boldsymbol{a}|c\rangle = 0$ with a scalar is antisymmetric and equals zero, as it is explained in [34].

The inner product $\langle\boldsymbol{a}|\boldsymbol{b}\rangle$ of two arbitrary vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3 = \mathsf{G}_1 \subset \mathsf{G}(\mathbb{R}^3)$ is obtained by the inner products of the basis vectors

$$\begin{aligned}\langle\boldsymbol{a}|\boldsymbol{b}\rangle = {} & (a_1b_2 + a_2b_1) \cdot \langle\boldsymbol{e}_1|\boldsymbol{e}_2\rangle + (a_1b_3 + a_3b_1) \cdot \langle\boldsymbol{e}_1|\boldsymbol{e}_3\rangle + (a_2b_3 + a_3b_2) \\ & \cdot \langle\boldsymbol{e}_2|\boldsymbol{e}_3\rangle + a_1b_1 \cdot \langle\boldsymbol{e}_1|\boldsymbol{e}_1\rangle + a_2b_2 \cdot \langle\boldsymbol{e}_2|\boldsymbol{e}_2\rangle + a_3b_3 \cdot \langle\boldsymbol{e}_3|\boldsymbol{e}_3\rangle .\end{aligned} \tag{1.5}$$

[6] The bilinear form $\langle\cdot|\cdot\rangle$ is also involved in the alternate definition of a geometric algebra as a quotient algebra of Equation (B.2) (Appendix B).

Table 2. Multiplication table for the geometric product of the particular geometric algebra $\mathsf{C}_{3,0,0} = \mathsf{G}(\mathbb{R}^3, *)$ of the positive definite three dimensional Euclidean space $(\mathbb{R}^3, *)$. The pseudoscalar I_3 is also indicated by $\mathsf{e}_{\mathsf{xyz}}$.

		G_0	G_1			G_2			G_3
		1	$\hat{\boldsymbol{e}}_\mathsf{x}$	$\hat{\boldsymbol{e}}_\mathsf{y}$	$\hat{\boldsymbol{e}}_\mathsf{z}$	E_xy	E_yz	E_zx	e_xyz
G_0	1	1	$\hat{\boldsymbol{e}}_\mathsf{x}$	$\hat{\boldsymbol{e}}_\mathsf{y}$	$\hat{\boldsymbol{e}}_\mathsf{z}$	E_xy	E_yz	E_zx	I_3
	$\hat{\boldsymbol{e}}_\mathsf{x}$	$\hat{\boldsymbol{e}}_\mathsf{x}$	1	E_xy	$-\mathsf{E}_\mathsf{zx}$	$\hat{\boldsymbol{e}}_\mathsf{y}$	I_3	$-\hat{\boldsymbol{e}}_\mathsf{z}$	E_yz
G_1	$\hat{\boldsymbol{e}}_\mathsf{y}$	$\hat{\boldsymbol{e}}_\mathsf{y}$	$-\mathsf{E}_\mathsf{xy}$	1	E_yz	$-\hat{\boldsymbol{e}}_\mathsf{x}$	$\hat{\boldsymbol{e}}_\mathsf{z}$	$-I_3$	$-\mathsf{E}_\mathsf{zx}$
	$\hat{\boldsymbol{e}}_\mathsf{z}$	$\hat{\boldsymbol{e}}_\mathsf{z}$	E_zx	$-\mathsf{E}_\mathsf{yz}$	1	I_3	$-\hat{\boldsymbol{e}}_\mathsf{y}$	$\hat{\boldsymbol{e}}_\mathsf{x}$	$-\mathsf{E}_\mathsf{xy}$
	E_xy	E_xy	$-\hat{\boldsymbol{e}}_\mathsf{y}$	$\hat{\boldsymbol{e}}_\mathsf{x}$	I_3	-1	$-\mathsf{E}_\mathsf{zx}$	E_yz	$-\hat{\boldsymbol{e}}_\mathsf{z}$
G_2	E_yz	E_yz	I_3	$-\hat{\boldsymbol{e}}_\mathsf{z}$	$\hat{\boldsymbol{e}}_\mathsf{y}$	E_zx	-1	$-\mathsf{E}_\mathsf{xy}$	$\hat{\boldsymbol{e}}_\mathsf{x}$
	E_zx	E_zx	$\hat{\boldsymbol{e}}_\mathsf{z}$	$-I_3$	$-\hat{\boldsymbol{e}}_\mathsf{x}$	$-\mathsf{E}_\mathsf{yz}$	E_xy	-1	$\hat{\boldsymbol{e}}_\mathsf{y}$
G_3	e_xyz	I_3	$-\mathsf{E}_\mathsf{yz}$	E_zx	E_xy	$\hat{\boldsymbol{e}}_\mathsf{z}$	$-\hat{\boldsymbol{e}}_\mathsf{x}$	$-\hat{\boldsymbol{e}}_\mathsf{y}$	-1

For the particular case of $\mathbb{R}^3$ with orthonormal basis $\{\hat{\boldsymbol{e}}_\mathsf{x}, \hat{\boldsymbol{e}}_\mathsf{y}, \hat{\boldsymbol{e}}_\mathsf{z}\}$ of elements $\hat{\boldsymbol{e}}_\mathsf{x} = (1,0,0)^\top$, $\hat{\boldsymbol{e}}_\mathsf{y} = (0,1,0)^\top$, and $\hat{\boldsymbol{e}}_\mathsf{z} = (0,0,1)^\top$ with Equation (1.2) and Equation (1.3), the scalar product of $\boldsymbol{a}$ and $\boldsymbol{b}$ simplifies to

$$\langle \boldsymbol{a} | \boldsymbol{b} \rangle = \boldsymbol{a} * \boldsymbol{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 \ .$$

Finally, the geometric product $\boldsymbol{ab}$ of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is given in terms of a commutative term and an anticommutative term by

$$\boldsymbol{ab} = \underbrace{\tfrac{1}{2} \cdot \big(\boldsymbol{ab} + \boldsymbol{ba}\big)}_{\text{commutative}} + \underbrace{\tfrac{1}{2} \cdot \big(\boldsymbol{ab} - \boldsymbol{ba}\big)}_{\text{anticommutative}} = \langle \boldsymbol{a} | \boldsymbol{b} \rangle + \boldsymbol{a} \wedge \boldsymbol{b} \ , \tag{1.6}$$

such that $\boldsymbol{ba} = \langle \boldsymbol{a} | \boldsymbol{b} \rangle - \boldsymbol{a} \wedge \boldsymbol{b}$.

The inverse of a vector $\boldsymbol{a}$ with respect to the geometric product[7] is denoted by $\boldsymbol{a}^{-1}$ and defined by

$$\boldsymbol{a}^{-1} := \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|^2} \quad \text{such that} \quad \boldsymbol{aa}^{-1} = \langle \boldsymbol{a} | \boldsymbol{a}^{-1} \rangle + \boldsymbol{a} \wedge \boldsymbol{a}^{-1} = 1. \tag{1.7}$$

Duality. The unit pseudoscalar I_d of a geometric algebra in dimension d is usually defined[8] as the d-blade $I_d := \hat{\boldsymbol{e}}_1 \hat{\boldsymbol{e}}_2 \cdots \hat{\boldsymbol{e}}_d$, a d-wise geometric product. For the case $d = 3$, the unit pseudoscalar I_3 and its inverse I_3^{-1} are thus defined by

$$\begin{aligned} I_3 &:= \hat{\boldsymbol{e}}_\mathsf{x} \hat{\boldsymbol{e}}_\mathsf{y} \hat{\boldsymbol{e}}_\mathsf{z} = \hat{\boldsymbol{e}}_\mathsf{y} \hat{\boldsymbol{e}}_\mathsf{z} \hat{\boldsymbol{e}}_\mathsf{x} = \hat{\boldsymbol{e}}_\mathsf{z} \hat{\boldsymbol{e}}_\mathsf{x} \hat{\boldsymbol{e}}_\mathsf{y} \\ I_3^{-1} &:= \hat{\boldsymbol{e}}_\mathsf{z} \hat{\boldsymbol{e}}_\mathsf{y} \hat{\boldsymbol{e}}_\mathsf{x} = \hat{\boldsymbol{e}}_\mathsf{y} \hat{\boldsymbol{e}}_\mathsf{x} \hat{\boldsymbol{e}}_\mathsf{z} = \hat{\boldsymbol{e}}_\mathsf{x} \hat{\boldsymbol{e}}_\mathsf{z} \hat{\boldsymbol{e}}_\mathsf{y} = -I_3 \end{aligned} \tag{1.8}$$

---

[7] The definition of the inverse [20] of a general multivector A reads $A^{-1} := \frac{\overline{A}}{A\overline{A}}$ by means of the Clifford conjugate $\overline{A}$ of Equation (C.3) in the appendix.

[8] A variant of the unit pseudoscalar is discussed in [10].

Table 3. Overview of k-blades of independent vectors in the three dimensional linear space.

Blade	*Algebraic*	*Geometric*	*Dimension*	*Unit DOF*	*Duality*
3-blade	pseudoscalar	space orientation	1	0	
2-blade	bivector	oriented plane	3	2	
1-blade	vector	direction	3	2	
0-blade	scalar	zerospace	1	0	

and summarized by the notation variant $I_3 = \mathsf{e}_{\mathsf{xyz}} := \hat{\boldsymbol{e}}_{\mathsf{x}}\,\hat{\boldsymbol{e}}_{\mathsf{y}}\,\hat{\boldsymbol{e}}_{\mathsf{z}}$. By means of I_3, the *dual* multivector to a multivector A is defined [23] as

$$A^\star := AI_3^{-1}\ .$$

For instance, the dual of the wedge product of Equation (1.4) is determined by

$$\begin{aligned}(\boldsymbol{a}\wedge\boldsymbol{b})^\star &= (a_1b_2 - a_2b_1)\cdot(\hat{\boldsymbol{e}}_{\mathsf{x}}\wedge\hat{\boldsymbol{e}}_{\mathsf{y}})\,\hat{\boldsymbol{e}}_{\mathsf{y}}\,\hat{\boldsymbol{e}}_{\mathsf{x}}\,\hat{\boldsymbol{e}}_{\mathsf{z}} + (a_2b_3 - a_3b_2)\\ &\quad\cdot(\hat{\boldsymbol{e}}_{\mathsf{y}}\wedge\hat{\boldsymbol{e}}_{\mathsf{z}})\,\hat{\boldsymbol{e}}_{\mathsf{z}}\,\hat{\boldsymbol{e}}_{\mathsf{y}}\,\hat{\boldsymbol{e}}_{\mathsf{x}} + (a_3b_1 - a_1b_3)\cdot(\hat{\boldsymbol{e}}_{\mathsf{z}}\wedge\hat{\boldsymbol{e}}_{\mathsf{x}})\,\hat{\boldsymbol{e}}_{\mathsf{x}}\,\hat{\boldsymbol{e}}_{\mathsf{z}}\,\hat{\boldsymbol{e}}_{\mathsf{y}} \\ &= (a_1b_2 - a_2b_1)\cdot\hat{\boldsymbol{e}}_{\mathsf{z}} + (a_2b_3 - a_3b_2)\cdot\hat{\boldsymbol{e}}_{\mathsf{x}} + (a_3b_1 - a_1b_3)\cdot\hat{\boldsymbol{e}}_{\mathsf{y}} = \boldsymbol{a}\times\boldsymbol{b}\end{aligned} \tag{1.9}$$

and transfers the bivector $\boldsymbol{a}\wedge\boldsymbol{b}$ into the cross product vector $\boldsymbol{a}\times\boldsymbol{b}$ perpendicular to $\boldsymbol{a}$ and $\boldsymbol{b}$. By means of these, the dual of the geometric product of two 3-vectors of Equation (1.6) reads

$$(\boldsymbol{ab})^\star = \langle\boldsymbol{a}|\boldsymbol{b}\rangle\cdot I_3^{-1} + (\boldsymbol{a}\times\boldsymbol{b})\ .$$

As a generalization of Equation (1.9), the dual of the wedge product is characterized by an inner product of a dualized term by

$$(\boldsymbol{a}\wedge\boldsymbol{b})^\star = (\boldsymbol{a}\wedge\boldsymbol{b})I_3^{-1} = \langle\boldsymbol{a}|\boldsymbol{b}I_3^{-1}\rangle = \langle\boldsymbol{a}|\boldsymbol{b}^\star\rangle.$$

The duality of blades for the case of three dimensional geometric algebras is illustrated in Table 3.

