ON BOURBAKI-BOUNDED SETS
ON QUASI-PSEUDOMETRIC SPACES

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Abstract. In metric spaces, a set is Bourbaki-bounded if and only if every real-valued uniformly continuous function on it is bounded. In this article, we study Bourbaki-boundedness on quasi-pseudometric spaces. It turns out that if a set is Bourbaki-bounded on a symmetrized quasi-pseudometric space, then it is Bourbaki-bounded in the quasi-metric space but the converse need not to be true. We show that an asymmetric normed space is Bourbaki-bounded if and only if it is bounded. Consequently, we prove that every real-valued semi-Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded. This article extends some results from Beer and Garrido’s paper [2] from the metric point of view to the context of quasi-metric spaces.

1. INTRODUCTION

The theory of Bourbaki-boundedness in metric spaces was introduced by Atsuji in [1] as a generalization of the concept of totally bounded metric spaces. However, the concept of Bourbaki-boundedness attracted a great interest of many scholars (see [5–7]). For instance in [6], the authors introduced new tools for the completeness of metric spaces, called Bourbaki-completeness and cofinal Bourbaki completeness.

In addition, Beer and Garrido [2] proved that in metric space \((X,d)\), a set \(B \subseteq X\) is Bourbaki-bounded if and only if \(f(B)\) is bounded in metric space \((Y,p)\) whenever \(f : (X,d) \to (Y,p)\) is uniformly continuous. Furthermore, they proved that \(B\) is Bourbaki-bounded if and only if \(f(B)\) is a member of the bornology of bounded subsets of \(\mathbb{R}\), where \(f : (X,d) \to (\mathbb{R},|.|)\) is Lipschitz in the small function.

In metric spaces, a family of Bourbaki-bounded sets sits between the family of totally bounded sets and the family of bounded sets. The concept of Bourbaki-boundedness in the framework of quasi-uniform spaces was introduced by Murdeshwar and Theckedath in [10]. They observed that if the set \([0,1]\) is equipped with the \(T_0\)-quasi-metric

\[
q(x, y) = \begin{cases} 
  y - x & \text{if } x \leq y \\
  1 & \text{if } x > y
\end{cases}
\]

and \(\mathcal{U}\) is the quasi-uniformity generated by \(q\) on \([0,1]\) and \(\mathcal{U}^{-1}\) is the conjugate quasi-uniformity of \(\mathcal{U}\) on \([0,1]\), then \(([0,1],\mathcal{U})\) and \(([0,1],\mathcal{U}^{-1})\) are Bourbaki-bounded.

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quasi-uniform spaces but \([0,1],U^s\) is the discrete uniform space which is not Bourbaki-bounded.

Moreover, we observe that for any two quasi-pseudometric spaces \((X,q)\) and \((Y,p)\), if the function \(\varphi : (X,q) \to (Y,p)\) is uniformly continuous, then \(\varphi : (X,q^s) \to (Y,p^s)\) is also uniformly continuous, but the converse is not true in general (see Example 3.3). All these are great motivations for generalizing the results about Bourbaki-boundedness from the symmetric point of view to asymmetric settings.

In this paper, we first revisit uniformly continuous and semi-Lipschitz functions in asymmetric settings, then we continue the study of the concept of Bourbaki-boundedness in a quasi-metric space. We show, for instance, that in a quasi-metric space \((X,q)\), if a set is \(q^s\)-Bourbaki-bounded, then it is \(q\)-Bourbaki-bounded (and \(q^t\)-Bourbaki-bounded) but the converse need not to be true (see Example 4.9). Furthermore, we characterize Bourbaki-bounded sets in terms of boundedness under uniformly continuous functions and semi-Lipschitz in the small functions (see Theorem 5.2).

2. Preliminaries

A function \(q : X \times X \to [0, \infty)\) on a set \(X\) will be called a quasi-pseudometric on \(X\) if, for any \(x,y,z \in X\),

1. \(q(x,x) = 0\);
2. \(q(x,y) \leq q(x,z) + q(z,y)\).

Furthermore, if we also have that
3. \(q(x,y) = 0 = q(y,x)\) implies \(x = y\), then the function \(q\) is called a \(T_0\)-quasi-metric (or quasi-metric) on \(X\), and the pair \((X,q)\) is a \(T_0\)-quasi-metric space.

If \(q\) is a quasi-pseudometric (\(T_0\)-quasi-metric) space on \(X\), then the function \(q^t : X \times X \to [0, \infty)\) defined by \(q^t(x,y) = q(y,x)\) for all \(x,y \in X\) is also a quasi-pseudometric (\(T_0\)-quasi-metric) on \(X\), often called the conjugate quasi-pseudometric of \(q\).

The symmetrized quasi-pseudometric of \(q\) is the function \(q^s : X \times X \to [0, \infty)\) given by \(q^s(x,y) = \max\{q(x,y),q(y,x)\}\) for all \(x,y \in X\). It is easy to see that \(q^s\) is a pseudometric (metric) on \(X\).

Let \((X,q)\) be a quasi-pseudometric space. Given that \(x \in X, \delta > 0\) and \(F \subseteq X\), we define \(\text{dist}_q(x,F)\) by \(\text{dist}_q(x,F) := \inf_{f \in F} q(x,f)\) and \(\text{dist}_q^t(x,F)\) by \(\text{dist}_q^t(x,F) = \text{dist}_q(F,x) = \inf_{f \in F} q(f,x)\). In addition, we have

\[
D_q(x,\delta) := \{y \in X : q(x,y) < \delta\} \quad \text{and} \quad D_q[x,\delta] := \{y \in X : q(x,y) \leq \delta\}.
\]

Moreover,
\[
\text{dist}_q^s(x,F) := \max\{\text{dist}_q(x,F),\text{dist}_q^t(x,F)\} = \inf_{f \in F} q^s(x,f).
\]

**Definition 2.1.** One says that a subset \(A\) of a quasi-pseudometric space \((X,q)\) is \(q\)-bounded if there exists \(x \in X\) and \(r > 0\) and \(s > 0\) such that \(A \subseteq D_q(x,r) \cap D_q^t(x,s)\).

