Math. Appl. **11** (2022), 181–190
DOI: 10.13164/ma.2022.14

ON μ-PROXIMITY SPACES

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Abstract. The purpose of this paper is to introduce the notion of a μ-proximity base and to explore some properties and results of μ-proximity spaces. We show that the collection of all μ-proximities constitutes a complete lattice.

1. Introduction

Proximity is a more exquisite structure than topology and gives a lucid and conceptual approach to many topological problems. So, after the introduction by Efremovič in 1951, many authors worked on proximity spaces. Smirnov [13, 14] and Naimpally [8,9] did the most significant work in proximity spaces. Sharma [11] studied the notion of proximity bases. After that, Singh *et al.* [12] defined proximity bases in soft setting. Recently, Mukherjee *et al.* [7] introduced the notion of a μ-proximity space (a proximity space on generalized topological spaces).

In the present paper, we introduce the notion of a μ-proximity base and investigate the properties and results of μ-proximity spaces. We recall some basic definitions and results in Section 2. In Section 3, we continue to study the properties and results of μ-proximity spaces. It is shown that every $\mathcal{G}$-topological group [5] is a μ-proximity space. In the last section, we define a μ-proximity base and obtain an algorithm to generate a μ-proximity space. Further, it is demonstrated that the collection of μ-proximities always has supremum and infimum. Thus, it is concluded that the collection of μ-proximities constitutes a complete lattice. A necessary and sufficient condition for a map to be δ_μ-continuous is given in terms of a μ-proximity base.

2. Preliminaries

We recall some basic definitions and results in this section.

Definition 2.1. [9] A binary relation δ on $\mathcal{P}(X)$ is called a proximity on X if the following axioms are satisfied for all A, B, C in $\mathcal{P}(X)$:

(i) $(\emptyset, A) \notin \delta$;
(ii) If $A \cap B \neq \emptyset$, then $(A, B) \in \delta$;
(iii) If $(A, B) \in \delta$, then $(B, A) \in \delta$;
(iv) $(A, B \cup C) \in \delta$ if and only if $(A, B) \in \delta$ or $(A, C) \in \delta$;

MSC (2020): primary 54E05; secondary 54A05.
Keywords: μ-proximity, δ_μ-continuous, μ-proximity base.

(v) If $(A, B) \notin \delta$, then there exists a subset C of X such that $(A, C) \notin \delta$ and $(X \backslash C, B) \notin \delta$.

The pair (X, δ) is called a proximity space.

Definition 2.2. [11] Let X be a non-empty set. A proximity base on X is a binary relation β on $\mathcal{P}(X)$ satisfying the following axioms for all A, B, C in $\mathcal{P}(X)$:

(i) $(\emptyset, A) \notin \beta$;
(ii) If $A \cap B \neq \emptyset$, then $(A, B) \in \beta$;
(iii) If $(A, B) \in \beta$, then $(B, A) \in \beta$;
(iv) If $(A, B) \in \beta$ and $A \subseteq A^*$, $B \subseteq B^*$, then $(A^*, B^*) \in \beta$;
(v) If $(A, B) \notin \beta$, then there exists a subset C of X such that $(A, C) \notin \beta$ and $(X \backslash C, B) \notin \beta$.

Definition 2.3. [2] A collection μ of subsets of a set X is said to be a generalized topology (or GT) on X if $\emptyset \in \mu$ and if $U_i \in \mu$ for $i \in I \neq \emptyset$, then $\bigcup_{i \in I} U_i \in I$. The pair (X, μ) is called a generalized topological space (or GTS) and the members of μ are called a μ-open sets. If $A \in \mu$, then $X \backslash A$ is called μ-closed set.

Definition 2.4. [6] Let β be a collection of subsets of a nonempty set X. Then, β is said to be a base for a GT μ on X if $\mu = \{\cup \beta' : \beta' \subseteq \beta\}$.

Definition 2.5. [2] Let (X, μ) and (Y, ν) be two generalized topological spaces. Then, a function $f : (X, \mu) \longrightarrow (Y, \nu)$ is said to be (μ, ν)-continuous if, for any $U \in \nu$, we have $f^{-1}(U) \in \mu$.

Definition 2.6. [10] A GTS (X, μ) is said to be μT_0 if, for any pair of distinct points in X, there is a μ-open set containing one point but not the other.

Definition 2.7. [10] A GTS (X, μ) is said to be μT_1 if, for any pair of distinct points x, y in X, there are μ-open sets U_x and V_y containing x and y, respectively, such that $y \notin U_x$ and $x \notin U_y$.

Definition 2.8. [10] A GTS (X, μ) is said to be μT_2 if, for any pair of distinct points x, y in X, there are μ-open sets U_x and V_y containing x and y, respectively, such that $U_x \cap V_y = \emptyset$.

Definition 2.9. [7] A GTS (X, μ) is called μ-completely regular if, for any μ-closed set F and for any $x \notin F$, there exists a μ-continuous map $f : (X, \mu) \longrightarrow (\mathbb{R}, \nu)$ such that $f(x) = 0$ and $f(F) = \{1\}$, where ν is the usual GT on $\mathbb{R}$ generated by the base $\beta = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$.

Definition 2.10. [4] A GTS (X, μ) is called μ-normal if, for every pair F, G of disjoint μ-closed sets, there exists a pair U, V of disjoint μ-open sets such that $F \subset U$ and $G \subset V$.