References. A selection of relevant literature is listed chronologically to direct to more information on the subjects. The historical development of vector analysis in the interplay of mathematics and the physical sciences is treated by Crowe [5]. The volume by Hestenes and Sobczyk [12] is certainly one of the most fundamental textbooks on the subject of geometric algebra. Vold introduces geometric algebra from a physical point of view in the tutorial [34]. The textbook by Lounesto [20] provides a thorough exposition of geometric algebras, including the theme of matrix representations. Sobczyk [30] treats geometric algebras for Non-Euclidean spaces. Hestenes [11] presents interconnections of geometric algebra with rigid body kinematics. The textbook by Meinrenken [25] applies geometric algebra to the study of Lie groups and Lie algebras. MacDonald [23] considers geometric algebra in combination with vector and matrix calculus. In the introduction by Hitzer [15], several geometric algebras are presented in comparison. The representation of dual quaternions by means of matrix algebra is documented by Thomas [32]. The exposition by Gunn [9] treats fundamentals of kinematics and dynamics by means of the projective model from a geometric algebra standpoint. Aristidou and Lasenby [1] present an efficient iterative kinematics solver based on conformal geometric algebra. The analysis by Klawitter [16] considers projective line geometry by means of geometric algebra. In the extensive paper by Li, Huang et al. [19], the theory of oriented lines and screws is approached from a perspective of geometric algebra. Dorst [8] studies classes of projective line transforms by means of geometric algebra and relates to the previous works. The textbook by Vaz and da Rocha [33] presents Clifford algebras, their classifications, and representations systematically. Macdonald [22] surveys the foundations of geometric algebra in a compact and systematic manner. The survey by Descamps [6] introduces geometric algebra and provides a rich historical account. Sobczyk [31] treats geometric algebra in combination with the theory of matrices. The volume by Bayro-Corrochano [2] portrays applications of geometric algebra to problems in robotics. An instance-driven introduction to geometric algebra is given in the lecture notes by Renaud [27]. The book by Hildenbrandt [13] accompanies an optimization software for computational geometric algebra.

Contributions. Geometric algebra is used to analyze the role of matrix products for three-dimensional vectors in this article: the concepts of the inner product, the wedge product, and the geometric product motivate to introduce three matrix products with similar properties. The three matrix expressions are analyzed both, with respect to matrix theory, geometric algebra, and tensor algebra from an algebraic point of view, and, as linear maps featuring distinct geometric images from a geometric point of view. By this approach, connections of geometric algebra with matrix-vector calculus and with matrix and tensor theory are identified.

Structure. The remainder of the paper is organized as follows: in Section 2, the three matrix products and their fundamental algebraic properties are presented. In Section 3, the matrix products are approached from an algebraic point of view and the interrelations to concepts of geometric algebra and further algebraic products are outlined. In Section 4, the matrix products are approached from a geometric point of view by considering the product matrices as endomorphisms of the three dimensional vector space. In Section 5, an overview is briefly discussed by assembling products concepts of matrix algebra and vector calculus for 1-blades, 2-blades, and 3-blades. Final conclusions are drawn in Section 6. The main document is supplemented by five appendices: Appendix A reports bilinear and quadratic forms. Appendix B states three definitions of multilinear algebra. Appendix C assembles involutions for geometric algebra. Appendix D contains detailed identities for the introduced products. Appendix E provides three geometric vector operations in terms of matrix-vector calculus.

2. Matrix Products

Three bivariate matrix products are introduced with dedicated terms and symbols in this section. The first product, denoted by $\boldsymbol{a} \circledast \boldsymbol{b}$, is a symmetric square matrix and named the *'wheel product'*. The second product, denoted by $\boldsymbol{a} \rtimes\!\!\!\ltimes \boldsymbol{b}$, is an antisymmetric square matrix and named the *'curl product'*. The third product, denoted by $\boldsymbol{a} \circ \boldsymbol{b}$, is a square matrix, neither symmetric nor antisymmetric in general, and named the *'full product'*.

Preparation. As a technical preparation, three univariate matrix products associated to a single vector $\boldsymbol{n} \in \mathbb{R}^3$ are introduced at first. The *'cross matrix'* $\boldsymbol{n}^{\otimes}$ is defined to indicate the antisymmetric matrix associated to a 3-vector by

$$\boldsymbol{n}^{\otimes} := \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} \in \mathsf{so}(3) \ . \tag{2.1}$$

In context of Lie theory, the cross matrix is understood $\boldsymbol{n}^{\otimes} = \mathrm{ad}(\boldsymbol{n})$ as the *adjoint representation* of the Lie algebra $(\mathbb{R}^3, \times)$ [17]. Two further symmetric product matrices associated to a vector $\boldsymbol{n}$ are introduced. The *'radial square'* $\boldsymbol{n}^{\circledcirc}$ and the *'axial square'* $\boldsymbol{n}^{\oslash}$ are defined as follows:

$$\boldsymbol{n}^{\circledcirc} := \boldsymbol{n}^{\otimes} * \boldsymbol{n}^{\otimes} = -(\boldsymbol{n}^{\otimes} \cdot \boldsymbol{n}^{\otimes}) = \begin{pmatrix} n_2^2 + n_3^2 & -n_1 n_2 & -n_1 n_3 \\ -n_1 n_2 & n_1^2 + n_3^2 & -n_2 n_3 \\ -n_1 n_3 & -n_2 n_3 & n_1^2 + n_2^2 \end{pmatrix} \in \mathsf{sym}(3) \tag{2.2}$$

$$\boldsymbol{n}^{\oslash} := (\boldsymbol{n} * \boldsymbol{n}) \cdot \boldsymbol{I} - \boldsymbol{n}^{\circledcirc} = \boldsymbol{n} \otimes \boldsymbol{n} = \begin{pmatrix} n_1^2 & n_1 n_2 & n_1 n_3 \\ n_1 n_2 & n_2^2 & n_2 n_3 \\ n_1 n_3 & n_2 a_3 & n_3^2 \end{pmatrix} \in \mathsf{sym}(3) \ . \tag{2.3}$$

The 'cross matrix' $\boldsymbol{n}^{\otimes}$ can be interpreted as an *'imaginary unit'* of the plane $\boldsymbol{n}^{\perp}$, or as a $\pi/2$-rotation around $\boldsymbol{n}$, via the formal congruence

$$\boldsymbol{n}^{\otimes} \cong \sqrt{-(\boldsymbol{n}^{\circledcirc})} \ ,$$

and the 'radial square' $\boldsymbol{n}^{\circledcirc}$ can be interpreted as a 0-rotation or as an *identity* of the plane $\boldsymbol{n}^{\perp}$. The significance of the univariate products for geometric vector operations is further illustrated in Appendix E.

Definitions. The *wheel product* of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is defined, in accordance with [4] and identical to Equation (1.1), as the matrix

$$\boldsymbol{a} \circledast \boldsymbol{b} = \mathrm{wheel}(\boldsymbol{a}, \boldsymbol{b}) := -(\boldsymbol{b}^{\otimes} \cdot \boldsymbol{a}^{\otimes} + \boldsymbol{a}^{\otimes} \cdot \boldsymbol{b}^{\otimes}) = \boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} + \boldsymbol{a}^{\otimes} * \boldsymbol{b}^{\otimes} \ . \tag{2.4}$$

The matrix of the wheel product is symmetric and has the entries

$$\boldsymbol{a} \circledast \boldsymbol{b} = \begin{pmatrix} 2 \cdot (a_2 b_2 + a_3 b_3) & -a_2 b_1 - a_1 b_2 & -a_3 b_1 - a_1 b_3 \\ -a_1 b_2 - a_2 b_1 & 2 \cdot (a_1 b_1 + a_3 b_3) & -a_3 b_2 - a_2 b_3 \\ -a_1 b_3 - a_3 b_1 & -a_2 b_3 - a_3 b_2 & 2 \cdot (a_1 b_1 + a_2 b_2) \end{pmatrix} \in \mathsf{sym}(3) \ , \tag{2.5}$$

the determinant is cubic and the trace is linear in terms of the entries of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Both are determined as

$$\begin{aligned} \det(\boldsymbol{a} \circledast \boldsymbol{b}) &= (\boldsymbol{a} * \boldsymbol{b}) \cdot \|\boldsymbol{a} \times \boldsymbol{b}\|^2 \\ \mathrm{trace}(\boldsymbol{a} \circledast \boldsymbol{b}) &= 4 \cdot (\boldsymbol{a} * \boldsymbol{b}) \ . \end{aligned} \tag{2.6}$$

By means of the 'radial square' from Equation (2.2), the wheel product is characterized via the 'quadratic' identities

$$\boldsymbol{a} \circledast \boldsymbol{b} = (\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} - \boldsymbol{a}^{\circledcirc} - \boldsymbol{b}^{\circledcirc} = \tfrac{1}{2} \cdot \big((\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} - (\boldsymbol{a} - \boldsymbol{b})^{\circledcirc} \big) \ ,$$

as matrix variant of the vector identities (Equation (A.3))

$$\boldsymbol{a} * \boldsymbol{b} = \tfrac{1}{2} \cdot \big(\|\boldsymbol{a} + \boldsymbol{b}\|^2 - \|\boldsymbol{a}\|^2 - \|\boldsymbol{b}\|^2 \big) = \tfrac{1}{4} \cdot \big(\|\boldsymbol{a} + \boldsymbol{b}\|^2 - \|\boldsymbol{b} - \boldsymbol{a}\|^2 \big) \ ,$$

Table 4. Six products in comparison: symbols, terminology, and references.

Term	*Label*	*Name*	*Ref.*
$\langle \boldsymbol{a} \vert \boldsymbol{b} \rangle$	I	inner product	(1.5)
$\boldsymbol{a} \wedge \boldsymbol{b}$	E	wedge product	(1.4)
$\boldsymbol{a}\boldsymbol{b}$	G	geometric product	(1.6)

Term	*Label*	*Name*	*Ref.*
$\boldsymbol{a} \circledast \boldsymbol{b}$	W	wheel product	(2.5)
$\boldsymbol{a} \rtimes \boldsymbol{b}$	C	curl product	(2.8)
$\boldsymbol{a} \circ \boldsymbol{b}$	F	full product	(2.10)

the law of cosines and the parallelogram rule.[9] The *curl product* of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is defined as the matrix

$$\boldsymbol{a} \rtimes \boldsymbol{b} = \operatorname{curl}(\boldsymbol{a}, \boldsymbol{b}) := +(\boldsymbol{a}^{\otimes} \cdot \boldsymbol{b}^{\otimes} - \boldsymbol{b}^{\otimes} \cdot \boldsymbol{a}^{\otimes}) = \boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} - \boldsymbol{a}^{\otimes} * \boldsymbol{b}^{\otimes} \; . \tag{2.7}$$

The matrix equals $\boldsymbol{a} \rtimes \boldsymbol{b} = (\boldsymbol{a} \times \boldsymbol{b})^{\otimes}$ and is antisymmetric[10]. The entries are

$$\boldsymbol{a} \rtimes \boldsymbol{b} = \begin{pmatrix} 0 & -(a_1 b_2 - a_2 b_1) & a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 & 0 & -(a_2 b_3 - a_3 b_2) \\ -(a_3 b_1 - a_1 b_3) & a_2 b_3 - a_3 b_2 & 0 \end{pmatrix} \in \mathsf{so}(3) \; , \tag{2.8}$$

the determinant is zero, and the $\operatorname{rank}(\boldsymbol{a} \rtimes \boldsymbol{b})$ equals two for independent $\boldsymbol{a}$ and $\boldsymbol{b}$. Finally, the *full product* of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ is defined as the matrix

$$\boldsymbol{a} \circ \boldsymbol{b} = \operatorname{full}(\boldsymbol{a}, \boldsymbol{b}) := -2 \cdot (\boldsymbol{b}^{\otimes} \cdot \boldsymbol{a}^{\otimes}) = +2 \cdot (\boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes}) \; . \tag{2.9}$$

The square matrix of the full product is neither symmetric nor antisymmetric and has the entries

$$\boldsymbol{a} \circ \boldsymbol{b} = 2 \cdot \begin{pmatrix} a_2 b_2 + a_3 b_3 & -a_1 b_2 & -a_1 b_3 \\ -a_2 b_1 & a_1 b_1 + a_3 b_3 & -a_2 b_3 \\ -a_3 b_1 & -a_3 b_2 & a_1 b_1 + a_2 b_2 \end{pmatrix} \in \mathsf{gl}(3) \; , \tag{2.10}$$

the determinant is zero and the rank equals two for independent $\boldsymbol{a}$ and $\boldsymbol{b}$. The trace of the matrix of the full product is $\operatorname{trace}(\boldsymbol{a} \circ \boldsymbol{b}) = 4 \cdot (\boldsymbol{a} * \boldsymbol{b})$, identical to the wheel product's trace.[11]

Interrelations. Obviously, the wheel product $\boldsymbol{a} \circledast \boldsymbol{b}$, the curl product $\boldsymbol{a} \rtimes \boldsymbol{b}$, and the full product $\boldsymbol{a} \circ \boldsymbol{b}$ fulfill the identity

$$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \circledast \boldsymbol{b} + \boldsymbol{a} \rtimes \boldsymbol{b} \; , \tag{2.11}$$

resembling the structure of the geometric product identity in Equation (1.6). As an outline of this observation, the three 'geometric algebra products' of Section 1 are opposed to the three 'matrix algebra products' of Section 2 in Table 4.

---

[9]Here, $\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) = \|\boldsymbol{b} - \boldsymbol{a}\|^2$ and $\|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 = \frac{1}{2} \cdot (\|\boldsymbol{a} + \boldsymbol{b}\|^2 + \|\boldsymbol{b} - \boldsymbol{a}\|^2)$.

[10]The application of $\boldsymbol{a} \rtimes \boldsymbol{b}$ on a vector $\boldsymbol{c}$ as a matrix-vector product $(\boldsymbol{a} \rtimes \boldsymbol{b}) \cdot \boldsymbol{c}$ matches the vector triple product $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$, characterized by the identities by Graßmann and Jacobi (Equation (E.9) and Equation (E.10)). The matrix-vector application is also considered in Section 3.3 and Section 4.