Note that the above definition is slightly different from the one given in \([14]\). In the sense of \([14]\), a subset \(A\) of \(X\) can be \(q\)-bounded and not necessarily \(q^t\)-bounded.
Obviously, in our context, a subset $A$ is $q$-bounded if and only if it is $q^t$-bounded. But $q$-boundedness does not imply $q^s$-boundedness in the context of [14]. Moreover, if $q$ is an extended quasi-pseudometric on $X$ (i.e., the distance between two points can be $\infty$), then a subset $B$ of $X$ can be included in $D_q(x, \varepsilon)$ for some $x \in X$, $\varepsilon > 0$ but its diameter $\text{diam}(B) = \sup \{q(y, z) : y, z \in B\} = \infty$ (see [14, p. 2022]).

**Definition 2.2.** [9, Definition 1.1] Let $X$ be a nonempty set. A family $\mathcal{B}$ of subsets of $X$ is called a bornology on $X$, provided the following conditions are satisfied:

(i) $\mathcal{B}$ forms a cover of $X$, i.e., $X = \bigcup_{B \in \mathcal{B}} B$;

(ii) $\mathcal{B}$ is hereditary under inclusion, i.e., whenever $B \in \mathcal{B}$ and $A$ is a subset of $X$ contained in $B$, then $A \in \mathcal{B}$;

(iii) $\mathcal{B}$ is stable under finite union, i.e., if $B_1, B_2, \ldots, B_n \in \mathcal{B}$, then we get $\bigcup_{i=1}^n B_i \in \mathcal{B}$.

Given a bornology $\mathcal{B}$ and the set $X$, a pair $(X, \mathcal{B})$ is called a bornological universe.

Let $(X, q)$ be a quasi-pseudometric space. It has been observed in [9,11] that the collection $\mathcal{B}_q(X)$ of all $q$-bounded subsets of $X$ forms a bornology on $X$ and this bornology is called the quasi-metric bornology determined by $q$. Furthermore, we have:

$$\mathcal{B}_q^s(X) = \mathcal{B}_q(X)$$

and bornologies $\mathcal{B}_q(X)$ and $\mathcal{B}_q^t(X)$ are equivalent. These observations came from the paper of Olela Otafudu et al. [9].

**Definition 2.3.** ([13, Definition 5]) A quasi-pseudometric space $(X, q)$ is $q$-totally bounded if, for any $\varepsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in X$ such that for any $x \in X$, $q^t(x, x_i) < \varepsilon$ for some $i \in \{1, \ldots, n\}$.

Let $\mathcal{T}_q(X)$ be the collection of all $q$-totally bounded subsets of $X$. One can easily see that

(i) the singleton $\{x\} \in \mathcal{T}_q(X)$ whenever $x \in X$;

(ii) if $B \in \mathcal{T}_q(X)$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{T}_q(X)$;

(iii) for any $A, B \in \mathcal{T}_q(X)$, it follows that $A \cup B \in \mathcal{T}_q(X)$.

Consequently, the collection $\mathcal{T}_q(X)$ forms a bornology on $X$ that we call the bornology of $q$-totally bounded sets.

**Remark 2.4.** Let $(X, q)$ be a quasi-pseudometric space. The following

$$\mathcal{T}_q(X) = \mathcal{T}_q^s(X) = \mathcal{T}_q^t(X)$$

is just a consequence of Definition 2.3.

An asymmetric norm on a real vector space $X$ is a function $||.|| : X \to [0, \infty)$ satisfying the following conditions:

(1) $||x|| = ||-x|| = 0$ implies $x = 0$; 

(2) $||ax|| = a||x||$;
Continuous is uniformly continuous. Furthermore, from the definition of \( \epsilon \)
and \( \delta \) defined by
\[
\phi(x,y) = |x-y| \quad \text{for any } x,y \in X.
\]
An asymmetric norm \( \| \cdot \| \) on \( X \) induces a quasi-metric \( q_{\| \cdot \|} \) on \( X \) defined by
\[
q_{\| \cdot \|}(x,y) = \|x - y\| \quad \text{for any } x,y \in X.
\]
If \( (X, \| \cdot \|) \) is a normed lattice space, then the function \( \| \cdot \| \) defined by \( \|x\| = |x^+| \), where \( x^+ = \max\{x,0\} \), is an asymmetric norm on \( X \).

3. Uniformly continuous and semi-Lipschitz functions

**Definition 3.1.** ([4, p. 146]) Let \( (X, q) \) and \( (Y, p) \) be quasi-pseudometric spaces. A function \( \varphi : (X, q) \to (Y, p) \) is called **quasi-uniformly continuous** (or uniformly continuous) if, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( q(x,y) \leq \delta \), then \( p(\varphi(x), \varphi(y)) < \epsilon \) for all \( x,y \in X \).

**Lemma 3.2.** Let \( (X, q) \) and \( (Y, p) \) be quasi-pseudometric spaces. If the function \( \varphi : (X, q) \to (Y, p) \) is uniformly continuous, then the function \( \varphi : (X, q^*) \to (Y, p^*) \) is uniformly continuous.