We say that a GTS (X, μ) is μT_4 if it is μT_1 and μ-normal.

Definition 2.11. [5] A nonempty set X is said to be a $\mathcal{G}$-topological group (or μ-topological group) if it is a group and also a μT_2 GTS such that the multiplication and inverse maps are μ-continuous.

Definition 2.12. [7] A μ-proximity on a set X is a binary relation δ_μ on the power set $\mathcal{P}(X)$ which satisfies the following axioms for all $A, B, C \in \mathcal{P}(X)$:

(i) $(A, B) \in \delta_\mu$ if and only if $(B, A) \in \delta_\mu$;
(ii) If $(A, B) \in \delta_\mu$ with $A \subseteq A^*$ and $B \subseteq B^*$, then $(A^*, B^*) \in \delta_\mu$;
(iii) $(\{x\}, \{x\}) \in \delta_\mu$, for all $x \in X$;
(iv) If $(A, B) \notin \delta_\mu$, then there exists some $E \subseteq X$ such that $(A, E) \notin \delta_\mu$ and $(X \backslash E, B) \notin \delta_\mu$.

The pair (X, δ_μ) is called a μ-proximity space.

Definition 2.13. [7] Let (X, δ_μ) be a μ-proximity space. A subset B is said to be a δ_μ-neighbourhood of A if $(A, X \backslash B) \notin \delta_\mu$. It is denoted by $A \ll_\mu B$.

Proposition 2.14. [7] *Let (X, δ_μ) be a μ-proximity space and A be a subset of X. Define " A is δ_μ-closed if and only if $x \in A$ whenever $(\{x\}, A) \in \delta_\mu$". Then, the collection of the complements of all δ_μ-closed sets forms a generalized topology (GT) $\mu = \mathcal{T}(\delta_\mu)$ on X.*

Definition 2.15. [7] Let δ_μ be a μ-proximity on GTS (X, μ). Then, δ_μ is compatible with μ if $\mathcal{T}(\delta_\mu) = \mu$.

Proposition 2.16. [7] *Let (X, δ_μ) be a μ-proximity space and $\mu = \mathcal{T}(\delta_\mu)$. Then, the μ-closure $C_\mu(A)$ of A in X is given by $C_\mu(A) = \{x : (\{x\}, A) \in \delta_\mu\}$.*

Corollary 2.17. [7] *Let (X, δ_μ) be a μ-proximity space and $\mu = \mathcal{T}(\delta_\mu)$. Then, $U \in \mu$ if and only if $(\{x\}, X \backslash U) \notin \delta_\mu$ for every $x \in U$.*

Theorem 2.18. [7] *Let (X, μ) be a μ-completely regular GTS. Then, the relation δ_{μ_f} defined by " $(A, B) \notin \delta_{\mu_f}$ if and only if there exists a μ-continuous function $g : (X, \mu) \longrightarrow [0, 1]$ such that $g(A) = \{0\}$ and $g(B) = \{1\}$, where $[0, 1]$ is endowed with subspace GT induced by usual GT ν on $\mathbb{R}$", is a compatible μ-proximity on X.*

3. On μ-proximity space

Let $\mathcal{F}_\mu$ be the family of all μ-proximities on a set X. Define a relation on $\mathcal{F}_\mu$ as: $\delta_{\mu_1} > \delta_{\mu_2}$ if and only if $(A, B) \in \delta_{\mu_1}$ implies $(A, B) \in \delta_{\mu_2}$ for all $A, B \subset X$. Then note that $>$ is a partial order relation on $\mathcal{F}_\mu$.

Definition 3.1. Let $\delta_{\mu_1}, \delta_{\mu_2}$ be two μ-proximities on X. Then, δ_{μ_1} is said to be finer than δ_{μ_2} (equivalently, δ_{μ_2} is coarser than δ_{μ_1}) if $\delta_{\mu_1} > \delta_{\mu_2}$.

Note that discrete and indiscrete proximities on a set X are also the finest and the coarsest μ-proximities on X, respectively. Thus, for any μ-proximity δ_μ, we have $\delta_{dis} > \delta_\mu > \delta_{ind}$, where δ_{dis} and δ_{ind} are discrete and indiscrete proximities, respectively.

Proposition 3.2. *Let δ_{μ_1} and δ_{μ_2} be two μ-proximities on a set X. If $\delta_{\mu_1} \geq \delta_{\mu_2}$, then $\mu_2 \subseteq \mu_1$, where μ_1 and μ_2 are generalized topologies on X generated by δ_{μ_1} and δ_{μ_2}, respectively.*

Proof. Let $U \in \mu_2$. Then, $(\{x\}, X \backslash U) \notin \delta_{\mu_2}$ for each $x \in U$. Thus, $(\{x\}, X \backslash U) \notin \delta_{\mu_1}$ for each $x \in U$. Hence, $U \in \mu_1$. □

Example 3.3. Let (X, δ_μ) be any μ-proximity space. Then, $\delta_{dis} > \delta_\mu > \delta_{ind}$. Therefore, by Proposition 3.2, we have $\mathcal{T}_{ind} \subset \mu \subset \mathcal{T}_{dis}$, where $\mathcal{T}_{ind}$, μ and $\mathcal{T}_{dis}$ are indiscrete, generalized and discrete topologies on X, respectively.