[11]With this property, the full product is closely related to the *Killing form*, a bilinear product for Lie algebras [28]: for the three-dimensional case with $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ the Killing form is $\operatorname{Killing}(\boldsymbol{a}^{\otimes}, \boldsymbol{b}^{\otimes}) = \operatorname{trace}\left(\boldsymbol{a}^{\otimes} \cdot \boldsymbol{b}^{\otimes}\right) = -2 \cdot \left(\boldsymbol{a} * \boldsymbol{b}\right)$. A related form for skew-symmetric matrices, defined as $\operatorname{sim}\left(\boldsymbol{a}^{\otimes}, \boldsymbol{b}^{\otimes}\right) := \boldsymbol{a} * \boldsymbol{b} = -\frac{1}{2} \cdot \operatorname{Killing}(\boldsymbol{a}, \boldsymbol{b})$, has been introduced [4] to measure the similarity of vectors via their adjoint representations.

Table 5. A summary of algebraic properties of the three matrix products. For the nonzero vectors $\boldsymbol{a} \neq \boldsymbol{0} \neq \boldsymbol{b}$, independence is assumed with $\boldsymbol{a} \times \boldsymbol{b} \neq \boldsymbol{0}$.

Name	*Wheel product*	*Curl product*	*Full product*	*Ref*
Symbol	$\boldsymbol{a} \circledast \boldsymbol{b}$	$\boldsymbol{a} \bowtie \boldsymbol{b}$	$\boldsymbol{a} \circ \boldsymbol{b}$	2.
Trace	$\mathrm{trace}(\boldsymbol{a} \circledast \boldsymbol{b}) = 4 \cdot (\boldsymbol{a} * \boldsymbol{b})$	$\mathrm{trace}(\boldsymbol{a} \bowtie \boldsymbol{b}) = 0$	$\mathrm{trace}(\boldsymbol{a} \circ \boldsymbol{b}) = 4 \cdot (\boldsymbol{a} * \boldsymbol{b})$	2.
Rank	$\mathrm{rank}(\boldsymbol{a} \circledast \boldsymbol{b}) = 3$	$\mathrm{rank}(\boldsymbol{a} \bowtie \boldsymbol{b}) = 2$	$\mathrm{rank}(\boldsymbol{a} \circ \boldsymbol{b}) = 2$	2.
Determ.	cubic in $\boldsymbol{a}$ and $\boldsymbol{b}$	$\det(\boldsymbol{a} \bowtie \boldsymbol{b}) = 0$	$\det(\boldsymbol{a} \circ \boldsymbol{b}) = 0$	(2.6)
Swap	$\boldsymbol{b} \circledast \boldsymbol{a} = \boldsymbol{a} \circledast \boldsymbol{b} = (\boldsymbol{a} \circledast \boldsymbol{b})^\top$	$\boldsymbol{b} \bowtie \boldsymbol{a} = -(\boldsymbol{a} \bowtie \boldsymbol{b}) = (\boldsymbol{a} \bowtie \boldsymbol{b})^\top$	$\boldsymbol{b} \circ \boldsymbol{a} = (\boldsymbol{a} \circ \boldsymbol{b})^\top$	(3.6)
Monov.	$\boldsymbol{a} \circledast \boldsymbol{a} = 2 \cdot \boldsymbol{a}^{\circledcirc}$	$\boldsymbol{a} \bowtie \boldsymbol{a} = 0 \cdot \boldsymbol{I}$	$\boldsymbol{a} \circ \boldsymbol{a} = 2 \cdot \boldsymbol{a}^{\circledcirc}$	T7
Lie alg.	$\mathsf{sym}(3)$	$\mathsf{so}(3)$	$\mathsf{gl}(3)$	2.

Due to the symmetry of the wheel product and antisymmetry of the curl product, Equation (2.11) is rephrased as the identity

$$2 \cdot (\boldsymbol{B} * \boldsymbol{A}) = \underbrace{\tfrac{2}{2} \cdot \big(\boldsymbol{B} * \boldsymbol{A} + \boldsymbol{A} * \boldsymbol{B}\big)}_{\text{symmetric}} + \underbrace{\tfrac{2}{2} \cdot \big(\boldsymbol{B} * \boldsymbol{A} - \boldsymbol{A} * \boldsymbol{B}\big)}_{\text{antisymmetric}} ,$$

in formal coherence to Equation (1.6) by letting the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ be defined by $\boldsymbol{A} := \boldsymbol{a}^{\otimes} \in \mathsf{so}(3)$ and $\boldsymbol{B} := \boldsymbol{b}^{\otimes} \in \mathsf{so}(3)$. Furthermore, the wheel product and the curl product are obtained from the full product by

$$\boldsymbol{a} \circledast \boldsymbol{b} = \tfrac{1}{2} \cdot (\boldsymbol{a} \circ \boldsymbol{b} + \boldsymbol{b} \circ \boldsymbol{a}) \qquad \boldsymbol{a} \bowtie \boldsymbol{b} = \tfrac{1}{2} \cdot (\boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{b} \circ \boldsymbol{a}) .$$

With $\boldsymbol{A}$ and $\boldsymbol{B}$, Equations (2.4), (2.7), and (2.9) are rephrased compactly as

$$\begin{aligned} \boldsymbol{a} \circledast \boldsymbol{b} &= \boldsymbol{B} * \boldsymbol{A} + \boldsymbol{A} * \boldsymbol{B} \\ \boldsymbol{a} \bowtie \boldsymbol{b} &= \boldsymbol{B} * \boldsymbol{A} - \boldsymbol{A} * \boldsymbol{B} \\ \boldsymbol{a} \circ \boldsymbol{b} &= 2 \cdot (\boldsymbol{B} * \boldsymbol{A}) . \end{aligned}$$

For the sake of brevity, fundamental algebraic properties of the three distinct matrix products are presented in compact form in Table 5.

3. Algebraic Context

In this section, the algebraic context of the matrix products of the previous Equations (2.4), (2.7), and (2.9) is approached from different directions. In Section 3.1, the correspondences of the matrix products with concepts of commutative and anticommutative products are indicated. In Section 3.2, the three matrix products are expressed via dyadic products. In Section 3.3, the full product is analyzed in more detail. Finally, overviews of interrelations are assembled in Section 3.4.

3.1. Symmetric and Antisymmetric

Dot Product and Wedge Product. The concepts of the three geometric algebra products of Section 1 are compared with the concepts of the three matrix algebra products of Section 2. The following correspondences are obtained for wheel product and for the curl product:

$$\boldsymbol{a} * \boldsymbol{b} \leftrightarrow \boldsymbol{a} \circledast \boldsymbol{b} \quad : \quad \boldsymbol{a} * \boldsymbol{b} = \tfrac{1}{4} \cdot \operatorname{trace}\big(\boldsymbol{a} \circledast \boldsymbol{b}\big) \tag{3.1}$$

$$\boldsymbol{a} \wedge \boldsymbol{b} \leftrightarrow \boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b} \quad : \quad \boldsymbol{a} \wedge \boldsymbol{b} = (\boldsymbol{a} \times \boldsymbol{b})^{\star} \mathrel{\widehat{=}} (\boldsymbol{a} \times \boldsymbol{b})^{\otimes} = \boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b} \tag{3.2}$$

For the geometric product $\boldsymbol{ab} = \langle \boldsymbol{a} | \boldsymbol{b} \rangle + \boldsymbol{a} \wedge \boldsymbol{b}$ of three-dimensional vectors (Equation (1.6)) the correspondences

$$\boldsymbol{ab} \leftrightarrow (\boldsymbol{a} \circledast \boldsymbol{b}, \boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b}) \quad : \quad \boldsymbol{ab} \mathrel{\widehat{=}} \tfrac{1}{4} \cdot \operatorname{trace}(\boldsymbol{a} \circledast \boldsymbol{b}) + \boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b}$$

$$\boldsymbol{ab} \leftrightarrow \boldsymbol{a} \circ \boldsymbol{b} \quad : \quad \boldsymbol{ab} \mathrel{\widehat{=}} \tfrac{1}{4} \cdot \operatorname{trace}(\boldsymbol{a} \circ \boldsymbol{b}) + \tfrac{1}{2} \cdot \big((\boldsymbol{a} \circ \boldsymbol{b}) - (\boldsymbol{a} \circ \boldsymbol{b})^{\top} \big)$$

$$\boldsymbol{ab} \leftrightarrow \boldsymbol{B} * \boldsymbol{A} \quad : \quad \boldsymbol{ab} \mathrel{\widehat{=}} \tfrac{1}{2} \cdot \operatorname{trace}(\boldsymbol{B} * \boldsymbol{A}) + \big((\boldsymbol{B} * \boldsymbol{A}) - (\boldsymbol{B} * \boldsymbol{A})^{\top} \big)$$

are obtained by letting the matrices $\boldsymbol{A} \coloneqq \boldsymbol{a}^{\otimes}$ and $\boldsymbol{B} \coloneqq \boldsymbol{b}^{\otimes}$.

Lie Product and Jordan Product. The antisymmetric curl product $\boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ equals the Lie bracket of their adjoint representation $\boldsymbol{A} = \boldsymbol{a}^{\otimes} = \operatorname{ad}(\boldsymbol{a}) \in \mathsf{so}(3)$ and $\boldsymbol{B} = \boldsymbol{b}^{\otimes} = \operatorname{ad}(\boldsymbol{b}) \in \mathsf{so}(3)$ with

$$\boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b} = (\boldsymbol{a} \times \boldsymbol{b})^{\otimes} = \big[\, \boldsymbol{A},\, \boldsymbol{B} \,\big] = \operatorname{Lie}(\boldsymbol{A}, \boldsymbol{B}) ,$$

where the Lie Bracket is defined [29] for antisymmetric matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{so}(d)$ as

$$\big[\, \boldsymbol{A},\, \boldsymbol{B} \,\big] = \operatorname{Lie}(\, \boldsymbol{A},\, \boldsymbol{B} \,) \coloneqq \boldsymbol{A} \cdot \boldsymbol{B} - \boldsymbol{B} \cdot \boldsymbol{A} ,$$

and also named the *commutator* of the two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. In contrast, the symmetric wheel product $\boldsymbol{a} \circledast \boldsymbol{b}$ matches – up to a constant factor of minus two – the so-called Jordan product of their adjoint matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathsf{so}(3)$

$$\boldsymbol{a} \circledast \boldsymbol{b} = -2 \cdot \operatorname{Jordan}(\boldsymbol{A}, \boldsymbol{B}) ,$$

that is defined [3, 24] for two square matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d \times d}$ as

$$\operatorname{Jordan}(\, \boldsymbol{A},\, \boldsymbol{B} \,) \coloneqq \tfrac{1}{2} \cdot (\boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B} \cdot \boldsymbol{A}) .$$

3.2. Dyadic Product

The dyadic product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, defined by $\boldsymbol{a} \otimes \boldsymbol{b} \coloneqq \boldsymbol{a} \cdot \boldsymbol{b}^{\top}$, is a bilinear tensor product (Equation (B.1) in Appendix B). It is a square matrix $\mathbb{R}^{d \times d}$ expressed as the weighted sum

$$\boldsymbol{a} \otimes \boldsymbol{b} = \sum_k \sum_l a_k \cdot b_l \cdot \big(\hat{\boldsymbol{e}}_k \otimes \hat{\boldsymbol{e}}_l\big)$$

via the dyads of the vector pairs of an orthonormal basis. For dimension $d = 3$, the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ permits to rephrase the three introduced matrix products by means of the identities

$$\boldsymbol{a} \circledast \boldsymbol{b} = 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - (\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) \tag{3.3}$$

$$\boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b} = \boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b} \tag{3.4}$$

$$\boldsymbol{a} \circ \boldsymbol{b} = 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - 2 \cdot (\boldsymbol{a} \otimes \boldsymbol{b}) . \tag{3.5}$$

With Equation (3.3), the symmetry of the matrix and commutativity of the wheel product can be directly deduced. Equation (3.4) reveals that the curl product is an antisymmetric matrix and an anticommutative product. Equation (3.5) indicates

Table 6. A tabular comparison of Equations (3.3), (3.4), and (3.5) together with rank information from Table 5.

			Diagonal	*Dyad*	*Dyad*
		Rank	3	1	1
	Rank	*Symbol*	$(\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I}$	$\boldsymbol{a} \otimes \boldsymbol{b}$	$\boldsymbol{b} \otimes \boldsymbol{a}$
Wheel product	3	$\boldsymbol{a} \circledast \boldsymbol{b}$	$+2$	-1	-1
Curl product	2	$\boldsymbol{a} \bowtie \boldsymbol{b}$	0	-1	$+1$
Full product	2	$\boldsymbol{a} \circ \boldsymbol{b}$	$+2$	-2	0

that the matrix of the full product is neither symmetric nor antisymmetric and the product is neither commutative nor anticommutative.