**Proof.** Let \( \epsilon > 0 \). There exists \( \delta > 0 \) such that if
\[
q(x,y) < \delta \quad \text{then} \quad p(\varphi(x), \varphi(y)) \leq \epsilon \quad \text{for all} \quad x,y \in X.
\]
Furthermore, from the definition of \( q^* \), it follows easily that
\[
q(x,y) \leq q^*(x,y) < \delta \quad \text{then} \quad p(\varphi(x), \varphi(y)) \leq \epsilon \quad (3.1)
\]
and
\[
q(y,x) \leq q^*(y,x) < \delta \quad \text{then} \quad p(\varphi(y), \varphi(x)) \leq \epsilon. \quad (3.2)
\]
For all \( x,y \in X \), we have \( p^*(\varphi(x), \varphi(y)) < \epsilon \) from (3.1) and (3.2).
Therefore, the function \( \varphi : (X, q^*) \to (Y, p^*) \) is uniformly continuous. \( \square \)

**Example 3.3.** We equip \( X = \mathbb{R}_+ = [0, \infty) \) with the quasi-metric \( q \) defined by
\[
q(x,y) = (y-x)^+ \quad \text{for any} \quad x,y \in [0, \infty)
\]
and equip \( Y = \mathbb{R} \) with the \( T_0 \)-quasi-metric
\[
p(x,y) = |y-x| \quad \text{for any} \quad x,y \in \mathbb{R}.
\]
Then, while \( f(x) = -\sqrt{x} \) is uniformly continuous from \((\mathbb{R}_+, |\cdot|)\) into \((\mathbb{R}, |\cdot|)\), it is not uniformly continuous from \((\mathbb{R}_+, q)\) into \((\mathbb{R}, p)\).

**Proof.** One can easily see that \( f \) is uniformly continuous from \((\mathbb{R}_+, |\cdot|)\) into \((\mathbb{R}, |\cdot|)\). To show that \( f \) is not uniformly continuous from \((\mathbb{R}_+, q)\) into \((\mathbb{R}, p)\), let \( \epsilon = 1 \) and \( \delta > 0 \) be chosen arbitrarily.

Then, for any \( x > 1 \),
\[
q(x,0) = (0-x)^+ = 0 < \delta,
\]
but
\[
p(f(x), f(0)) = (f(0) - f(x))^+ = (\sqrt{x})^+ = \sqrt{x} > 1 = \epsilon.
\]
\( \square \)
Remark 3.4. It can be noted from Example 3.3 that the converse of Lemma 3.2 does not hold in general. So, uniform continuity of the function $\varphi : (X, q^s) \to (Y, p^s)$ does not imply the uniform continuity of the function $\varphi : (X, q) \to (Y, p)$.

Let $(X, q)$ be a quasi-metric space and $(Y, ||.)$ be an asymmetric normed space. Then, a function $\varphi : (X, q) \to (Y, ||.)$ is called a semi-Lipschitz if there exists $k \geq 0$ such that
\[
||\varphi(x) - \varphi(y)|| \leq kq(x, y) \quad \text{for all } x, y \in X.
\]
A number $k$ satisfying (3.3) is called semi-Lipschitz constant for $\varphi$. For more details about semi-Lipschitz functions, we recommend the reader to see for instance [3, 12].

Definition 3.5. Let $(X, q)$ be a quasi-metric space and $(Y, ||.)$ be an asymmetric normed space. Then:

(a) A function $\varphi : (X, q) \to (Y, ||.)$ is called locally semi-Lipschitz, provided that, for all $x \in X$, there exists $\delta(x) > 0$ such that $\varphi|_{D_q(x, \delta(x))}$ is semi-Lipschitz.

(b) A function $\varphi : (X, q) \to (Y, ||.)$ is called uniformly locally semi-Lipschitz, provided that, for all $x \in X$, there exists $\delta > 0$ (does not depend on $x$) such that $\varphi|_{D_q(x, \delta)}$ is semi-Lipschitz.

(c) A function $\varphi : (X, q) \to (Y, ||.)$ is called semi-Lipschitz in the small if there exists $\delta > 0$ and $k \geq 0$ such that $q(x, y) < \delta$, then
\[
||\varphi(x) - \varphi(y)|| \leq kq(x, y).
\]

We omit the proof of the following since it is a direct application of the definition of $q^s$.

Lemma 3.6. Let $(X, q)$ be a quasi-metric space and $(Y, ||.)$ be an asymmetric normed space. If a function $\varphi : (X, q) \to (Y, ||.)$ is locally semi-Lipschitz, then $\varphi : (X, q^s) \to (Y, ||.)$ is locally semi-Lipschitz.

The following lemma is easy to prove; therefore, we leave it to the reader.

Lemma

normed space. If function $\varphi : (X, q) \to (Y, ||.)$ is semi-Lipschitz in the small, then $\varphi : (X, q) \to (Y, ||.)$ is uniformly continuous.

Remark 3.8. Let $(X, q)$ be a quasi-metric space and $(Y, ||.)$ be an asymmetric normed space.

1. If a function $\varphi : (X, q) \to (Y, ||.)$ is locally semi-Lipschitz, then $\varphi|_{D_q(x, \delta_x)}$ is continuous whenever $x \in X$ and for some $\delta_x > 0$.

2. Let $F \subseteq X$. If $\varphi_i : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz restricted to $F$ for $i = \{1, 2, \ldots, n\}$, then $\max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is semi-Lipschitz restricted to $F$, where $u$ is the standard quasi-metric defined on $\mathbb{R}$ defined by $u(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$.