Example 3.4. Let δ_1, δ_2 be two proximities on $\mathbb{R}$ defined as:
$(A, B) \in \delta_1$ if and only if $d(A, B) = 0$, where d is the usual metric on $\mathbb{R}$
$(A, B) \in \delta_2$ if and only if $Cl(A) \cap Cl(B) \neq \emptyset$.
Then, δ_1, δ_2 are μ-proximities on $\mathbb{R}$ such that $\mu(\delta_1) = \mu(\delta_2)$. If $A = \{n : n \in \mathbb{N}\}$ and $B = \{n - \frac{1}{n} : n \in \mathbb{N}\}$, then $(A, B) \in \delta_1$ but $(A, B) \notin \delta_2$. Therefore, $\delta_1 \neq \delta_2$.

Corollary 3.5. *Every proximity base on a set X is a μ-proximity on X.*

Observe that the intersection of proximities may not be a proximity, but the intersection of μ-proximities is always a μ-proximity.

Theorem 3.6. *Let (X, μ_0) be a μT_4 GTS. Then, the relation δ_{μ_0} defined by $(A, B) \in \delta_{\mu_0}$ if and only if $C_{\mu_0}(A) \cap C_{\mu_0}(B) \neq \emptyset$, is the largest compatible μ-proximity on X.*

Proof. To prove δ_{μ_0} is a μ-proximity, note that axioms $(i), (ii)$ and (iii) easily follow. So, for axiom (iv), let $(A, B) \notin \delta_{\mu_0}$. Then, $C_{\mu_0}(A) \cap C_{\mu_0}(B) = \emptyset$. Since X is μ-normal, there exists a pair U, V of disjoint μ-open sets such that $C_{\mu_0}(A) \subset U$ and $C_{\mu_0}(B) \subset V$. Thus, $C_{\mu_0}(A) \cap (X \backslash U) = \emptyset$, which implies $(A, X \backslash U) \notin \delta_{\mu_0}$. Similarly, $(X \backslash V, B) \notin \delta_{\mu_0}$. Since $V \subset X \backslash U$, $(A, V) \notin \delta_{\mu_0}$. Hence, there is a subset V such that $(A, V) \notin \delta_{\mu_0}$ and $(X \backslash V, B) \notin \delta_{\mu_0}$.

Now, it is to show that δ_{μ_0} is compatible with μ_0. Let $x \in C_{\mu_0}(A)$, which implies $C_{\mu_0}(x) \cap C_{\mu_0}(A) \neq \emptyset$. Therefore, $(\{x\}, A) \in \delta_{\mu_0}$. Conversely, let $(\{x\}, A) \in \delta_{\mu_0}$. Then, $C_{\mu_0}(x) \cap C_{\mu_0}(A) \neq \emptyset$. Since X is μT_1, $C_{\mu_0}(x) = x$. Thus, $x \in C_{\mu_0}(A)$.

Let δ_μ be an arbitrary μ-proximity compatible with μ_0. Let $(A, B) \in \delta_{\mu_0}$, which implies $C_{\mu_0}(A) \cap C_{\mu_0}(B) \neq \emptyset$. Therefore, $(C_{\mu_0}(A), C_{\mu_0}(B)) \in \delta_\mu$. Thus, $(A, B) \in \delta_\mu$. Hence, $\delta_{\mu_0} > \delta_\mu$. ☐

Corollary 3.7. *In a μT_4 GTS (X, μ), the μ-proximities δ_{μ_0} and δ_{μ_f}, which are defined in Theorem 3.6 and 2.18, respectively, are equivalent.*

Proof. Let $(A, B) \notin \delta_{\mu_f}$. Then, $A \subset f^{-1}\{0\}$ and $B \subset f^{-1}\{1\}$. Therefore, $C_\mu(A) \subset f^{-1}\{0\}$ and $C_\mu(B) \subset f^{-1}\{1\}$. Therefore, $C_\mu(A) \cap C_\mu(B) = \emptyset$, which implies $(A, B) \notin \delta_{\mu_0}$. On the other hand, suppose that $(A, B) \notin \delta_{\mu_0}$. Then, $C_\mu(A) \cap C_\mu(B) = \emptyset$. Since X is μT_4, there exists a μ-continuous map $f : (X, \mu) \longrightarrow [0, 1]$ such that $f(C_\mu(A)) = \{0\}$ and $f(C_\mu(B)) = \{1\}$. Therefore, $f(A) = \{0\}$ and $f(B) = \{1\}$, which implies $(A, B) \notin \delta_{\mu_f}$. ☐

Theorem 3.8. *If a μ-completely regular space (X, μ) has a compatible μ-proximity defined as $(A, B) \in \delta_\mu$ if and only if $C_\mu(A) \cap C_\mu(B) \neq \emptyset$, then X is μ-normal.*

Proof. Let A and B be disjoint μ-closed sets. Then, $(A, B) \notin \delta_\mu$. Therefore, there exists a subset U such that $(A, U) \notin \delta_\mu$ and $(X \backslash U, B) \notin \delta_\mu$. So, $A \subset i_\mu(X \backslash U)$ and $B \subset i_\mu(U)$. Since $i_\mu(U)$ and $i_\mu(X \backslash U)$ are disjoint, X is μ-normal. ☐

Proposition 3.9. *Let (X, μ) be a μT_4 GTS and δ_μ be the μ-proximity as defined in Theorem 3.5 such that $\mu^* \subseteq \mu$, where μ^* is GT generated by an arbitrary μ-proximity δ^*_μ. Then, $\delta_\mu > \delta^*_\mu$.*