Since all three products are expressed in terms of dyadic products and a symmetric diagonal matrix with the scalar entry $\boldsymbol{a} * \boldsymbol{b}$, the property of 'transpositority' of the dyadic product, $\boldsymbol{b} \otimes \boldsymbol{a} = (\boldsymbol{a} \otimes \boldsymbol{b})^\top$, is also feasible for the three products

$$\boldsymbol{b} \diamond \boldsymbol{a} = (\boldsymbol{a} \diamond \boldsymbol{b})^\top \quad \text{for} \quad \diamond \in \left\{ \circledast, \bowtie, \circ \right\} . \tag{3.6}$$

As a reformulation of Equation (3.5), the dyadic product $\boldsymbol{a} \otimes \boldsymbol{b}$ is related to the full product via the equations

$$\begin{aligned} \boldsymbol{a} \otimes \boldsymbol{b} &= \big(\boldsymbol{a} * \boldsymbol{b}\big) \cdot \boldsymbol{I} - \tfrac{1}{2} \cdot \big(\boldsymbol{a} \circ \boldsymbol{b}\big) \\ &= \tfrac{1}{4} \cdot \operatorname{trace}(\boldsymbol{a} \circ \boldsymbol{b}) \cdot \boldsymbol{I} - \tfrac{1}{2} \cdot (\boldsymbol{a} \circ \boldsymbol{b}) . \end{aligned} \tag{3.7}$$

As another reformulation of Equation (3.5), the sum of the full product and the dyadic product is determined as

$$\big(\boldsymbol{a} * \boldsymbol{b}\big) \cdot \boldsymbol{I} = \tfrac{1}{2} \cdot \big(\boldsymbol{a} \circ \boldsymbol{b}\big) + \big(\boldsymbol{a} \otimes \boldsymbol{b}\big) ,$$

which simplifies for identical arguments $\boldsymbol{a} = \boldsymbol{b}$, according to Equation (2.3), to

$$\big(\boldsymbol{a} * \boldsymbol{a}\big) \cdot \boldsymbol{I} = \tfrac{1}{2} \cdot \big(\boldsymbol{a} \circ \boldsymbol{a}\big) + \big(\boldsymbol{a} \otimes \boldsymbol{a}\big) = \boldsymbol{a}^{\otimes} * \boldsymbol{a}^{\otimes} + \boldsymbol{a}^{\oslash} = \boldsymbol{a}^{\circledcirc} + \boldsymbol{a}^{\oslash} .$$

In Table 6, Equations (3.3) to (3.5) are outlined together with the rank information from Table 5.

3.3. Full Product

Invertibility. In contrast to inner product and wedge product, the geometric product of a geometric algebra has the particular property of *invertibility.* Given a geometric product $\boldsymbol{ab}$ and $\boldsymbol{b}$, the vector $\boldsymbol{a}$ can be reconstructed, given $\boldsymbol{ab}$ and $\boldsymbol{a}$, the vector $\boldsymbol{b}$ can be reconstructed [30]. Formally, the reconstructions are obtained by multiplications with the inverses $\boldsymbol{b}^{-1}$ and $\boldsymbol{a}^{-1}$ (Equation (1.7)):

$$\begin{array}{ccccccccc} \boldsymbol{a} & = & \boldsymbol{a}_\parallel + \boldsymbol{a}_\perp & = & \langle \boldsymbol{a} | \boldsymbol{b} \rangle \boldsymbol{b}^{-1} & + & (\boldsymbol{a} \wedge \boldsymbol{b}) \boldsymbol{b}^{-1} & = & (\boldsymbol{ab}) \boldsymbol{b}^{-1} \\ \boldsymbol{b} & = & \boldsymbol{b}_\parallel + \boldsymbol{b}_\perp & = & \boldsymbol{a}^{-1} \langle \boldsymbol{a} | \boldsymbol{b} \rangle & + & \boldsymbol{a}^{-1} (\boldsymbol{a} \wedge \boldsymbol{b}) & = & \boldsymbol{a}^{-1} (\boldsymbol{ab}) \end{array} \tag{3.8}$$

A matrix-vector computation is stated in Equation (E.8). If, in place of the geometric product $\boldsymbol{ab}$, the full product $\boldsymbol{a} \circ \boldsymbol{b}$ is given as primary information, one

Table 7. Comparison of nine product terms for two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, once for the independent case $\boldsymbol{a} \not\parallel \boldsymbol{b}$ and once for the limiting case $\boldsymbol{a} \rightarrow \boldsymbol{b}$. The nine products are assorted to three categories.

Category	*Name*	$\boldsymbol{a} \not\parallel \boldsymbol{b}$	$\rightarrow$	$\boldsymbol{a} = \boldsymbol{b}$
Geometric algebra	scalar product	$\langle \boldsymbol{a} \vert \boldsymbol{b} \rangle$	$\rightarrow$	$\Vert \boldsymbol{a} \Vert^2$
	wedge product	$\boldsymbol{a} \wedge \boldsymbol{b}$	$\rightarrow$	0
	geometric product	$\boldsymbol{a}\boldsymbol{b}$	$\rightarrow$	$\Vert \boldsymbol{a} \Vert^2$
Matrix algebra	wheel product	$\boldsymbol{a} \circledast \boldsymbol{b}$	$\rightarrow$	$2 \cdot \boldsymbol{a}^{\circledcirc}$
	curl product	$\boldsymbol{a} \rtimes \boldsymbol{b}$	$\rightarrow$	$0 \cdot \boldsymbol{I}$
	full product	$\boldsymbol{a} \circ \boldsymbol{b}$	$\rightarrow$	$2 \cdot \boldsymbol{a}^{\circledcirc}$
Classic concepts	dot product	$\boldsymbol{a} * \boldsymbol{b}$	$\rightarrow$	$\boldsymbol{a} * \boldsymbol{a}$
	cross product	$\boldsymbol{a} \times \boldsymbol{b}$	$\rightarrow$	$0 \cdot \boldsymbol{1}$
	dyadic product	$\boldsymbol{a} \otimes \boldsymbol{b}$	$\rightarrow$	$\boldsymbol{a}^{\oslash}$

vector, given the other of a pair $(\boldsymbol{a}, \boldsymbol{b})$, is inversely obtained by means of these identities:

$$\boldsymbol{a} = \boldsymbol{a}_{\parallel} + \boldsymbol{a}_{\perp} = \tfrac{1}{4} \cdot \operatorname{trace}(\boldsymbol{a} \circledast \boldsymbol{b}) \cdot \boldsymbol{b}^{-1} - (\boldsymbol{a} \rtimes \boldsymbol{b}) \cdot \boldsymbol{b}^{-1}$$
$$\boldsymbol{b} = \boldsymbol{b}_{\parallel} + \boldsymbol{b}_{\perp} = \tfrac{1}{4} \cdot \operatorname{trace}(\boldsymbol{a} \circledast \boldsymbol{b}) \cdot \boldsymbol{a}^{-1} + (\boldsymbol{a} \rtimes \boldsymbol{b}) \cdot \boldsymbol{a}^{-1}$$

Associativity. One of the axioms for a geometric product is the *associativity* $A(BC) = (AB)C$, as stated in Section 1. For the case of the full product, the analogue requirement, formally stated as

$$\boldsymbol{a} \circ (\boldsymbol{b} \circ \boldsymbol{c}) = (\boldsymbol{a} \circ \boldsymbol{b}) \circ \boldsymbol{c} \tag{3.9}$$

is not defined: the operator of the full product maps two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ into a square matrix, as $\circ : (\mathbb{R}^3, \mathbb{R}^3) \rightarrow \mathbb{R}^{3\times3}$. The outer operators in Equation (3.9) then require to multiply vectors and matrices, which is not compatible with the definition of the full product. Still, by means of the conventions

$$(\boldsymbol{a} \circ \boldsymbol{b}) \circ \boldsymbol{c} := (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c} - (\boldsymbol{a} \rtimes \boldsymbol{b}) \cdot \boldsymbol{c}$$
$$\boldsymbol{a} \circ (\boldsymbol{b} \circ \boldsymbol{c}) := (\boldsymbol{b} * \boldsymbol{c}) \cdot \boldsymbol{a} + (\boldsymbol{b} \rtimes \boldsymbol{c}) \cdot \boldsymbol{a}$$

an associativity for vector terms is obtained. The evaluation of the first expression

$$\begin{aligned}(\boldsymbol{a} \circ \boldsymbol{b}) \circ \boldsymbol{c} &= (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c} - (\boldsymbol{a} \rtimes \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c} - (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} \\ &= (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c} - (\boldsymbol{a} * \boldsymbol{c}) \cdot \boldsymbol{b} + (\boldsymbol{b} * \boldsymbol{c}) \cdot \boldsymbol{a}\end{aligned}$$

then equals the evaluation of the second expression

$$\boldsymbol{a} \circ (\boldsymbol{b} \circ \boldsymbol{c}) = (\boldsymbol{b} * \boldsymbol{c}) \cdot \boldsymbol{a} + (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} = (\boldsymbol{b} * \boldsymbol{c}) \cdot \boldsymbol{a} - (\boldsymbol{a} * \boldsymbol{c}) \cdot \boldsymbol{b} + (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{c}\,.$$

The Graßmann identities for vector triple products are stated in Equation (E.9).

3.4. Summary

The three matrix products introduced in Section 2 are opposed to inner product, wedge product, and geometric product – as concepts from geometric algebra – as well as to the dyadic product – as a concept of vector calculus – in Section 3.1 and Section 3.2. In Section 3.3, the full product is compared with the concept of the

Table 8. A summary of interrelations of six products and their properties.

	First summand		*Second summand*		*Compound sum*	*Ref*
Multilinear algebra	0-blade		2-blade		2-multivector	T3
Algebraic element	scalar		bivector		formal sum	T3
Geometric algebra	inner product		wedge product		geometric product	T4
Mapping properties	commutative		anticommutative		associative	
Blade identity	$\langle \boldsymbol{a} \| \boldsymbol{b} \rangle$	$+$	$\boldsymbol{a} \wedge \boldsymbol{b}$	$=$	$\boldsymbol{ab}$	(1.6)
Relation	$\boldsymbol{a} * \boldsymbol{b} = \frac{1}{4} \cdot \mathrm{trace}(\boldsymbol{a} \circledast \boldsymbol{b})$		$\boldsymbol{a} \wedge \boldsymbol{b} \,\widehat{=}\, \boldsymbol{a} \bowtie \boldsymbol{b}$		—	(3.1, 3.2)
Matrix identity	$\boldsymbol{a} \circledast \boldsymbol{b}$	$+$	$\boldsymbol{a} \bowtie \boldsymbol{b}$	$=$	$\boldsymbol{a} \circ \boldsymbol{b}$	(2.11)
Matrix properties	symmetry		antisymmetry		bilinearity	(3.6)
Matrix algebra	wheel product		curl product		full product	T4
Abstract algebra	$-2 \cdot$ Jordan product		Lie product		—	3.1.

geometric product. An outline of the different concepts is provided by ordering nine products for vector pairs in three categories in Table 7.

Another outline of the connections of the three matrix products to related concepts is given in Table 8. In the central block of the table, correspondences between geometric algebra and matrix algebra are shown. In the upper block, information on geometric algebra of Table 3 is summarized. In the lower block, connections to abstract algebra are indicated.

A further conclusion can be drawn by observing the identity of the wheel product of Equation (1.1) with the generating set of the ideal used for defining a geometric algebra as a quotient algebra of the tensor algebra (Equation (B.3) in Appendix B). This identity yields a direct argument for the fact that, while the curl product $\boldsymbol{a} \bowtie \boldsymbol{b} = \boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b} \in \mathsf{so}(3)$ is a matrix representation for the bivector $\boldsymbol{a} \wedge \boldsymbol{b}$ (Equation (3.2) and Table 12), which is an element of the geometric algebra $\mathsf{C}_{3,0,0} \cong \mathsf{G}(\mathbb{R}^3, *)$), the wheel product $\boldsymbol{a} \circledast \boldsymbol{b} \in \mathsf{sym}(3)$ does not provide a matrix instance for an element of the geometric algebra, but for the element $2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - (\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})$ of the tensor algebra $\mathsf{T}(\mathbb{R}^3)$ (Equation (3.3) and Table 11). In consequence, the matrix of the full product

$$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \circledast \boldsymbol{b} + \boldsymbol{a} \bowtie \boldsymbol{b} = 2 \cdot (\boldsymbol{a} * \boldsymbol{b} \cdot \boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{b}) = -2 \cdot (\boldsymbol{b}^{\otimes} \cdot \boldsymbol{a}^{\otimes}) \in \mathsf{gl}(3)$$

represents an element of the tensor algebra $\mathsf{T}(\mathbb{R}^3)$ and corresponds to the matrix $\boldsymbol{a} \bowtie \boldsymbol{b} = \left[\boldsymbol{a}^{\otimes}, \boldsymbol{b}^{\otimes} \right] \in \mathsf{so}(3)$ representing the bivector element $\boldsymbol{a} \wedge \boldsymbol{b}$ in the geometric algebra $\mathsf{G}(\mathbb{R}^3, *)$. This observation is resembled by the correspondence between the identity $\mathsf{so}(3) = \mathsf{gl}(3)/\mathsf{sym}(3)$ for matrix algebras – exemplified by the equation $\boldsymbol{a} \bowtie \boldsymbol{b} = \boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{a} \circledast \boldsymbol{b}$ for matrix instances for a pair of vectors – and the definition $\mathsf{G}(\mathbb{R}^3) = \mathsf{T}(\mathbb{R}^3)/((\, \boldsymbol{a} \circledast \boldsymbol{b} \,))$ of a geometric algebra in Equation (B.3).

4. Geometric Context

In this section, the three matrix products, introduced in Section 2, are analyzed from a geometric point of view. The images of an orthogonal basis under the three matrices are considered in Section 4.1 for this purpose. In Section 4.2, the eigenvalues and eigenvectors of the three matrices are documented.