4. Some first results

Definition 4.1. Let $(X, q)$ be a quasi-pseudometric space and $\delta > 0$. For any $\emptyset \neq F \subseteq X$, we define the $\delta$- enlargement $D_q(F, \delta)$ of $F$ by
\[
D_q(F, \delta) := \{x \in X : \text{dist}(F, x) < \delta\} = \bigcup_{f \in F} D_q(f, \delta)
\]
and
\[ D_q^t(F, \delta) := \{ x \in X : \text{dist}^t(F, x) < \delta \} = \bigcup_{f \in F} D_q^t(f, \delta). \]

Furthermore,
\[ D_q^t(F, \delta) = \max \left\{ D_q(F, \delta), D_q^t(F, \delta) \right\} = \bigcup_{f \in F} D_q^t(f, \delta). \]

**Lemma 4.2.** Let \((X, q)\) be a quasi-pseudometric space and \(\epsilon, \delta > 0\). We have
\[ D_q(D_q(F, \epsilon), \delta) \subseteq D_q(F, \epsilon + \delta). \]

**Proof.** Let \(y \in D_q(D_q(F, \epsilon), \delta) = \bigcup_{v \in D_q(F, \epsilon)} D_q(v, \delta)\). Then, there exists \(v \in D_q(F, \epsilon)\) such that \(y \in D_q(v, \delta)\). It follows that there exists \(f \in F\) such that \(q(f, v) < \epsilon\) and \(q(v, y) < \delta\). Moreover,
\[ q(f, y) \leq q(f, v) + d(v, y) < \epsilon + \delta. \]

Hence, \(y \in D_q(f, \epsilon + \delta)\). Thus, \(y \in D_q(F, \epsilon + \delta)\). \(\Box\)

Let \((X, q)\) be a quasi-pseudometric space and \(\delta > 0\). If \(x \in X\) and \(n = 0, 1, 2, \ldots\), we define the sets \(D_q^n(x, \delta)\) by
\[ D_q^0(x, \delta) := \{ x \} \quad \text{and} \quad D_q^{n+1}(x, \delta) := D_q(D_q^n(x, \delta), \delta). \]

The next remark follows by induction.

**Remark 4.3.** Let \((X, q)\) be a quasi-pseudometric space and \(\delta > 0\). For any \(x \in X\) and \(n = 0, 1, 2, \ldots\), we have
\[ D_q^n(x, \delta) \subseteq D_q^{n+1}(x, \delta). \]

Furthermore, note that from Lemma 4.2 we get
\[ D_q^n(x, \delta) \subseteq D_q(x, n\delta). \]

**Definition 4.4.** (compare [8, Definition 2.1]) Let \((X, q)\) be a quasi-pseudometric space. For any given \(x, y \in X\) and \(\delta > 0\), a \(\delta\)-chain of length \(n\) from \(x\) to \(y\) in \((X, q)\) is a finite sequence of points \(x_0, x_1, \ldots, x_n\) such that \(x = x_0, x_n = y\) and \(q(x_{i-1}, x_i) < \delta\) for any \(i\) with \(1 \leq i \leq n\).

**Remark 4.5.** Let \((X, q)\) be a quasi-pseudometric space and \(\delta > 0\). If there exists a \(\delta\)-chain of length \(n\) from \(x\) to \(y\) in \((X, q)\), then there exists a \(\delta\)-chain of length \(n\) from \(y\) to \(x\) in \((X, q')\) whenever \(x, y \in X\).

The following is a consequence of Remark 4.5 and the definition of \(D_q^n(y, \delta)\).

**Remark 4.6.** Let \(\delta > 0\) be a positive real number and \(x\) and \(y\) be points in a quasi-pseudometric space \((X, q)\). It is easy to check that there exists \(\delta\)-chain of length \(n\) from \(x\) to \(y\) if and only if \(y \in D_q^n(x, \delta)\) if and only if \(x \in D_q^n(y, \delta)\).

Let \((X, q)\) be a quasi-pseudometric space and \(\delta > 0\). For any \(x, y \in X\) and \(\delta > 0\), we define the relation \(\asymp_{\delta}\) on \(X\) by \(x \asymp_{\delta} y\) if there exists a \(\delta\)-chain of some length from \(x\) to \(y\).

**Lemma 4.7.** For any quasi-pseudometric space \((X, q)\) and any \(\delta > 0\), the relation \(\asymp_{\delta}\) is a quasiorder on \(X\).
Let \((X, d)\) be a quasi-pseudometric space. For \(x \in X\), we define the set \(x_{\geq \delta}\) by
\[
x_{\geq \delta} = \bigcup_{n=0}^{\infty} D^n_q(x, \delta).
\]

**Remark 4.8.** Let \((X, q)\) be a quasi-pseudometric space. For any \(\delta > 0\) and \(x, y \in X\), it is easy to see that if \((x_i)_{i=0}^{n}\) is a \(\delta\)-chain in \((X, q^s)\) of length \(n\) from \(x\) to \(y\), then \((x_i)_{i=0}^{n}\) is also a \(\delta\)-chain in \((X, q)\) and in \((X, q^t)\) of length \(n\) from \(x\) to \(y\). Then, with regard to Remark 4.6, we have
\[
D^n_{q^s}(x, \delta) \subseteq D^n_q(x, \delta) \tag{4.1}
\]
and
\[
D^n_{q^s}(x, \delta) \subseteq D^n_q(x, \delta) \tag{4.2}
\]
The following example shows that the inclusions (4.1) and (4.2) cannot be reversed.