Proof. Let $(A,B) \notin \delta^*_\mu$. Then, $(C_{\mu^*}(A), C_{\mu^*}(B) \notin \delta^*_\mu$. Since $\mu^* \subset \mu$, $C_\mu(A) \subset C_{\mu^*}(A)$ for any set A. So, $(C_\mu(A), C_\mu(B)) \notin \delta^*_\mu$, which implies $C_\mu(A) \cap C_\mu(B) = \emptyset$. Thus, $(A,B) \notin \delta_\mu$. □

Theorem 3.10. *Let A be a subset of the μ-proximity space (X, δ_μ). Then, $C_\mu(A) = \bigcap\{U : A \ll_\mu U\}$.*

Proof. Let $A \ll_\mu U$. Then, $C_\mu(A) \ll_\mu U$. Therefore, $C_\mu(A) \subset \bigcap\{U : A \ll_\mu U\}$. On the other hand, if $x \notin C_\mu(A)$, then $(\{x\}, A) \notin \delta_\mu$. Therefore, there exists a subset U such that $(\{x\}, U) \notin \delta_\mu$ and $(X \backslash U, A) \notin \delta_\mu$. Thus, there is a δ_μ-neighbourhood U of A such that $x \notin U$. Hence, $x \notin \bigcap\{U : A \ll_\mu U\}$. □

Proposition 3.11. *For a μ-proximity δ_μ on X, the following statements are equivalent:*

(i) *If $(P,Q) \notin \delta_\mu$, then there exists some $E \subseteq X$ such that $(P,E) \notin \delta_\mu$ and $(X \backslash E, Q) \notin \delta_\mu$.*
(ii) *If $(P,Q) \notin \delta_\mu$, then there exist subsets R and S such that $(P,R) \notin \delta_\mu$ and $(S,Q) \notin \delta_\mu$ with $R \cup S = X$.*
(iii) *If $(P,Q) \notin \delta_\mu$, then there exist subsets R and S such that $(P, X \backslash R) \notin \delta_\mu$ and $(X \backslash S, Q) \notin \delta_\mu$ with $R \cap S = \emptyset$.*

Proof. (i) $\Rightarrow$ (ii). If $E = R$ and $X \backslash E = S$, then (ii) holds.

(ii) $\Rightarrow$ (iii). If $R' = X \backslash R$ and $S' = X \backslash S$, then $(P, X \backslash R') \notin \delta_\mu$ and $(X \backslash S', Q) \notin \delta_\mu$ with $R' \cap S' = \emptyset$.

(iii) $\Rightarrow$ (i). If $X \backslash R = S$, then (i) holds. □

Definition 3.12. A μ-proximity space (X, δ_μ) is said to be μ-separated if $(\{x\}, \{y\}) \notin \delta_\mu$ whenever $x \neq y$ for all $x, y \in X$.

Theorem 3.13. *A μ-proximity space is μ-separated if and only if the generated GT μ is μT_0.*

Proof. Let X be μ-separated and $x, y \in X$. If $x \neq y$, then $(\{x\}, \{y\}) \notin \delta_\mu$. So, there exists a subset U such that $(\{x\}, U) \notin \delta_\mu$ and $(X \backslash U, \{y\}) \notin \delta_\mu$. Therefore, U is a μ-open set which contains y but not x. Thus, μ is μT_0.

Conversely, let μ be μT_0 and $x \neq y$ in X. Then, there is a μ-open set U containing x but not y. Therefore, $(\{x\}, X \backslash U) \notin \delta_\mu$ and $y \in X \backslash U$. Thus, $(\{x\}, \{y\}) \notin \delta_\mu$. □

Corollary 3.14. *Let (X, δ_μ) be a μ-separated μ-proximity space. Then, $x \ll_\mu X \backslash \{y\}$ if and only if $x \neq y$.*

Definition 3.15. Let (X, δ_μ) and $(Y, \delta'_{\mu'})$ be two μ-proximity spaces. Then, a map $f : X \longrightarrow Y$ is said to be δ_μ-continuous if $(f(A), f(B)) \in \delta'_{\mu'}$ whenever $(A,B) \in \delta_\mu$ for all subsets A, B of X.

Proposition 3.16. *Let (X, δ_μ) and $(Y, \delta'_{\mu'})$ be two μ-proximity spaces. Then, a map $f : X \longrightarrow Y$ is δ_μ-continuous if and only if $f^{-1}(C) \ll_\mu f^{-1}(D)$ whenever $C \ll_\mu D$.*

Proof. Proof is obvious. □

Theorem 3.17. *Every δ_μ-continuous map is (μ, μ')-continuous.*

Proof. Let $f : X \longrightarrow Y$ be a δ_μ-continuous map and $x \in C_\mu(A)$. Then, $(\{x\}, A) \in \delta_\mu$. Therefore, $(f(x), f(A)) \in \delta'_{\mu'}$ as f is δ_μ-continuous. So, $f(x) \in C_{\mu'}(f(A))$. Thus, $f(C_\mu(A)) \subset C_{\mu'}(f(A))$. Hence, f is (μ, μ')-continuous. □

The converse of the above theorem may not be true as the following example shows.