4.1. Images of an Orthogonal Basis

For illustrating the geometry of the matrix products, they are considered as linear mappings between three-dimensional spaces $\mathbb{R}^3$. For each of the three product matrices, the wheel product $\boldsymbol{a} \circledast \boldsymbol{b}$, the curl product $\boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b}$, and the full product $\boldsymbol{a} \circ \boldsymbol{b}$, we assume the basis of the vector spaces $\mathbb{R}^3$ given by the three independent vectors $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b}\}$. Thus, the images of the basis vectors are obtained by sparse matrix-vector multiplications. They further provide insight into the geometric properties of the three product matrices in terms of the given vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Preparation. As introduced in Equation (E.3) in the appendix, the rejection identities are stated as a technical tool

$$\tau_{\boldsymbol{n}}(\boldsymbol{x}) = \boldsymbol{n}^{\circledcirc} \cdot \boldsymbol{x} = \boldsymbol{n} \times \boldsymbol{x} \times \boldsymbol{n} \ , \tag{4.1}$$

by means of the 'radial square' of Equation (2.2).

Wheel product. The wheel product $\boldsymbol{a} \circledast \boldsymbol{b}$ with rank three (Table 5) is a full-dimensional automorphism of $\mathbb{R}^3$ for independent $\boldsymbol{a}$ and $\boldsymbol{b}$. For convenience, let the column representation of the matrix of the wheel product be denoted by

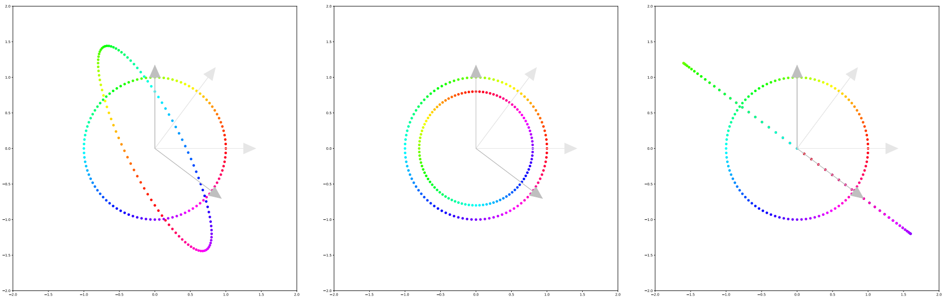
$$\boldsymbol{W} = \left(\, \boldsymbol{w}_1 \, \middle| \, \boldsymbol{w}_2 \, \middle| \, \boldsymbol{w}_3 \, \right) := \boldsymbol{a} \circledast \boldsymbol{b} \ .$$

It is straightforward to analyze the image of the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b}\}$ under the automorphism $W : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{p} \mapsto \boldsymbol{W} \cdot \boldsymbol{p}$ of the wheel product that reads

$$W : \mathbb{R}^3 \to \mathbb{R}^3 \ : \quad \left\{ \begin{array}{ccccc} \boldsymbol{a} & \longmapsto & \boldsymbol{W} \cdot \boldsymbol{a} & = & -\tau_{\boldsymbol{a}}(\boldsymbol{b}) \\ \boldsymbol{b} & \longmapsto & \boldsymbol{W} \cdot \boldsymbol{b} & = & -\tau_{\boldsymbol{b}}(\boldsymbol{a}) \\ \boldsymbol{a} \times \boldsymbol{b} & \longmapsto & \boldsymbol{W} \cdot (\boldsymbol{a} \times \boldsymbol{b}) & = & 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) \ , \end{array} \right. \tag{4.2}$$

employing the rejection map τ of Equation (4.1) twice. The image of the mapping W is illustrated in the plane $\operatorname{span}(\boldsymbol{a}, \boldsymbol{b}) \cong (\boldsymbol{a} \times \boldsymbol{b})^{\perp}$ by indicating the plane's unit circle and its image in colors encoding the argument of a point on a unit circle. The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are indicated in light gray in Figure 1a. Within the plane $(\boldsymbol{a} \times \boldsymbol{b})^{\perp}$, the linear map W distorts the unit circle to an ellipse with principal axes along $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$. In coherence with this interpretation, the vectors $\tau_{\boldsymbol{a}}(\boldsymbol{b})$ and $\tau_{\boldsymbol{b}}(\boldsymbol{a})$ (indicated in dark gray in Figure 1a) span the plane symmetrically to the ellipse. In addition to the distortion, the image in the plane $(\boldsymbol{a} \times \boldsymbol{b})^{\perp}$ is characterized by a linear scaling including the factor of $\sin(\vartheta_{ab})$ via Equation (E.6). The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is invariant under W but is determined by a scaling including the factor $\cos(\vartheta_{ab})$. Since the determinant of $\boldsymbol{W}$ can be negative, the orientation of the ellipse can be opposite to the orientation of the circle.

Curl product. The curl product $\boldsymbol{a} \rtimes\!\!\ltimes \boldsymbol{b}$ with deficient rank two is an endomorphism of $\mathbb{R}^3$ with kernel of dimension one along the direction $\boldsymbol{a} \times \boldsymbol{b}$. For convenience, let the column representation of the matrix of the curl product be



(a) Image of wheel product. **(b)** Image of curl product. **(c)** Image of full product.

Figure 1. Image of the unit circle in the plane $(\boldsymbol{a} \times \boldsymbol{b})^\perp$ under the matrices $\boldsymbol{W}$, $\boldsymbol{C}$, and $\boldsymbol{F}$ of wheel product, curl product, and full product for vectors $\boldsymbol{a} = (1,0,0)^\top$ and $\boldsymbol{b} = \frac{1}{5} \cdot (4,3,0)^\top$. For the sake of readability, the arrows of $\boldsymbol{a}$ and $\boldsymbol{b}$ and the $\tau_{\boldsymbol{a}}(\boldsymbol{b})$ and $\tau_{\boldsymbol{b}}(\boldsymbol{a})$ are stretched by the factor 5/4.

denoted by

$$\boldsymbol{C} = \left(\boldsymbol{c}_1 \,\middle|\, \boldsymbol{c}_2 \,\middle|\, \boldsymbol{c}_3 \right) := \boldsymbol{a} \ltimes \boldsymbol{b} \,,$$

and indicate the rank deficiency by

$$\boldsymbol{C} \subset (\boldsymbol{a} \times \boldsymbol{b})^\perp \quad \cong \quad \boldsymbol{c}_i \in (\boldsymbol{a} \times \boldsymbol{b})^\perp \quad \text{for} \quad i = 1,2,3 \,.$$

It is straightforward to analyze the image of the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b}\}$ under the endomorphism $C : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{p} \mapsto \boldsymbol{C} \cdot \boldsymbol{p}$ of the curl product which reads

$$C : \mathbb{R}^3 \to \mathbb{R}^3 \;:\; \begin{cases} \boldsymbol{a} & \longmapsto & \boldsymbol{C} \cdot \boldsymbol{a} & = & +\tau_{\boldsymbol{a}}(\boldsymbol{b}) \\ \boldsymbol{b} & \longmapsto & \boldsymbol{C} \cdot \boldsymbol{b} & = & -\tau_{\boldsymbol{b}}(\boldsymbol{a}) \\ \boldsymbol{a} \times \boldsymbol{b} & \longmapsto & \boldsymbol{C} \cdot (\boldsymbol{a} \times \boldsymbol{b}) & = & \boldsymbol{0} \,, \end{cases} \tag{4.3}$$

employing the rejection map τ of Equation (4.1) twice with opposite signs. The image of the unit circle under C is indicated in Figure 1b. The image is characterized as another circle that is rotated by $\pi/2$ and scaled proportional to the factor of $\sin(\vartheta_{ab})$.

Full product. The full product $\boldsymbol{a} \circ \boldsymbol{b}$ of Equation (2.10) with deficient rank two is an endomorphism of $\mathbb{R}^3$ with kernel of dimension one along the direction $\boldsymbol{b}$. For convenience, let the column representation of the matrix of the full product be denoted by

$$\boldsymbol{F} = \left(\boldsymbol{f}_1 \,\middle|\, \boldsymbol{f}_2 \,\middle|\, \boldsymbol{f}_3 \right) := \boldsymbol{a} \circ \boldsymbol{b} \,,$$

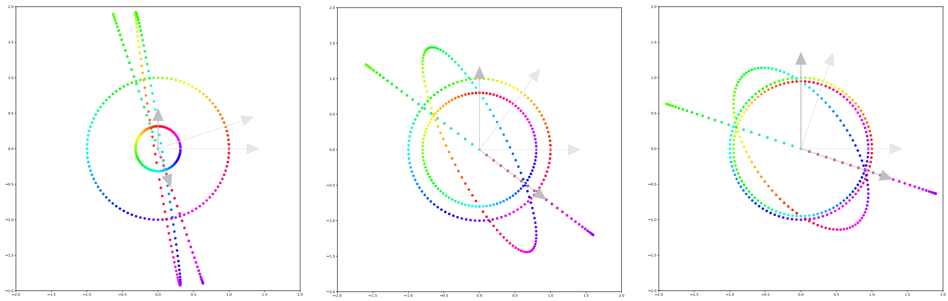
and indicate the rank deficiency by

$$\boldsymbol{F} \subset \boldsymbol{b}^\perp \quad \cong \quad \boldsymbol{f}_i \in \boldsymbol{b}^\perp \quad \text{for} \quad i = 1,2,3 \,.$$

It is straightforward to analyze the image of the basis $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b}\}$ under the endomorphism $F : \mathbb{R}^3 \to \mathbb{R}^3, \boldsymbol{p} \mapsto \boldsymbol{F} \cdot \boldsymbol{p}$ of the full product which reads

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \;:\; \begin{cases} \boldsymbol{a} & \longmapsto & \boldsymbol{F} \cdot \boldsymbol{a} & = & \boldsymbol{0} \\ \boldsymbol{b} & \longmapsto & \boldsymbol{F} \cdot \boldsymbol{b} & = & -2 \cdot \tau_{\boldsymbol{b}}(\boldsymbol{a}) \\ \boldsymbol{a} \times \boldsymbol{b} & \longmapsto & \boldsymbol{F} \cdot (\boldsymbol{a} \times \boldsymbol{b}) & = & 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot (\boldsymbol{a} \times \boldsymbol{b}) \end{cases}$$

as the sum of the images of W of Equation (4.2) and C of Equation (4.3), thus employing the rejection map τ of Equation (4.1) once. The image of the unit circle



(a) Almost collinear vectors with $\boldsymbol{b} = (1/\sqrt{10}, 3/\sqrt{10}, 0)^\top$.

(b) Plots identical to Figure 1 with $\boldsymbol{b} = (4/5, 3/5, 0)^\top$.

(c) Almost orthogonal vectors with $\boldsymbol{b} = (3/\sqrt{10}, 1/\sqrt{10}, 0)^\top$.

Figure 2. Overlay of the images of the wheel product, the curl product, and the full product in a common coordinate system for different vectors $\boldsymbol{b}$.

of the plane $(\boldsymbol{a} \times \boldsymbol{b})^\perp$ is illustrated in Figure 1c. The planar circle is mapped onto a finite segment of the linear subspace $\mathrm{span}(\tau_{\boldsymbol{b}}(\boldsymbol{a})) \subset \boldsymbol{b}^\perp$. With Equation (E.6), the distance of the mapped point from the origin depends linearly on $\sin(\vartheta_{ab})$. The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is scaled linearly with $\cos(\vartheta_{ab}) \propto \boldsymbol{a} * \boldsymbol{b}$.

Summary. In the central plot of Figure 2, the illustrations of Figure 1 are combined in one coordinate system. The plots on the left and on the right of Figure 2 illustrate analogue constructions for different pairs of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. On the left, the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ approach almost collinear directions in Figure 2a; on the right, the vectors approach almost perpendicular directions in Figure 2c. The characteristics of the linear maps of the matrices $\boldsymbol{W}$, $\boldsymbol{C}$, and $\boldsymbol{F}$ as discussed before are summarized in Table 9. For convenience, the image is characterized separately for the normal direction $\boldsymbol{a} \times \boldsymbol{b}$, for the plane $(\boldsymbol{a} \times \boldsymbol{b})^\perp$, and for image of the unit circle in that plane.

Table 9. Description of the geometric effects of the three products.

	Wheel product	*Curl product*	*Full product*
Matrix rank	$\mathrm{rank}(\boldsymbol{W}) = 3$	$\mathrm{rank}(\boldsymbol{C}) = 2$	$\mathrm{rank}(\boldsymbol{F}) = 2$
Normal $\mathrm{span}(\boldsymbol{a} \times \boldsymbol{b})$	$\cos(\vartheta_{ab})$-scale	$(\boldsymbol{a} \times \boldsymbol{b})^\perp$-projection	$\cos(\vartheta_{ab})$-scale
Plane $\mathrm{span}(\boldsymbol{a}, \boldsymbol{b})$	$\sin(\vartheta_{ab})$-scale orthogonal scale	$\sin(\vartheta_{ab})$-scale $(\pi/2)$-turn	$\sin(\vartheta_{ab})$-scale $\tau_{\boldsymbol{b}}(\boldsymbol{a})$-filter
Unit circle image	$(\boldsymbol{a} \pm \boldsymbol{b})$-ellipse	$(\pi/2)$-circle	$\mathrm{span}(\tau_{\boldsymbol{b}}(\boldsymbol{a}))$-segment

4.2. Eigenvalues and Eigenvectors

For the sake of brevity, the constraints of unit vectors $\|\boldsymbol{a}\| = \|\boldsymbol{b}\| = 1$ and linear independence $|\boldsymbol{a} * \boldsymbol{b}| \neq 1$ are assumed to document the eigenvalues and eigenvectors of the three linear maps. The matrix of the wheel product $\boldsymbol{W} = \boldsymbol{a} \circledast \boldsymbol{b}$ has the

Table 10. Overview of products in matrix and scalar form, associated to linear independent vectors. The product matrices associated to a single vector in the table rows of 'univariate-linear' order (1L) and 'univariate-quadratic' order (1Q) are introduced in the beginning of Section 2 and treated in more detail in Appendix E. The case 'bivariate-linear' products of order (2L) contains two subrows in the table. The full product of Equation (2.11) is a rank-two tensor and the dyadic product of Equation (3.7) is a rank-one tensor.