**Example 4.9.** Let \(X\) be a set of four points \(\{1, 2, 3, 4\}\). If we equip \(X\) with the \(T_0\)-quasi-metric \(q\) defined by the distance matrix
\[
Q = \begin{pmatrix}
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
2 & 1 & 1 & 0
\end{pmatrix}
\]
that is, \(q(i, j) = q_{i,j}\) whenever \(i, j \in X\), the symmetrized metric \(q^s\) of \(q\) is induced by the matrix
\[
Q^s = \begin{pmatrix}
0 & 1 & 2 & 2 \\
1 & 0 & 1 & 2 \\
2 & 1 & 0 & 1 \\
2 & 2 & 1 & 0
\end{pmatrix}
\]
Let \(\delta = 2\). If \((f_i)_{i=0}^{2} := (4, 2, 1)\) is a sequence in \(X\), then we have
\[
q(f_0, f_1) = q(4, 2) = 1 = q(f_1, f_2) = q(2, 1) < \delta.
\]
Hence, the sequence \((f_i)_{i=0}^{2} := (4, 2, 1)\) is a \(\delta\)-chain in \((X, q)\) of length 2 from 4 to 1. But the same sequence \((f_i)_{i=0}^{2} := (4, 2, 1)\) is not a \(\delta\)-chain in \((X, q^s)\) of length 2 from 4 to 1 because \(q^s(f_0, f_1) = q^s(4, 2) = 2 \geq \delta\).

The motivation of the following definition comes from [11, Definition 1.5] and Remark 4.6.

**Definition 4.10.** (compare [8, Definition 2.1 and Remark 2.2]) Let \((X, q)\) be a quasi-pseudometric space and \(F \subseteq X\). We say that \(F\) is \(q\)-Bourbaki-bounded if, for any \(\delta > 0\), there exists a finite subset \(\{f_1, f_2, \ldots, f_k\}\) of \(X\) and for some positive integer \(n\) such that
\[
F \subseteq \bigcup_{i=1}^{k} D^n_q(f_i, \delta).
\]
The converse of the next lemma does not hold from Example 4.9.

**Lemma 4.11.** Let \((X, q)\) be a quasi-pseudometric space and \(F \subseteq X\). If \(F\) is \(q^s\)-Bourbaki-bounded, then \(F\) is \(q\)-Bourbaki-bounded and \(q^t\)-Bourbaki-bounded.
Proposition 4.12. Let \((X, q)\) be a quasi-pseudo-metric space. If \(F\) is a subset of \(X\) and for \(\delta > 0\), then we have

(a) If \(F\) is \(q\)-totally bounded, then \(F\) is \(q\)-Bourbaki-bounded.

(b) If \(F\) is \(q\)-Bourbaki-bounded and \(q^t\)-Bourbaki-bounded, then \(F\) is \(q^s\)-bounded.

(c) If \(F\) is \(q\)-Bourbaki-bounded, then \(\text{Cl}_{\tau(q)} F\) is also \(q\)-Bourbaki-bounded.

Proof. (a) It is immediate.

(b) Let \(\delta\) and \(\epsilon\) be positive real numbers. From the \(q\)-Bourbaki-boundedness of \(F\) and \(q^t\)-Bourbaki-boundedness, there exist finite sets \(\{x_1, x_2, \ldots, x_r\} \subseteq X\) and \(\{y_1, \ldots, y_l\} \subseteq X\). Let \(\{z_1, \ldots, z_k\} = \{x_1, x_2, \ldots, x_r\} \cup \{y_1, \ldots, y_l\}\). Then, there exists some positive integer \(n\) such that

\[
F \subseteq \bigcup_{i=1}^{r} D^n_{q}(z_i, \delta) \text{ and } F \subseteq \bigcup_{i=1}^{l} D^n_{q^t}(z_i, \epsilon),
\]

where \(r, l \leq k\). We now show that \(F\) is \(q^s\)-bounded.

Indeed,

\[
F \subseteq \bigcup_{i=1}^{r} D^n_{q}(z_i, \delta) \cap \bigcup_{i=1}^{l} D^n_{q^t}(z_i, \epsilon)
\]

\[
= \bigcup_{i=1}^{r} \bigcup_{i=1}^{l} D^n_{q}(z_i, \delta) \cap D^n_{q^t}(z_i, \epsilon)
\]

\[
\subseteq \bigcup_{i=1}^{r} \bigcup_{i=1}^{l} D_{q}(z_i, n\delta) \cap D_{q^t}(z_i, n\epsilon).
\]

Thus, \(F\) is \(q^s\)-bounded.

(c) Follows since \(F\) is included in the union of \(q\)-open balls. \(\square\)

Remark 4.13. Let \((X, q)\) be a quasi-metric space. In the sequel, we denote by \(\mathcal{BB}_q(X)\) the collection of all \(q\)-Bourbaki-bounded subsets in \((X, q)\). We observe that \(\mathcal{BB}_q(X)\) forms a bornology on \(X\) that we call the bornology of \(q\)-Bourbaki-bounded sets in \((X, q)\).

Remark 4.14. If \((X, q)\) is a quasi-pseudo-metric space, then we have the following inclusions:

\[
\mathcal{T}_q(X) \subseteq \mathcal{BB}_q(X) \subseteq \tau(q^s) \subseteq \mathcal{B}_q(X) \subseteq \mathcal{BB}_q(X).
\]

The first two inclusions of (4.3) can be found in [2] and [5], for example, and the last inclusion of (4.3) is a consequence of Proposition 4.12. For more details about connections between bornology of \(q^s\)-totally bounded sets, bornology of \(q^s\)-Bourbaki-bounded sets and bornology of \(q^s\)-bounded sets, we recommend, for instance, [2, 5, 7, 8].