Example 3.18. Let δ_1 and δ_2 be two μ-proximities on $\mathbb{R}$ as defined in Example 3.4. Then, the identity map $I : (\mathbb{R}, \delta_1) \longrightarrow (\mathbb{R}, \delta_2)$ is μ-continuous. If $A = \{n : n \in \mathbb{N}\}$ and $B = \{n - \frac{1}{n} : n \in \mathbb{N}\}$, then $(A, B) \in \delta_1$ but $(A, B) \notin \delta_2$. Therefore, it is not δ_μ-continuous.

Theorem 3.19. *Let $f : X \longrightarrow (Y, \delta'_{\mu'})$ be a map where $(Y, \delta'_{\mu'})$ is a μ-proximity space. Then, the coarsest μ-proximity δ_{μ_0} on X which makes f a δ_μ-continuous map is given by $(A, B) \notin \delta_{\mu_0}$ if and only if there exists a subset C of Y such that $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash B)$.*

Proof. Firstly, it is to prove that δ_{μ_0} is a μ-proximity on X.

(i) Let $(A, B) \notin \delta_{\mu_0}$. Then, there is a subset C such that $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash B)$. Let $P = Y\backslash f(A)$. Then, $(f(B), Y\backslash P) \notin \delta'_{\mu'}$ because $f(B) \subset (Y\backslash C)$ and $(f(A), Y\backslash C) \notin \delta'_{\mu'}$. Also, $f^{-1}(P) \subset (X\backslash A)$. Thus, by the definition of δ_{μ_0}, $(B, A) \notin \delta_{\mu_0}$.

(ii) Let $(A, B) \notin \delta_{\mu_0}$, $P \subseteq A$ and $Q \subseteq B$. Then, there is a subset C such that $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash B)$. Therefore, $(f(P), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash Q)$. Thus, $(P, Q) \notin \delta_{\mu_0}$.

(iii) Obviously, $(\{x\}, \{x\}) \in \delta_{\mu_0}$, for all $x \in X$.

(iv) Let $(A, B) \notin \delta_{\mu_0}$. Then, there is a subset C such that $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash B)$. Since $\delta'_{\mu'}$ is a μ-proximity and $(f(A), Y\backslash C) \notin \delta'_{\mu'}$, there exists a subset P of Y such that $(f(A), P) \notin \delta'_{\mu'}$ and $(Y\backslash P, Y\backslash C) \notin \delta'_{\mu'}$. Let $E = f^{-1}(P)$. Then, $(A, E) \notin \delta_{\mu_0}$ as $(f(A), P) \notin \delta'_{\mu'}$ and $f^{-1}(Y\backslash P) = (X\backslash E)$. Also, $f(X\backslash E) \subset (Y\backslash P)$ and $(Y\backslash P, Y\backslash C) \notin \delta'_{\mu'}$ with $f^{-1}(C) \subset (X\backslash B)$, therefore, $(X\backslash E, B) \notin \delta_{\mu_0}$.

Now, it is to show that f is δ_μ-continuous. Let $(f(A), f(B)) \notin \delta'_{\mu'}$. Then, $f(A) \ll_\mu Y\backslash f(B)$. Therefore, there exists a subset C of Y such that $f(A) \ll_\mu C \ll_\mu Y\backslash f(B)$. Thus, $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset X\backslash B$. Hence, $(A, B) \notin \delta_{\mu_0}$.

It only remains to show that δ_{μ_0} is the coarsest μ-proximity on X which makes f a δ_μ-continuous map. So, let δ_{μ_1} be any μ-proximity on X which makes f a δ_μ-continuous map. Then, $\delta_{\mu_1} > \delta_{\mu_0}$ because, if $(A, B) \notin \delta_{\mu_0}$, then there exists a subset C of Y such that $(f(A), Y\backslash C) \notin \delta'_{\mu'}$ and $f^{-1}(C) \subset (X\backslash B)$; therefore, $B \subset X\backslash f^{-1}(C)$ and $(A, X\backslash f^{-1}(C)) \notin \delta_{\mu_1}$ as f is δ_μ-continuous; thus, $(A, B) \notin \delta_{\mu_1}$. □

Corollary 3.20. *Let $f : (X, \delta_{\mu_0}) \longrightarrow (Y, \delta'_{\mu'})$ be a map where δ_{μ_0} is the μ-proximity as defined in Theorem 3.19. Then, $f^{-1}(\mu') \subset \mu_0$, where μ' and μ_0 are generated GT's.*

Proposition 3.21. *Let (X,δ_μ) be a μ-proximity space and $Y \subset X$. For $A,B \subset Y$, let $(A,B) \in \delta_{\mu_Y}$ if and only if $(A,B) \in \delta_\mu$. Then, (Y,δ_{μ_Y}) is a μ-proximity space.*

Proof. Axioms $(i),(ii),(iii)$ of μ-proximity are obvious. So, for axiom (iv), let $(A,B) \notin \delta_{\mu_Y}$. Then, $(A,B) \notin \delta_\mu$. Therefore, there exists a subset E^* of X such that $(A,E^*) \notin \delta_\mu$ and $(X\backslash E^*,B) \notin \delta_\mu$. Let $E = E^* \cap Y$. Then, $E \subset E^*$ and, therefore, $Y\backslash E \subset X\backslash E^*$. Thus, $(A,E) \notin \delta_\mu$ and $(Y\backslash E,B) \notin \delta_\mu$. Hence, $(A,E) \notin \delta_{\mu_Y}$ and $(Y\backslash E,B) \notin \delta_{\mu_Y}$. □

The μ-proximity as defined in the above theorem on the subset Y is called subspace μ-proximity (induced μ-proximity) on Y, and the pair (Y,δ_{μ_Y}) is called μ-proximity subspace.