Blade			*Matrix product terms*		*Scalar product terms*	
Grade	*Tuple*	*Rep.*	*Order*	*Matrix*	*Scalar*	*Geom.*
3	triple $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$	$\boldsymbol{X}$	3Q	$\boldsymbol{X} * \boldsymbol{X}$	$\det(\boldsymbol{X} * \boldsymbol{X})$	$(\mathrm{vol})^2$
			3L	$\boldsymbol{X}$	$\det(\boldsymbol{X}) = (\boldsymbol{a} \times \boldsymbol{b}) * \boldsymbol{c}$	vol
2	pair $\{\boldsymbol{a}, \boldsymbol{b}\}$	$\boldsymbol{Y}$	2Q	$\boldsymbol{Y} * \boldsymbol{Y}$	$\det(\boldsymbol{Y} * \boldsymbol{Y})$	$(\mathrm{vol})^2$
			2L	$\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{a} \circledast \boldsymbol{b} + \boldsymbol{a} \rtimes \boldsymbol{b}$	$\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert$	$\lvert \mathrm{vol} \rvert$
				$\boldsymbol{a} \otimes \boldsymbol{b} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - \frac{1}{2} \cdot (\boldsymbol{a} \circ \boldsymbol{b})$	$\boldsymbol{a} * \boldsymbol{b}$	sim
1	single $\{\boldsymbol{a}\}$	$\boldsymbol{a}$	1Q	$\boldsymbol{a}^{\oslash} = \lVert \boldsymbol{a} \rVert^2 \cdot \boldsymbol{I} - \boldsymbol{a}^{\circledcirc}$	$\lVert \boldsymbol{a} \rVert^2 = \boldsymbol{a} * \boldsymbol{a}$	$(\mathrm{vol})^2$
			1L	$\boldsymbol{a}^{\otimes} \cong \sqrt{-\boldsymbol{a}^{\circledcirc}}$	$\lVert \boldsymbol{a} \rVert = \sqrt{\boldsymbol{a} * \boldsymbol{a}}$	$\lvert \mathrm{vol} \rvert$

eigenvalues

$$\lambda_1 = 2 \cdot \boldsymbol{a} * \boldsymbol{b}, \qquad \lambda_2 = \boldsymbol{a} * \boldsymbol{b} - 1, \qquad \lambda_3 = \boldsymbol{a} * \boldsymbol{b} + 1$$

with eigenvectors $\boldsymbol{v}_1 = \boldsymbol{a} \times \boldsymbol{b}$, as well as $\boldsymbol{v}_2 = \boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{v}_3 = \boldsymbol{a} - \boldsymbol{b}$. The matrix of the curl product $\boldsymbol{C} = \boldsymbol{a} \rtimes \boldsymbol{b}$ has the eigenvalues

$$\lambda_1 = 0, \qquad \lambda_2 = +\mathrm{i}, \qquad \lambda_3 = -\mathrm{i}$$

with eigenvectors $\boldsymbol{v}_1 = \boldsymbol{a} \times \boldsymbol{b}$, as well as $\boldsymbol{v}_2$ complex and $\boldsymbol{v}_3$ complex. The matrix of the full product $\boldsymbol{F} = \boldsymbol{a} \circ \boldsymbol{b}$ has the eigenvalues

$$\lambda_1 = 0, \qquad \lambda_2 = 2 \cdot \boldsymbol{a} * \boldsymbol{b}, \qquad \lambda_3 = 2 \cdot \boldsymbol{a} * \boldsymbol{b}$$

with eigenvectors $\boldsymbol{v}_2 = \boldsymbol{a}$, as well as $\boldsymbol{v}_2 \in \boldsymbol{b}^{\perp}$ and $\boldsymbol{v}_3 \in \boldsymbol{b}^{\perp}$.

5. More Products

By means of Table 10, full product $\boldsymbol{a} \circ \boldsymbol{b}$, wheel product $\boldsymbol{a} \circledast \boldsymbol{b}$, and curl product $\boldsymbol{a} \rtimes \boldsymbol{b}$ from Section 2 are set into a broader context of scalar products and matrix products. As indicated by the order (2L), wheel product, curl product, and full product, as well as the dyadic product all are bilinear matrix products. Next to these matrix products, the scalar product terms $\boldsymbol{a} * \boldsymbol{b}$ and $\lVert \boldsymbol{a} \times \boldsymbol{b} \rVert$ that depend linearly on the two arguments $\boldsymbol{a}$ and $\boldsymbol{b}$ are listed. Above, for the order (2Q), quadratic bivariate product terms in matrix form, the Gramian matrix, and in scalar form, the Gramian determinant are stated. Similarly the table states products terms for a triple of vectors in the rows of order (3L) and (3Q). The rows of order (1L) and (1Q) contain matrix and scalar product terms associated to a single vector.

6. Conclusions

The article demonstrates that the geometric algebra concepts of inner product, wedge product, and geometric product can be rendered for the case of three-dimensional space by means of matrices, named as wheel product, curl product, and full product, in a uniform way. All three matrix products are expressible as linear combinations of dyadic products and a diagonal matrix. In particular, the properties of symmetry, antisymmetry, transpositority, and associativity are compared. Relations of the matrix products to the Jordan product, the Lie bracket, and the Killing form, as well as to the concept of a geometric algebra as a quotient of a tensor algebra are indicated. The geometry of the three matrix products is analyzed and illustrated by examples. The context of the bivariate matrix products with univariate and trivariate products of linear and quadratic order is outlined. In the future it will be interesting to see if more geometric use cases or algebraic generalizations can be identified for the defined matrix products. The presented connections may contribute to link geometric algebra closer with other disciplines of theoretical and applied research.

Appendix A. Bilinear and Quadratic Forms

Let $\mathbb{V}$ denote a $\mathbb{R}$-vector space of finite dimension d, such that $\mathbb{V} = \mathbb{R}^d$ can be assumed. A bivariate map $\langle \cdot | \cdot \rangle : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{R}$ is called a *symmetric bilinear form* if, for all $\lambda \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{V}$, the map $\langle \cdot | \cdot \rangle$ fulfills

$$\begin{aligned} \langle \boldsymbol{a} | \boldsymbol{b} \rangle &= \langle \boldsymbol{b} | \boldsymbol{a} \rangle \\ \langle \lambda \cdot \boldsymbol{a} | \boldsymbol{b} \rangle &= \lambda \cdot \langle \boldsymbol{a} | \boldsymbol{b} \rangle \\ \langle \boldsymbol{a} + \boldsymbol{b} | \boldsymbol{c} \rangle &= \langle \boldsymbol{a} | \boldsymbol{c} \rangle + \langle \boldsymbol{b} | \boldsymbol{c} \rangle \ . \end{aligned} \tag{A.1}$$

A univariate map $Q : \mathbb{V} \longrightarrow \mathbb{R}$ is called a *quadratic form* if, for all $\lambda \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}$, the map Q fulfills

$$\begin{aligned} Q(\lambda \cdot \boldsymbol{a}) &= \lambda^2 \cdot Q(\boldsymbol{a}) \\ Q(\boldsymbol{a} + \boldsymbol{b}) + Q(\boldsymbol{b} - \boldsymbol{a}) &= 2 \cdot \big(Q(\boldsymbol{a}) + Q(\boldsymbol{b}) \big) \ . \end{aligned} \tag{A.2}$$

The connections between the bilinear form of Equation (A.1) and the quadratic form of Equation (A.2) are given via

$$\begin{aligned} \langle \boldsymbol{a} | \boldsymbol{b} \rangle &:= \frac{1}{2} \cdot \big(Q(\boldsymbol{a} + \boldsymbol{b}) - Q(\boldsymbol{a}) - Q(\boldsymbol{b}) \big) \\ Q(\boldsymbol{a}) &:= \langle \boldsymbol{a} | \boldsymbol{a} \rangle. \end{aligned} \tag{A.3}$$

For the positive-definite standard product $\langle \boldsymbol{a} | \boldsymbol{b} \rangle = \boldsymbol{a} * \boldsymbol{b}$, the quadratic form equals $Q(\boldsymbol{a}) = \boldsymbol{a} * \boldsymbol{a}$, matching the usual two-norm via $\sqrt{Q(\boldsymbol{a})} = \|\boldsymbol{a}\|_2 = \sqrt{\sum_i a_i^2}$.

Appendix B. Tensor, Exterior, and Geometric Algebras

The *tensor algebra* associated to a d-dimensional $\mathbb{R}$-vector space $\mathbb{V} \cong \mathbb{R}^d$ is defined [25] as the direct sum of all k-tensor powers

$$\begin{aligned}\mathsf{T}(\mathbb{V}) := \bigoplus_{k=0}^{\infty} \mathsf{T}^k(\mathbb{V}) &= \mathsf{T}^0(\mathbb{V}) \oplus \mathsf{T}^1(\mathbb{V}) \oplus \mathsf{T}^2(\mathbb{V}) \oplus \mathsf{T}^3(\mathbb{V}) \oplus \dots \\ &= \mathbb{R} \oplus \mathbb{V} \oplus (\mathbb{V} \otimes \mathbb{V}) \oplus (\mathbb{V} \otimes \mathbb{V} \otimes \mathbb{V}) \oplus \dots ,\end{aligned} \tag{B.1}$$

where the k-th tensor power has dimension $\dim(\mathsf{T}^k(\mathbb{V})) = d^k$. A (Graßmann) *exterior algebra* $\bigwedge(\mathbb{V})$ is defined [25] as the quotient

$$\bigwedge(\mathbb{V}) := \mathsf{T}(\mathbb{V}) \,/\, \mathcal{I}_{\text{ext}}(\mathbb{V})$$

where $\mathcal{I}_{\text{ext}}(\mathbb{V}) \subseteq \mathsf{T}(\mathbb{V})$ is the ideal

$$\mathcal{I}_{\text{ext}}(\mathbb{V}) = \big(\big(\ \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}\ :\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}\ \big)\big)$$

generated by all $\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}$ or, for real vector spaces, by all $\boldsymbol{a} \otimes \boldsymbol{a}$ for $\boldsymbol{a} \in \mathbb{V}$. The dimension of an exterior algebra is $\dim(\bigwedge(\mathbb{V})) = \sum_{k=0}^{d} \binom{d}{k} = 2^d$. A (Clifford) *geometric algebra* $\mathsf{G}(\mathbb{V}; \langle\cdot|\cdot\rangle)$ is defined [25] as the quotient

$$\mathsf{G}(\mathbb{V}; \langle\cdot|\cdot\rangle) := \mathsf{T}(\mathbb{V}) \,/\, \mathcal{I}_{\text{geo}}(\mathbb{V}; \langle\cdot|\cdot\rangle), \tag{B.2}$$

where $\mathcal{I}_{\text{geo}}(\mathbb{V}; \langle\cdot|\cdot\rangle) \subseteq \mathsf{T}(\mathbb{V})$ is the ideal

$$\mathcal{I}_{\text{geo}}(\mathbb{V}; \langle\cdot|\cdot\rangle) = \big(\big(\, \boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a} - 2 \cdot \langle \boldsymbol{a}|\boldsymbol{b}\rangle \cdot \boldsymbol{I}\ :\ \boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}\ \big)\big) \tag{B.3}$$

generated by all $\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a} - 2 \cdot \langle \boldsymbol{a}|\boldsymbol{b}\rangle \cdot \boldsymbol{I}$, for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{V}$ or, for real vector spaces, by all $\boldsymbol{a} \otimes \boldsymbol{a} - Q(\boldsymbol{a}) \cdot \boldsymbol{I}$ for $\boldsymbol{a} \in \mathbb{V}$. For a trivial bilinear form $\langle\cdot|\cdot\rangle = 0$, the geometric algebra reduces to the exterior algebra $\mathsf{G}(\mathbb{V}; \langle\cdot|\cdot\rangle) = \bigwedge(\mathbb{V})$.