5. Bornologies and semi-Lipschitz functions

In this last section, we intend to characterize \(q\)-Bourbaki-bounded sets in terms of uniformly continuous functions and \(\tau(q^s)\)-compact set in terms of semi-Lipschitz functions.
In the sequel, we equip \( \mathbb{R} \) with its usual \( T_0 \)-quasi-metric \( u \) given by
\[
u(x, y) = (x - y)^+ = \max\{x - y, 0\}\ 	ext{whenever } x, y \in \mathbb{R}.
\]

**Theorem 5.1.** Let \((X, q)\) be a quasi-metric space and \(\emptyset \neq F \subseteq X\). Then, the following conditions are equivalent:

1. \(cl_{\tau(q)}(F)\) is \(\tau(q)\)-compact;
2. if \((Y, ||.||)\) is an asymmetric normed space and \(\varphi : (X, q) \to (Y, ||.||)\) is continuous, then \(\varphi(F) \in \mathcal{B}_{q||.||}(Y)\);
3. if \((Y, ||.||)\) is an asymmetric normed space and \(\varphi : (X, q) \to (Y, ||.||)\) is locally semi-Lipschitz, then \(\varphi(F) \in \mathcal{B}_{q||.||}(Y)\);
4. if \(\varphi : (X, q) \to (\mathbb{R}, u)\) is locally semi-Lipschitz, then \(\varphi(F)\) is a \(u\)-bounded set of real numbers.

**Proof.** (1) \(\Rightarrow\) (2) Suppose that \(cl_{\tau(q)}(F)\) is \(\tau(q)\)-compact and \(\varphi : (X, q) \to (Y, ||.||)\) is continuous. Then, \(\varphi(cl_{\tau(q)}(F))\) is \(\tau(||.||)\)-compact. Thus, \(\varphi(F)\) is \(q||.||\)-bounded.

(2) \(\Rightarrow\) (3) Follows from the continuity of locally semi-Lipschitz functions and (3) \(\Rightarrow\) (4) follows without doubt.

(4) \(\Rightarrow\) (1) Suppose that \(cl_{\tau(q)}(F)\) is not \(\tau(q)\)-compact. Then, we can find a sequence \((f_n)_{n \in \mathbb{N}}\) in \(F\) with \(f_j \neq f_i\) for \(i \neq j\) and the sequence \((f_n)_{n \in \mathbb{N}}\) in \(F\) does not converge with respect to \(\tau(q)\).

For any \(n \in \mathbb{N}\), let \(\mu_n := q(f_n, \{f_j : j \neq n\}) > 0\) and \(\epsilon_n := \left\{ \frac{1}{n}, \frac{\mu_n}{3} \right\}\).

It follows that the family \(\{D_q(f_n, \epsilon_n) : n \in \mathbb{N}\}\) is such that whenever \(i \neq k\), we have \(D_q(f_i, \epsilon_i) \neq D_q(f_k, \epsilon_k)\) and \(\epsilon_i + \epsilon_k < \max\{\mu_i, \mu_k\}\).

For any \(n \in \mathbb{N}\), let \(\phi_n : (X, q) \to (\mathbb{R}, u)\) be a function defined by
\[
\phi_n(x) := n - \frac{n}{\epsilon_n} q(f_n, x) \ 	ext{for any } x \in X.
\]

Then, for any \(x, y \in X\), we have two cases. The case \(u(\phi_n(x), \phi_n(y)) = 0\) is obvious. Otherwise, we have
\[
u(\phi_n(x), \phi_n(y)) = \phi_n(x) - \phi_n(y) = \left[ n - \frac{n}{\epsilon_n} q(f_n, x) \right] - \left[ n - \frac{n}{\epsilon_n} q(f_n, y) \right]
\]
\[
= \frac{n}{\epsilon_n} \left[ q(f_n, y) - q(f_n, x) \right]
\]
\[
\leq \frac{n}{\epsilon_n} \left[ q(f_n, x) + q(x, y) - q(f_n, x) \right]
\]
\[
= \frac{n}{\epsilon_n} q(x, y).
\]

Hence, \(\phi_n : (X, q) \to (\mathbb{R}, u)\) is \(k\)-semi-Lipschitz with \(k = \frac{n}{\epsilon_n}\). Observe that \(\phi_n(x) > 0\) if and only if \(q(f_n, x) < \epsilon_n\) whenever \(n \in \mathbb{N}\).

Let \(\varphi : (X, q) \to (\mathbb{R}, u)\) be defined by
\[
\varphi(x) = \begin{cases} 
\phi_n(x) & \text{if } x \in D_q(f_n, \epsilon_n) \\
0 & \text{otherwise}.
\end{cases}
\]

Since \(\varphi(f_n) = \phi_n(f_n) = n - \frac{n}{\epsilon_n} q(f_n, f_n) = n\), it follows that \(\varphi(F)\) is \(u\)-unbounded.
To complete the proof, we need to show that $\varphi$ is locally semi-Lipschitz. Let us consider an arbitrary point $x_0 \in X$. Since $\epsilon_n < \frac{1}{n}$ for any $n \in \mathbb{N}$ and the sequence $(f_n)$ does not converge to $x_0$, there exists $\delta > 0$ such that $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$ for at most finitely many $n$. Let us say $n_1, n_2, \ldots, n_k$.

Case 1. If $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) = \emptyset$, then $\varphi \mid_{D_q(x_0, \delta)} = 0$.

Case 2. If $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$, then whenever $q(x_0, x) < \delta$, we have
$$\varphi(x) = \max \{0, \phi_{n_1}(x), \phi_{n_2}(x), \ldots, \phi_{n_k}(x)\}.$$ Either way, $\varphi \mid_{D_q(x_0, \delta)}$ is semi-Lipschitz.