Theorem 3.22. *Let (Y,δ_{μ_Y}) be a μ-proximity subspace of (X,δ_μ). Then, $\mu_Y = \mu|_Y$.*

Proof. Note that $A \in \mu_Y$ if and only if $(\{x\},Y\backslash A) \notin \delta_{\mu_Y}$ for every $x \in A$ if and only if $(\{x\} \cap Y,(X\backslash A) \cap Y) \notin \delta_\mu$ for every $x \in A$ if and only if $A \in \mu|_Y$. □

Proposition 3.23. *Every $\mathcal{G}$-topological group $((X,.),\mathcal{G})$ induces a μ-proximity on X given by $(A,B) \in \delta_\mu$ if and only if, for each $U \in \mathcal{N}$, $UA \cap B \neq \emptyset$, where $\mathcal{N}$ is the neighbourhood system of identity.*

Proof. (i) Let $(A,B) \in \delta_\mu$. Then, $UA \cap B \neq \emptyset$ for each $U \in \mathcal{N}$. Since for each $U \in \mathcal{N}$ there exists $U^{-1} \in \mathcal{N}$ such that $UU^{-1}A \subset A$, $A \cap U^{-1}B \neq \emptyset$ for each $U \in \mathcal{N}$. Thus, $(B,A) \in \delta_\mu$.

(ii) Let $(A,B) \in \delta_\mu$ and $A \subset A^*, B \subset B^*$. Then, $UA^* \cap B^* \neq \emptyset$ for each $U \in \mathcal{N}$. Therefore, $(A^*,B^*) \in \delta_\mu$.

(iii) $(\{x\},\{x\}) \in \delta_\mu$ for each $x \in X$.

(iv) Let $(A,B) \notin \delta$. Then, there exists some $U \in \mathcal{N}$ such that $UA \cap B = \emptyset$. Let $E = U^{-1}B$. Then, $A \cap UE = A \cap U(U^{-1}B) \subset A \cap B$. Therefore, $A \cap UE = \emptyset$. Thus, $(A,E) \notin \delta_\mu$. Also, $U^{-1}B \cap X\backslash E = \emptyset$. Therefore, $(X\backslash E,B) \notin \delta_\mu$.

Hence, δ_μ is a μ-proximity. □

4. μ-PROXIMITY BASE

Definition 4.1. Let X be a nonempty set. A μ-proximity base on X is a binary relation β_μ on the power set $\mathcal{P}(X)$ of X which satisfies the following axioms for all subsets A,B of X:

(i) If $A \cap B \neq \emptyset$, then $(A,B) \in \beta_\mu$;

(ii) If $(A,B) \notin \beta_\mu$, then there exists some $E \subseteq X$ such that $(A,E) \notin \beta_\mu$ and $(X\backslash E,B) \notin \beta_\mu$.

Definition 4.2. A μ-proximity base β_μ is said to be μ-separated if it satisfies the following axiom:

"if x,y are two distinct elements of X such that $(\{x\},\{y\}) \in \beta_\mu$, then there exist subsets P and Q containing x and y, respectively, such that either $(P,Q) \notin \beta_\mu$ or $(Q,P) \notin \beta_\mu$".

Theorem 4.3. *Let β_μ be a μ-proximity base on a set X and let $\delta_\mu(\beta_\mu)$ be a binary relation on $\mathcal{P}(X)$ which is defined as $(A,B) \in \delta_\mu(\beta_\mu)$ if and only if for any sets $A \subseteq A^*$ and $B \subseteq B^*$, both (A^*,B^*) and (B^*,A^*) are the elements of β_μ. Then, $\delta_\mu(\beta_\mu)$ is the coarsest μ-proximity on X finer than β_μ. Moreover, $\delta_\mu(\beta_\mu)$ is μ-separated if and only if β_μ is μ-separated.*

Proof. Obviously, $\delta_\mu(\beta_\mu) \geq \beta_\mu$. Now, it is to prove that $\delta_\mu(\beta_\mu)$ is a μ-proximity on X. Axiom (i) is obvious. Let $(A,B) \in \delta_\mu(\beta_\mu)$ with $A \subseteq A^{**}$ and $B \subseteq B^{**}$. Suppose A^* and B^* are any sets such that $A^{**} \subseteq A^*$ and $B^{**} \subseteq B^*$. Then, $A \subseteq A^*$ and $B \subseteq B^*$. Since $(A,B) \in \delta_\mu(\beta_\mu)$ with $A \subseteq A^*$ and $B \subseteq B^*$, both (A^*,B^*) and (B^*,A^*) are the elements of β_μ. Thus, $(A^{**},B^{**}) \in \delta_\mu(\beta_\mu)$. For axiom (iii), Let A and B be the subsets of X containing x. Then, $A \cap B \neq \emptyset$. Therefore, both (A,B) and (B,A) are the elements of β_μ. Thus, $(\{x\},\{x\}) \in \delta_\mu(\beta_\mu)$ for all $x \in X$. To prove the last axiom, let $(A,B) \notin \delta_\mu(\beta_\mu)$. Then, by the definition of $\delta_\mu(\beta_\mu)$, there exist sets A^* and B^* such that $A \subseteq A^*$ and $B \subseteq B^*$ with either $(A^*,B^*) \notin \beta_\mu$ or $(B^*,A^*) \notin \beta_\mu$.