Table 11. An overview of identities between different concepts of matrix algebra for the wheel product for pairs of standard unit vectors of the three-dimensional space and two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Wheel	*Sum of dyads*	*Difference of powers*	*Matrix*
$\hat{\boldsymbol{e}}_{\mathsf{y}} \circledast \hat{\boldsymbol{e}}_{\mathsf{z}} =$	$-(\hat{\boldsymbol{e}}_{\mathsf{z}} \otimes \hat{\boldsymbol{e}}_{\mathsf{x}} + \hat{\boldsymbol{e}}_{\mathsf{x}} \otimes \hat{\boldsymbol{e}}_{\mathsf{z}})$	$= (\hat{\boldsymbol{e}}_{\mathsf{y}} + \hat{\boldsymbol{e}}_{\mathsf{z}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{y}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{z}})^{\circledcirc} =$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$
$\hat{\boldsymbol{e}}_{\mathsf{z}} \circledast \hat{\boldsymbol{e}}_{\mathsf{x}} =$	$-(\hat{\boldsymbol{e}}_{\mathsf{z}} \otimes \hat{\boldsymbol{e}}_{\mathsf{x}} + \hat{\boldsymbol{e}}_{\mathsf{x}} \otimes \hat{\boldsymbol{e}}_{\mathsf{z}})$	$= (\hat{\boldsymbol{e}}_{\mathsf{z}} + \hat{\boldsymbol{e}}_{\mathsf{x}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{z}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{x}})^{\circledcirc} =$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
$\hat{\boldsymbol{e}}_{\mathsf{x}} \circledast \hat{\boldsymbol{e}}_{\mathsf{y}} =$	$-(\hat{\boldsymbol{e}}_{\mathsf{x}} \otimes \hat{\boldsymbol{e}}_{\mathsf{y}} + \hat{\boldsymbol{e}}_{\mathsf{y}} \otimes \hat{\boldsymbol{e}}_{\mathsf{x}})$	$= (\hat{\boldsymbol{e}}_{\mathsf{x}} + \hat{\boldsymbol{e}}_{\mathsf{y}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{x}})^{\circledcirc} - (\hat{\boldsymbol{e}}_{\mathsf{y}})^{\circledcirc} =$	$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\boldsymbol{a} \circledast \boldsymbol{b}$	$= 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - (\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) =$	$(\boldsymbol{a} + \boldsymbol{b})^{\circledcirc} - (\boldsymbol{a})^{\circledcirc} - (\boldsymbol{b})^{\circledcirc}$	Eq. (2.5)

Table 12. An overview of identities and correspondences between concepts of geometric algebra and matrix algebra for the cross matrices of standard vectors of three-dimensional space and of two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Wedge	*Dualized*	*Cross*	*Curl*	*Dyad difference*	*Matrix*	*Vector*
$\hat{\boldsymbol{e}}_y \wedge \hat{\boldsymbol{e}}_z =$	$(\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_z)^\star \mathrel{\widehat{=}}$	$(\hat{\boldsymbol{e}}_y \times \hat{\boldsymbol{e}}_z)^\otimes =$	$\hat{\boldsymbol{e}}_y \rtimes \hat{\boldsymbol{e}}_z =$	$\hat{\boldsymbol{e}}_z \otimes \hat{\boldsymbol{e}}_y - \hat{\boldsymbol{e}}_y \otimes \hat{\boldsymbol{e}}_z =$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} =$	$(\hat{\boldsymbol{e}}_x)^\otimes$
$\hat{\boldsymbol{e}}_z \wedge \hat{\boldsymbol{e}}_x =$	$(\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_x)^\star \mathrel{\widehat{=}}$	$(\hat{\boldsymbol{e}}_z \times \hat{\boldsymbol{e}}_x)^\otimes =$	$\hat{\boldsymbol{e}}_z \rtimes \hat{\boldsymbol{e}}_x =$	$\hat{\boldsymbol{e}}_x \otimes \hat{\boldsymbol{e}}_z - \hat{\boldsymbol{e}}_z \otimes \hat{\boldsymbol{e}}_x =$	$\begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} =$	$(\hat{\boldsymbol{e}}_y)^\otimes$
$\hat{\boldsymbol{e}}_x \wedge \hat{\boldsymbol{e}}_y =$	$(\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y)^\star \mathrel{\widehat{=}}$	$(\hat{\boldsymbol{e}}_x \times \hat{\boldsymbol{e}}_y)^\otimes =$	$\hat{\boldsymbol{e}}_x \rtimes \hat{\boldsymbol{e}}_y =$	$\hat{\boldsymbol{e}}_y \otimes \hat{\boldsymbol{e}}_x - \hat{\boldsymbol{e}}_x \otimes \hat{\boldsymbol{e}}_y =$	$\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$	$(\hat{\boldsymbol{e}}_z)^\otimes$
$\boldsymbol{a} \wedge \boldsymbol{b} =$	$(\boldsymbol{a} \times \boldsymbol{b})^\star \mathrel{\widehat{=}}$	$(\boldsymbol{a} \times \boldsymbol{b})^\otimes =$	$\boldsymbol{a} \rtimes \boldsymbol{b} =$	$\boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b}$	Eq. (2.8)	–

Appendix C. Involutions

A linear mapping $f : \mathsf{G} \to \mathsf{G}$ is called an *automorphism* or an *anti-automorphism* if it fulfills

$$f(AB) = f(A)f(B) \qquad\qquad f(AB) = f(B)f(A) \tag{C.1}$$

respectively. Such mapping f is called an *involution* of an *anti-involution* if the condition $f(f(A)) = A$ is further satisfied. The *first* main involution (or grade involution) $\overline{A}$ and the *second* main involution (or reversion[12]) $\underline{A}$ of a multivector $A = \sum_{k=0}^{d} \{A\}_k \in \mathsf{G}$ are defined [20, 21, 27] as

$$\begin{aligned} \overline{A} &= \operatorname{inv}(A) := \sum_{k=0}^{d} (-1)^k \cdot \{A\}_k \overset{d=3}{=} \{A\}_0 - \{A\}_1 + \{A\}_2 - \{A\}_3 \\ \underline{A} &= \operatorname{rev}(A) := \sum_{k=0}^{d} (-1)^{\lfloor k/2 \rfloor} \cdot \{A\}_k \overset{d=3}{=} \{A\}_0 + \{A\}_1 - \{A\}_2 - \{A\}_3 \,. \end{aligned} \tag{C.2}$$

The Clifford conjugate $\underline{\overline{A}}$ of an element $A \in \mathsf{G}$ is defined [20, 21, 27] as

$$\underline{\overline{A}} = \operatorname{con}(A) := \operatorname{inv}(\operatorname{rev}(A)) \overset{d=3}{=} \{A\}_0 - \{A\}_1 - \{A\}_2 + \{A\}_3 \,, \tag{C.3}$$

equal to $\underline{\overline{A}} = \sum_{k=0}^{d} (-1)^{\lfloor (k+1)/2 \rfloor} \cdot \{A\}_k$, see [27]. While the grade involution is an automorphism, the reverse and the conjugate are anti-automorphisms

$$\overline{AB} = \overline{A}\,\overline{B} \qquad\qquad \underline{AB} = \underline{B}\,\underline{A} \qquad\qquad \underline{\overline{AB}} = \underline{\overline{B}}\,\underline{\overline{A}} \,.$$

[12]For a g-blade $A = \{A\}_g = \boldsymbol{a}_1 \cdots \boldsymbol{a}_g$, the reversion $\underline{A} = \operatorname{rev}(A) = (-1)^{\lfloor g/2 \rfloor} \cdot A = \boldsymbol{a}_g \cdots \boldsymbol{a}_1$ reverses the factors of the geometric product [23].

Appendix D. Product Identities

Simple Product. For two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$, the *simple product* $\boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} = \frac{1}{2} \cdot \mathrm{full}(\boldsymbol{a}, \boldsymbol{b})$ features the matrix entries

$$\boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} = \begin{pmatrix} a_2 b_2 + a_3 b_3 & -a_1 b_2 & -a_1 b_3 \\ -a_2 b_1 & a_1 b_1 + a_3 b_3 & -a_2 b_3 \\ -a_3 b_1 & -a_3 b_2 & a_1 b_1 + a_2 b_2 \end{pmatrix} .$$

The simple product is not symmetric nor antisymmetric. However, the identities

$$\boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} = -(\boldsymbol{b}^{\otimes} \cdot \boldsymbol{a}^{\otimes}) = -(\boldsymbol{a}^{\otimes} \cdot \boldsymbol{b}^{\otimes})^{\top} = (\boldsymbol{a}^{\otimes} * \boldsymbol{b}^{\otimes})^{\top}$$

hold which characterize the 'transpositority' of Equation (3.6) (matrix transposition with $(\boldsymbol{A} \cdot \boldsymbol{B})^{\top} = \boldsymbol{B}^{\top} \cdot \boldsymbol{A}^{\top}$ is anti-involutive, Equation (C.1)). The simple product is related to the dyadic product $\boldsymbol{a} \otimes \boldsymbol{b} := \boldsymbol{a} \cdot \boldsymbol{b}^{\top}$ via the identity

$$\boldsymbol{b}^{\otimes} * \boldsymbol{a}^{\otimes} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{b} . \tag{D.1}$$

In the case of identical arguments, the simple product reduces to the 'radial second power' $\boldsymbol{a}^{\circledcirc}$ of Equation (2.2):

$$\boldsymbol{a}^{\otimes} * \boldsymbol{a}^{\otimes} = (\boldsymbol{a} * \boldsymbol{a}) \cdot \boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a} = (\boldsymbol{a} * \boldsymbol{a}) \cdot \boldsymbol{I} - \boldsymbol{a}^{\oslash} = \boldsymbol{a}^{\circledcirc}. \tag{D.2}$$

Wheel Product. The wheel product of Equation (2.4) is expressed, by means of Equation (D.1), via dyadic products as

$$\boldsymbol{a} \circledast \boldsymbol{b} = 2 \cdot (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{I} - (\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a}) .$$

In the case of swapped arguments, the symmetry of the wheel product is characterized by the chain of equations

$$\boldsymbol{a} \circledast \boldsymbol{b} = (\boldsymbol{a} \circledast \boldsymbol{b})^{\top} = (\boldsymbol{b} \circledast \boldsymbol{a})^{\top} = \boldsymbol{b} \circledast \boldsymbol{a} .$$

In the case of identical arguments, the wheel product simplifies, by means of Equation (D.2), to

$$\boldsymbol{a} \circledast \boldsymbol{a} = 2 \cdot ((\boldsymbol{a} * \boldsymbol{a}) \cdot \boldsymbol{I} - \boldsymbol{a} \otimes \boldsymbol{a}) = 2 \cdot ((\boldsymbol{a} * \boldsymbol{a}) \cdot \boldsymbol{I} - \boldsymbol{a}^{\oslash}) = 2 \cdot \boldsymbol{a}^{\circledcirc} .$$

In Table 11, the wheel product is exemplified for the case of standard vectors and two arbitrary vectors.

Curl Product. The curl product of Equation (2.7) is restated via dyadic products as

$$(\boldsymbol{a} \times \boldsymbol{b})^{\otimes} = \boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b}.$$

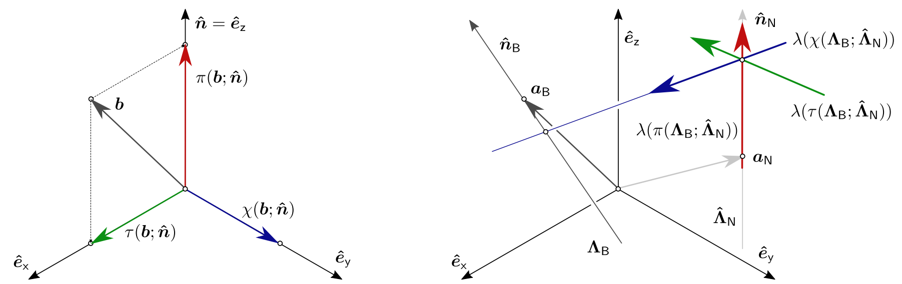
In the case of swapped arguments, the antisymmetry of the curl product is characterized by the chain of equations

$$\boldsymbol{b} \ltimes \boldsymbol{a} = (\boldsymbol{b} \times \boldsymbol{a})^{\otimes} = -(\boldsymbol{a} \times \boldsymbol{b})^{\otimes} = (\boldsymbol{a} \ltimes \boldsymbol{b})^{\top} = -(\boldsymbol{a} \ltimes \boldsymbol{b}) .$$

In the case of identical arguments, the curl product simplifies to the zero matrix:

$$(\boldsymbol{a} \times \boldsymbol{a})^{\otimes} = \boldsymbol{a} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{a} = 0 \cdot \boldsymbol{I}.$$

In Table 12, the curl product is exemplified for the case of standard vectors and two arbitrary vectors together with the corresponding terms from geometric algebra.



(a) Three operations for (linear) vectors. **(b)** Three operations for (affine) spears.

Figure 3. Examples of the geometric operations *projection*, *rejection*, and *complementation* for vectors (3a) and spears (3b): For the element $\boldsymbol{b}$ (gray), the projection (red), the rejection (green), and the complementation (blue) with respect to the unit element $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_z$ are indicated [4].

Appendix E. Geometric Operations

Three geometric vector operations are introduced together with alternate formulations according to [4]. An example for these operations is illustrated in Figure 3a.