We point out that one can also generalize [2, Theorem 3.3] in our context.

**Theorem 5.2.** Let $(X, q)$ be a quasi-metric space and $\emptyset \neq F \subseteq X$. Then, the following conditions are equivalent:

1. $F$ is $q$-Bourbaki-bounded;
2. if $(Y, \|\cdot\|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|\cdot\|)$ is uniformly continuous, then $\varphi(F) \in \mathbb{R}_{\|\cdot\|}(Y)$;
3. if $(Y, \|\cdot\|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|\cdot\|)$ is semi-Lipschitz in the small, then $\varphi(F) \in \mathbb{R}_{\|\cdot\|}(Y)$;
4. if $\varphi : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz in the small, then $\varphi(F)$ is a u-bounded set of real numbers.

**Proof.** (1) $\Rightarrow$ (2) We assume that $\varphi : (X, q) \to (Y, \|\cdot\|)$ is uniformly continuous. Then there exists $\delta > 0$ such that whenever $x, y \in X$ with $q(x, y) < \delta$, then
$$q_{\|\cdot\|}(\varphi(x), \varphi(y)) = \|\varphi(x) - \varphi(y)\| < 1.$$ (5.1) By the $q$-Bourbaki-boundedness of $F$, there exists $A := \{a_1, a_2, \ldots, a_m\} \subseteq X$ such that
$$F \subseteq \bigcup_{i=1}^{m} D_q^n(a_i, \delta)$$ for some positive integer $n$. If we take $f$ arbitrarily in $F$, then there exists $k$ with $1 \leq k \leq m$ such that $f \in D_q^n(a_k, \delta)$. Then, for some $k$ with $1 \leq k \leq m$, there exists a $\delta$-chain $\{f_0, f_1, \ldots, f_n\}$ with $f_0 = a_k$ and $f_n = f$ and
$$q(f_{i-1}, f_i) < \delta$$ whenever $i$ with $1 \leq i \leq m$.

For some $k$ with $1 \leq k \leq m$, it follows from the uniform continuity of $\varphi$ and inequality (5.1) that
$$q_{\|\cdot\|}(\varphi(f_{i-1}), \varphi(f_i)) < 1$$ whenever $i$ with $1 \leq i \leq m$.

Hence, for some $k$ with $1 \leq k \leq m$, we have
$$q_{\|\cdot\|}(\varphi(a_k), \varphi(f)) = q_{\|\cdot\|}(f_0, f_n) \leq q_{\|\cdot\|}(f_0, f_1) + q_{\|\cdot\|}(f_1, f_2) + \cdots + q_{\|\cdot\|}(f_{n-1}, f_n) < n.$$ Hence, $\varphi(f) \in \bigcup_{i=1}^{m} D_q^n(\varphi(a_i), n)$ for any $f \in F$. Thus, $\varphi(F) \subseteq D_q(\varphi(A), n)$.

Hence, $\varphi(F)$ is $q_{\|\cdot\|}$-bounded.

(2) $\Rightarrow$ (3) Follows from Lemma 3.7.

(3) $\Rightarrow$ (4) It is obvious.
(4) ⇒ (1) Suppose that $F$ is not $q$-Bourbaki-bounded. Then, there exists $\delta$ such that if $\{f_1, f_2, \ldots, f_k\} \subseteq X$ and $n$ is a positive integer, we have $F \not\subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$. We have two cases on the structure of $F$.

**Case 1.** If $f \in F$, there exists a positive integer $n$ such that

$F \cap D_q^n(f, \delta) = F \cap \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$.

Let $f_1$ be an arbitrary point of $F$. We choose a positive integer $n_1$ such that

$F \cap D_q^{n_1}(f_1, \delta) = F \cap \bigcup_{n=1}^{\infty} D_q^n(f_1, \delta)$.

Since $F$ is not $q$-Bourbaki-bounded, there exists $f_2 \in F$ such that $f_2 \notin D_q^{n_1}(f_1, \delta)$.

It follows that $\bigcup_{n=1}^{\infty} D_q^n(f_1, \delta) \neq \bigcup_{n=1}^{\infty} D_q^n(f_2, \delta)$ by the choice of $n_1$.

Choose another positive integer $n_2$ such that $n_2 > n_1$ and

$F \cap D_q^{n_2}(f_2, \delta) = F \cap \bigcup_{n=1}^{\infty} D_q^n(f_2, \delta)$.

Moreover, since $F \not\subseteq \bigcup_{n=0}^{\infty} D_q^n(f_j, \delta)$, we can find

$f_3 \in F \setminus \left[ \bigcup_{n=0}^{\infty} D_q^n(f_3, \delta) \cup \bigcup_{n=2}^{\infty} D_q^n(f_2, \delta) \right]$.

Continuing this procedure by induction, we can find a sequence $(f_j)$ with distinct terms in $F$ such that, for any $i \neq j$, we have $\bigcup_{n=1}^{\infty} D_q^n(f_i, \delta) \neq \bigcup_{n=1}^{\infty} D_q^n(f_j, \delta)$.

Therefore, we define a function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$\varphi(x) = \begin{cases} j & \text{if } x \asymp f \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}$

It follows that the function $\varphi$ is constant on $D_q(x, \delta)$ and it is unbounded on $F$ since $\varphi(f_j) = j$. Therefore, the function $\varphi$ is semi-Lipschitz in the small function.

**Case 2.** If there exists $f \in F$, then for all positive integers $n$, there exists $j \in \mathbb{N}$ such that

$F \cap D_q^n(f, \delta) \subset F \cap D_q^{n+j}(f, \delta)$.