Case (i). If $(A^*,B^*) \notin \beta_\mu$, then, by the definition of β_μ, there exists a subset E such that $(A^*,E) \notin \beta_\mu$ and $(X \backslash E, B^*) \notin \beta_\mu$. Therefore, by the definition of $\delta_\mu(\beta_\mu)$, $(A,E) \notin \delta_\mu(\beta_\mu)$. Similarly, $(X\backslash E, B) \notin \delta_\mu(\beta_\mu)$.

Case (ii). If $(B^*,A^*) \notin \beta_\mu$, then, by the definition of β_μ, there exists a subset E such that $(B^*,E) \notin \beta_\mu$ and $(X\backslash E, A^*) \notin \beta_\mu$. Therefore, by the definition of $\delta_\mu(\beta_\mu)$, $(B,E) \notin \delta_\mu(\beta_\mu)$. Similarly, $(X\backslash E, A) \notin \delta_\mu(\beta_\mu)$.

Therefore, $\delta_\mu(\beta_\mu)$ is a μ-proximity on X. Now, let δ_μ be any μ-proximity on X such that $\delta_\mu \geq \beta_\mu$. The next claim is that $\delta_\mu \geq \delta_\mu(\beta_\mu)$. Let $(A,B) \in \delta_\mu$. Then, $(A^*,B^*) \in \delta_\mu$ and $(B^*,A^*) \in \delta_\mu$ for any sets A^* and B^* with $A \subseteq A^*$ and $B \subseteq B^*$. Therefore, $(A^*,B^*) \in \beta_\mu$ and $(B^*,A^*) \in \beta_\mu$. Thus, by the definition of $\delta_\mu(\beta_\mu)$, $(A,B) \in \delta_\mu(\beta_\mu)$. Now, it only remains to show that $\delta_\mu(\beta_\mu)$ is μ-separated if and only if β_μ is μ-separated. So, let $\delta_\mu(\beta_\mu)$ be μ-separated and $x \neq y$ in X such that $(\{x\},\{y\}) \in \beta_\mu$. Then, $(\{x\},\{y\}) \notin \delta_\mu(\beta_\mu)$. So, by definition of $\delta_\mu(\beta_\mu)$, there exist subsets P and Q containing x and y, respectively, such that either $(P,Q) \notin \beta_\mu$ or $(Q,P) \notin \beta_\mu$. Thus, β_μ is μ-separated. Conversely, let β_μ be μ-separated and $x \neq y$ in X. To show $\delta_\mu(\beta_\mu)$ is μ-separated, it suffices to show that $(\{x\},\{y\}) \notin \beta_\mu$. So, let $(\{x\},\{y\}) \in \beta_\mu$. Then, there exist subsets P and Q containing x and y, respectively, such that either $(P,Q) \notin \beta_\mu$ or $(Q,P) \notin \beta_\mu$. Thus, by the definition of $\delta_\mu(\beta_\mu)$, $(\{x\},\{y\}) \notin \delta_\mu(\beta_\mu)$. □

Theorem 4.4. *Let $\mathcal{F} = \{\delta_{\mu_a} : a \in \mathcal{I}\}$ be a family of μ-proximities on a set X. Then, there exists a coarsest μ-proximity δ_μ on X such that δ_μ is finer than δ_{μ_a} for each $a \in \mathcal{I}$.*

Proof. Let $\delta_\mu = \bigcap_{a\in\mathcal{I}} \delta_{\mu_a}$. Then, δ_μ is a μ-proximity as the intersection of μ-proximities is also a μ-proximity. Also, $\delta_\mu \geq \delta_{\mu_a}$ for each $a \in \mathcal{I}$. Now, let δ'_μ be any μ-proximity on X such that $\delta'_\mu \geq \delta_{\mu_a}$ for each $a \in \mathcal{I}$. Then, $\delta'_\mu \geq \bigcap_{a\in\mathcal{I}} \delta_{\mu_a}$. Therefore, $\delta'_\mu \geq \delta_\mu$. □

Remark 4.5. In view of Theorem 4.4, the coarsest μ-proximity δ_μ which is finer than each δ_{μ_a} is denoted by $\sup\{\delta_{\mu_a} : a \in \mathcal{I}\}$.

Proposition 4.6. *If $\{\delta_{\mu_a} : a \in \mathcal{I}\}$ is a family of μ-proximities on a set X, then $\mu[\sup\{\delta_{\mu_a} : a \in \mathcal{I}\}] = \sup\{\mu(\delta_{\mu_a}) : a \in \mathcal{I}\}$.*

Proof. It is easy to see that $\sup\{\mu(\delta_{\mu_a}) : a \in \mathcal{I}\} \subseteq \mu[\sup\{\delta_{\mu_a} : a \in \mathcal{I}\}]$ as $\sup\{\delta_{\mu_a} : a \in \mathcal{I}\} \geq \delta_{\mu_a}$ for each $a \in \mathcal{I}$. On the other hand, let $U \in \mu[\sup\{\delta_{\mu_a} : a \in \mathcal{I}\}]$. Then, $(\{x\}, X\backslash U) \notin \delta_\mu$ for each $x \in U$, where $\delta_\mu = \sup\{\delta_{\mu_a} : a \in \mathcal{I}\}$. Therefore, there is some $a' \in \mathcal{I}$ such that $(\{x\}, X\backslash U) \notin \delta_{\mu_{a'}}$ for each $x \in U$. Therefore, $U \in \mu(\delta_{\mu_{a'}})$, which implies $U \in \sup\{\mu(\delta_{\mu_a}) : a \in \mathcal{I}\}$. □