Products and Angle Trigonometry. The scalar product and the norm of the cross product (Gibbs' product) are the 'cosine similarity' and the 'sine metric'

$$\boldsymbol{a} * \boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos(\vartheta_{ab}) \ , \tag{E.1}$$

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot |\sin(\vartheta_{ab})| \ . \tag{E.2}$$

Similar expressions for a 'bilinear application' of three matrices associated to the cross product $\boldsymbol{a} \times \boldsymbol{b}$ of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are reported for the sake of completeness:

$$\boldsymbol{a} * (\boldsymbol{a} \times \boldsymbol{b})^{\circledcirc} \cdot \boldsymbol{b} = \cos(\vartheta_{ab}) \cdot \sin^2(\vartheta_{ab}) \cdot \|\boldsymbol{a}\|^3 \cdot \|\boldsymbol{b}\|^3 = (\boldsymbol{a} * \boldsymbol{b}) \cdot \|\boldsymbol{a} \times \boldsymbol{b}\|^2$$

$$\boldsymbol{a} * (\boldsymbol{a} \times \boldsymbol{b})^{\otimes} \cdot \boldsymbol{b} = -\sin^2(\vartheta_{ab}) \cdot \|\boldsymbol{a}\|^2 \cdot \|\boldsymbol{b}\|^2 = -\|\boldsymbol{a} \times \boldsymbol{b}\|^2$$

$$\boldsymbol{a} * (\boldsymbol{a} \times \boldsymbol{b})^{\oslash} \cdot \boldsymbol{b} = 0$$

Rejection, Projection, Complementation. By means of the matrices $\boldsymbol{a}^{\otimes}$, $\boldsymbol{a}^{\circledcirc}$, and $\boldsymbol{a}^{\oslash}$, introduced in Equations (2.1), (2.2), and (2.3), the *rejection* $\tau(\boldsymbol{b}; \boldsymbol{a})$, the *projection* $\pi(\boldsymbol{b}; \boldsymbol{a})$, and the *complementation* $\chi(\boldsymbol{b}; \boldsymbol{a})$ of a vector $\boldsymbol{b}$ with respect to a reference vector $\boldsymbol{a}$ are defined by

$$\tau_{\boldsymbol{a}}(\boldsymbol{b}) = \tau(\boldsymbol{b}; \boldsymbol{a}) := \boldsymbol{a}^{\circledcirc} \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \times \boldsymbol{a} = \|\boldsymbol{a}\|^2 \cdot \boldsymbol{b} - \pi_{\boldsymbol{a}}(\boldsymbol{b}) \tag{E.3}$$

$$\pi_{\boldsymbol{a}}(\boldsymbol{b}) = \pi(\boldsymbol{b}; \boldsymbol{a}) := \boldsymbol{a}^{\oslash} \cdot \boldsymbol{b} = (\boldsymbol{a} * \boldsymbol{b}) \cdot \boldsymbol{a} = \|\boldsymbol{a}\|^2 \cdot \boldsymbol{b} - \tau_{\boldsymbol{a}}(\boldsymbol{b}) \tag{E.4}$$

$$\chi_{\boldsymbol{a}}(\boldsymbol{b}) = \chi(\boldsymbol{b}; \boldsymbol{a}) := \boldsymbol{a}^{\otimes} \cdot \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b} \tag{E.5}$$

The operations of rejection and projection depend on the norm $\|\boldsymbol{a}\|$ quadratically, the operation of complementation depends on the norm $\|\boldsymbol{a}\|$ linearly

$$\|\tau_{\boldsymbol{a}}(\boldsymbol{b})\| = \|\boldsymbol{a}\|^2 \cdot \|\boldsymbol{b}\| \cdot |\sin(\vartheta_{ab})| \tag{E.6}$$

$$\|\pi_{\boldsymbol{a}}(\boldsymbol{b})\| = \|\boldsymbol{a}\|^2 \cdot \|\boldsymbol{b}\| \cdot \cos(\vartheta_{ab}) \tag{E.7}$$

$$\|\chi_{\boldsymbol{a}}(\boldsymbol{b})\| = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot |\sin(\vartheta_{ab})|$$

in close relation to Equation (E.1) and Equation (E.2).

Orthogonal Decomposition. With Equation (E.6) and Equation (E.7), the orthogonal decomposition (Figure 3a) is obtained via the rejection and the projection of Equation (E.3) and Equation (E.4), via normalization with the squared norm

$$\begin{aligned}\boldsymbol{b}_\perp &:= \frac{\tau_a(\boldsymbol{b})}{\|\boldsymbol{a}\|^2} = (\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{a})\cdot\frac{1}{\|\boldsymbol{a}\|^2} = (\boldsymbol{a}^\otimes\cdot\boldsymbol{b}^\otimes - \boldsymbol{b}^\otimes\cdot\boldsymbol{a}^\otimes)\cdot\boldsymbol{a}^{-1} = (\boldsymbol{a}\rtimes\!\!\ltimes\boldsymbol{b})\cdot\boldsymbol{a}^{-1}\\ \boldsymbol{b}_\parallel &:= \frac{\pi_a(\boldsymbol{b})}{\|\boldsymbol{a}\|^2} = (\boldsymbol{b}-\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{a})\cdot\frac{1}{\|\boldsymbol{a}\|^2} = (\boldsymbol{a}^\oslash\cdot\boldsymbol{b})\cdot\frac{1}{\|\boldsymbol{a}\|^2}\\ &= \boldsymbol{a}\cdot(\boldsymbol{a}*\boldsymbol{b})\cdot\frac{1}{\|\boldsymbol{a}\|^2} = \boldsymbol{a}^{-1}\cdot(\boldsymbol{a}*\boldsymbol{b}).\end{aligned} \tag{E.8}$$

By means of the two correspondences

$$\begin{aligned}(\boldsymbol{a}\rtimes\!\!\ltimes\boldsymbol{b})\cdot\boldsymbol{a}^{-1} = (\boldsymbol{a}\times\boldsymbol{b})^\otimes\cdot\boldsymbol{a}^{-1} \mathrel{\widehat{=}} (\boldsymbol{a}\times\boldsymbol{b})^\star\boldsymbol{a}^{-1} = (\boldsymbol{a}\wedge\boldsymbol{b})\boldsymbol{a}^{-1} = \boldsymbol{a}^{-1}(\boldsymbol{a}\wedge\boldsymbol{b})\\ \boldsymbol{a}^{-1}\cdot(\boldsymbol{a}*\boldsymbol{b}) \mathrel{\widehat{=}} \boldsymbol{a}^{-1}\langle\boldsymbol{a}|\boldsymbol{b}\rangle\end{aligned}$$

the decomposition $\boldsymbol{b} = \boldsymbol{b}_\perp + \boldsymbol{b}_\parallel = \boldsymbol{a}^{-1}(\boldsymbol{a}\wedge\boldsymbol{b}) + \boldsymbol{a}^{-1}\langle\boldsymbol{a}|\boldsymbol{b}\rangle = \boldsymbol{a}^{-1}(\boldsymbol{a}\boldsymbol{b})$, from Equation (3.8), in terms of geometric algebra is derived.

Vector Triple Products. The *Graßmann identities* or the *BAC-CAB* and the *BAC-ABC* computation rules are the identities

$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{a}*\boldsymbol{c}) - \boldsymbol{c}\cdot(\boldsymbol{a}*\boldsymbol{b})\\ (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} = \boldsymbol{b}\cdot(\boldsymbol{a}*\boldsymbol{c}) - \boldsymbol{a}\cdot(\boldsymbol{b}*\boldsymbol{c})\end{aligned}\ . \tag{E.9}$$

In the case $\boldsymbol{a} = \boldsymbol{c}$, the vector triple product simplifies to

$$\boldsymbol{a}\times\boldsymbol{b}\times\boldsymbol{a} = (\boldsymbol{a}*\boldsymbol{a})\cdot\boldsymbol{b} - (\boldsymbol{a}*\boldsymbol{b})\cdot\boldsymbol{a} = \tau_{\boldsymbol{a}}(\boldsymbol{b})\ .$$

The vector triple products are related via the *Jacobi identity* that states

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) + \boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a}) + \boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{0}\ . \tag{E.10}$$

References

[1] A. Aristidou and J. Lasenby, *Inverse kinematics solutions using conformal geometric algebra*, in: L. Dorst and J. Lasenby (eds.), Guide to Geometric Algebra in Practice, Springer-Verlag, London, 2011.

[2] E. Bayro-Corrochano, *Geometric Algebra Applications Vol. II, Robot Modelling and Control*, Springer Cham, 2020.

[3] W. Bertram, *Jordan- and Lie geometries*, Arch. Math., Brno **49** (2013), 275–293.

[4] B. Bongardt, *The adjoint trigonometric representation of displacements and a closed-form solution to the IKP of General 3C chains*, J. Appl. Math. Mech. **100** (2020), e201900214, 26 pp.

[5] M. J. Crowe, *A History of Vector Analysis. The Evolution of the Idea of a Vectorial System*, University of Notre Dame Press, 1967.

[6] S. Xambó-Descamps, *From Leibniz's Characteristica Geometrica to contemporary geometric algebra*, Quaderns d'Història de l'Enginyeria **XVI** (2018), 109–141.

[7] L. Dorst, *The inner products of geometric algebra*, in: L. Dorst, C. Doran and J. Lasenby (eds.), Applications of Geometric Algebra in Computer Science and Engineering, 2002, pp. 35–46.

[8] L. Dorst, *3D oriented projective geometry through versors of* $R^{3,3}$, Adv. Appl. Clifford Algeb. **26** (2015), 1137–1172.

[9] C. Gunn, *On the homogeneous model of Euclidean geometry*, in: L. Dorst and J. Lasenby (eds.), Guide to geometric algebra in practice, Springer, London, 2011, pp. 297–327.
[10] C. Gunn, *Geometric algebras for Euclidean geometry*, Adv. Appl. Clifford Algeb. **27** (2016), 185–208.
[11] D. Hestenes, *New tools for computational geometry and rejuvenation of screw theory*, in: E. Bayro-Corrochano and G. Scheuermann (eds.), Geometric Algebra Computing in Engineering and Computer Science, Springer, London, 2010, pp. 3–33.
[12] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics*, Reidel/Kluwer, 1984.
[13] D. Hildenbrand, *The Power of Geomtric Algebra Computing*, CRC Press, 2021.
[14] D. Hildenbrand, J. Zamora and E. Bayro-Corrochano, *Inverse kinematics computation in computer graphics and robotics using conformal geometric algebra*, Adv. Appl. Clifford Algeb. **18** (2008), 699–713.
[15] E. Hitzer, *Introduction to Clifford's geometric algebra*, Journal of The Society of Instrument and Control Engineers **51** (2012), 338–350.
[16] D. Klawitter, *A Clifford algebraic approach to line geometry*, Adv. Appl. Clifford Algeb. **24** (2014), 713–736.
[17] W. Kühnel, *Matrizen und Lie-Gruppen. Eine geometrische Einführung*, Vieweg–Teubner, Wiesbaden, 2011.
[18] H. Li, L. Huang, L. Dong and C. Shao, *Elements of line geometry with geometric algebra*, in: S. X. Descamps, J. M. Parra Serra and R. G. Calvet (eds.), Early Proceedings of the AGACSE 2015 Conference, 2015, pp. 195–204.
[19] H. Li, L. Huang, C. Shao and L. Dong, *Three-dimensional projective geometry with geometric algebra*, arXiv:1507.06634, 2015.
[20] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed., Cambridge University Press, 2001.
[21] D. Lundholm and L. Svensson, *Clifford Algebra, Geometric Algebra, and Applications*, 2016.
[22] A. Macdonald, *A Survey of geometric algebra and geometric calculus*, Adv. Appl. Clifford Algeb. **27** (2017), 853–891.
[23] A. Macdonald, *Linear and Geometric Algebra*, CreateSpace, 2011.
[24] K. McCrimmon, *Jordan algebras and their applications*, Bull. Am. Math. Soc. **84** (1978), 612–627.
[25] E. Meinrenken, *Clifford Algebras and Lie Theory*, Springer, Berlin, 2013.
[26] H. Pottmann and J. Wallner, *Computational Line Geometry*, Mathematics and Visualization, Springer, 2001.
[27] P. Renaud, *Clifford Algebras Lecture Notes on Applications in Physics*, HAL Open Science, hal-03015551, 2020.
[28] J. M. Selig, *Geometric Fundamentals of Robotics*, 2nd ed., Springer, New York, 2005.
[29] M. R. Sepanski, *Compact Lie Groups*, Graduate Texts in Mathematics **235**, Springer, New York, 2007.
[30] G. Sobczyk, *Clifford geometric algebras in multlinear algebra and non-euclidean geometries*, in: J. Byrnes (ed.), Computational Noncommutative Algebra and Applications, NATO Science Series II: Mathematics, Physics and Chemistry **136**, Springer, Dordrecht, 2003, pp. 1–27.
[31] G. Sobczyk, *Matrix Gateway to Geometric Algebra, Spacetime and Spinors*, Garretstar, 2019.
[32] F. Thomas, *Approaching dual quaternions from matrix algebra*, IEEE Trans. Robot. **30** (2014), 1037–1048.
[33] J. Vaz Jr. and R. da Rocha Jr., *An Introduction to Clifford Algebras and Spinors*, Oxford University Press, 2016.
[34] T. G. Vold, *An introduction to geometric algebra with an application in rigid body mechanics*, Am. J. Phys. **61** (1993), 491–504.

Bertold Bongardt, Institut für Robotik und Prozessinformatik, Technische Universität Braunschweig, Mühlenpfordtstraße 23, 38106 Braunschweig, Germany
e-mail: `b.bongardt@tu-braunschweig.de`

Harald Löwe, Institut für Partielle Differentialgleichungen, Technische Universität Braunschweig, Universitätsplatz 2, 38106 Braunschweig, Germany
e-mail: `h.loewe@tu-braunschweig.de`