For $x \asymp f$, let $n(x)$ be the smallest positive integer $n$ such that

$x \in F \cap D_q^n(f, \delta)$.

We then define the function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$\varphi(x) = \begin{cases} (n(x) - 1)\delta + \text{dist}_q(x, D_q^{n(x)}(f, \delta)) & \text{if } x \neq f \text{ and } x \asymp f \\ 0 & \text{otherwise.} \end{cases}$
By definition, the function $\varphi$ is unbounded on $F$. We now have to show that if $x$ is not related to $y$ with respect to $\asymp \delta$ and $q(x, y) < \delta$, then for $k = 2$,

$$u(\varphi(x), \varphi(y)) \leq kq(x, y).$$

If either $x$ or $y$ is not related to $f$ with respect to $\asymp \delta$, then we have:

1. If $y \asymp \delta f$ but $x$ is not related to $f$ with respect to $\asymp \delta$, then

   $$u(\varphi(x), \varphi(y)) = 0 < 2q(x, y)$$

   since $\varphi(x) = 0$.

2. If $x \asymp \delta f$ but $y$ is not related to $f$ with respect to $\asymp \delta$, then

   $$u(\varphi(x), \varphi(y)) = (n(x) - 1)\delta + \text{dist}_q(x, D^n_q(x) - 1(f, \delta))$$

   $$\leq (n(x) - 1)\delta + q(x, y)$$

   $$= q(x, y) < 2q(x, y),$$

   since $f$ is not related to $y$ with respect to $\asymp \delta$, triangle inequality holds and $n(x) \leq 1$.

Now, if $x \asymp \delta f$ and $y \asymp \delta f$, then we have some cases on $n(x)$ and $n(y)$.

(a) If one of $n(x)$ or $n(y)$ is zero, then firstly, if $n(x) = 0$ and $n(y) \neq 0$, then we have $x = f$ and $0 < q(x, y) < \delta$, which implies that $y \in D^n_q(x, \delta)$, thus, $n(y) = 1$. Hence,

$$u(\varphi(x), \varphi(y)) = u(0, \varphi(y)) = 0 < 2q(x, y).$$

Secondly, if $n(y) = 0$ and $n(x) \neq 0$, then by similar arguments $n(x) = 1$. Thus,

$$u(\varphi(x), \varphi(y)) = u((1 - 1)\delta + \text{dist}_q(x, D^0_q(f, \delta)), 0)$$

$$= \text{dist}_q(x, \{y\}) = q(x, y) < 2q(x, y).$$

(b) If $n(x) = n(y) \geq 1$, then

$$u(\varphi(x), \varphi(y)) = \max\{\text{dist}_q(x, D^n_q(x) - 1(f, \delta)) - \text{dist}_q(y, D^n_q(x) - 1(f, \delta)), 0\}$$

$$\leq q(x, y) < 2q(x, y).$$

(c) If $n(x) > n(y) \geq 1$ (i.e. $n(x) = n(y) + j$ for some $j \geq 1$), then if $\varphi(x) \leq \varphi(y)$, then there is nothing to prove since $u(\varphi(x), \varphi(y)) = 0 < 2d(x, y)$.

Now, if $\varphi(x) > \varphi(y)$, then

$$u(\varphi(x), \varphi(y)) = \varphi(x) - \varphi(y)$$

$$= [(n(x) - 1)\delta + \text{dist}_q(x, D^n_q(x) - 1(f, \delta))] - [(n(y) - 1)\delta + \text{dist}_q(y, D^n_q(y) - 1(f, \delta))]$$

$$= [(n(y) + j - 1)\delta - [(n(y) - 1)\delta - \text{dist}_q(x, D^n(y) + j - 1(f, \delta))] - \text{dist}_q(y, D^n(y) - 1(f, \delta))]$$

Furthermore,

$$u(\varphi(x), \varphi(y)) = j\delta + [\text{dist}_q(x, D^n_q(y) - 1 + j(f, \delta))] - \text{dist}_q(y, D^n_q(y) - 1(f, \delta))$$

$$\leq j\delta + q(x, y) + \text{dist}_q(y, D^n(y) - 1 + j(f, \delta)) - \text{dist}_q(y, D^n(y) - 1(f, \delta)).$$

Since $n(y)$ is the smallest $n$ such that $y \in F \cap D^n_q(f, \delta)$, then

$$\text{dist}_q(y, D^n(y) - 1 + j(f, \delta)) = 0,$$
then we have
\[ u(\varphi(x), \varphi(y)) \leq j\delta + q(x, y) - \text{dist}_q(y, D^{n(y)-1}_q(f, \delta)). \]  
(5.2)

We claim that
\[ j\delta - q(x, y) \leq \text{dist}_q(y, D^{n(y)-1}_q(f, \delta)). \]  
(5.3)

Suppose otherwise that \( \text{dist}_q(y, D^{n(y)-1+j}_q(f, \delta)) < j\delta - q(x, y) \). Then,
\[
\begin{align*}
\text{dist}_q(x, D^{n(y)-1}_q(f, \delta)) &\leq q(x, y) + \text{dist}_q(y, D^{n(y)-1}_q(f, \delta)) \\
&< q(x, y) + j\delta - q(x, y) \\
&< j\delta.
\end{align*}
\]

So, \( x \in D^{n(y)-1+j}_q(f, \delta) \), which implies that \( n(x) \leq n(y) - 1 + j \) but this is a contradiction since \( n(x) > n(y) \).

Combining (5.2) and (5.3) we have
\[ u(\varphi(x), \varphi(y)) \leq j\delta + q(x, y) - j\delta + q(x, y) \leq 2q(x, y). \]

Therefore, the proof is complete.

\[ \square \]

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