Theorem 4.7. *Let $\mathcal{F} = \{\delta_{\mu_a} : a \in \mathcal{I}\}$ be a family of μ-proximities on a set X. Then, there exists a finest μ-proximity δ_μ on X such that δ_μ is coarser than each δ_{μ_a} for $a \in \mathcal{I}$.*

Proof. Let γ be the collection of μ-proximities on X which is defined as:

$$\gamma = \{\delta_{\mu_p} : \delta_{\mu_a} \geq \delta_{\mu_p} \text{ for each } a \in \mathcal{I}\}.$$

Clearly, γ is a nonempty collection. Let $\delta_\mu = \sup\{\delta_{\mu_p} : \delta_{\mu_p} \in \gamma\}$. Now, to show $\delta_{\mu_a} \geq \delta_\mu$. So, suppose $(A, B) \in \delta_{\mu_a}$ for some $a \in \mathcal{I}$. Then, $(A, B) \in \delta_{\mu_p}$ for each $\delta_{\mu_p} \in \gamma$. Therefore, $(A, B) \in \bigcap\{\delta_{\mu_p} : \delta_{\mu_p} \in \gamma\} = \sup\{\delta_{\mu_p} : \delta_{\mu_p} \in \gamma\} = \delta_\mu$. Thus, $\delta_{\mu_a} \geq \delta_\mu$, that is, $\delta_\mu \in \gamma$. Now, let δ'_μ be any μ-proximity on X such that $\delta_{\mu_a} \geq \delta'_\mu$ for each $a \in \mathcal{I}$. Then, $\delta'_\mu \in \gamma$. Now, it only remains to show that $\delta_\mu \geq \delta'_\mu$. If $(A, B) \in \delta_\mu$, then $(A, B) \in \delta_{\mu_p}$ for each $\delta_{\mu_p} \in \gamma$. Therefore, $(A, B) \in \delta'_\mu$. Thus, $\delta_\mu \geq \delta'_\mu$. □

Remark 4.8. In view of Theorem 4.7, the finest μ-proximity δ_μ which is coarser than each δ_{μ_a} is denoted by $\inf\{\delta_{\mu_a} : a \in \mathcal{I}\}$.

Corollary 4.9. *If $\{\delta_{\mu_a} : a \in \mathcal{I}\}$ is a family of μ-proximities on a set X, then $\mu[\inf\{\delta_{\mu_a} : a \in \mathcal{I}\}] \subseteq \inf\{\mu(\delta_{\mu_a}) : a \in \mathcal{I}\}$.*

Theorem 4.10. *The family of all μ-proximities on a nonempty set X forms a complete lattice under the ordering $\geq$.*

Theorem 4.11. *Let (X, δ_μ) and $(Y, \delta'_{\mu'})$ be two μ-proximity spaces and let $\beta'_{\mu'}$ be a μ-proximity base for the μ-proximity $\delta'_{\mu'}$. Then, a map $f : X \longrightarrow Y$ is δ_μ-continuous if and only if $(f^{-1}(A), f^{-1}(B)) \notin \delta_\mu$ whenever $(A, B) \notin \beta'_{\mu'}$ for all $A, B \subset Y$.*

Proof. Let $f : X \longrightarrow Y$ be a δ_μ-continuous map and $A, B \subset Y$. If $(A, B) \notin \beta'_{\mu'}$, then $(A, B) \notin \delta'_{\mu'}$. Therefore, $(f^{-1}(A), f^{-1}(B)) \notin \delta_\mu$.

Conversely, let $(A, B) \notin \delta'_{\mu'}$. Then, it is to show that $(f^{-1}(A), f^{-1}(B)) \notin \delta_\mu$. If $(A, B) \notin \beta'_{\mu'}$, then it holds by the hypothesis. On the other hand, if $(A, B) \in \beta'_{\mu'} - \delta'_{\mu'}$, then there exist subsets $A \subseteq A^*$ and $B \subseteq B^*$ such that either $(A^*, B^*) \notin \beta'_{\mu'}$ or $(B^*, A^*) \notin \beta'_{\mu'}$.

Case (i). If $(A^*, B^*) \notin \beta'_{\mu'}$, then $(f^{-1}(A^*), f^{-1}(B^*)) \notin \delta_\mu$ using the hypothesis. δ_μ is a μ-proximity and $A \subseteq A^*, B \subseteq B^*$, therefore, $(f^{-1}(A), f^{-1}(B)) \notin \delta_\mu$.

Case (ii). If $(B^*, A^*) \notin \beta'_{\mu'}$, then $(f^{-1}(B^*), f^{-1}(A^*)) \notin \delta_\mu$ using the hypothesis. δ_μ is a μ-proximity and $A \subseteq A^*, B \subseteq B^*$, therefore, $(f^{-1}(B), f^{-1}(A)) \notin \delta_\mu$. Therefore, $(f^{-1}(A), f^{-1}(B)) \notin \delta_\mu$. □

Theorem 4.12. *Let X be a nonempty set and $\mathcal{F}$ be a nonempty family of maps such that $\mathcal{F} = \{f : f : X \longrightarrow (Y_f, \delta_{\mu_f})\}$. Then, there exists a coarsest μ-proximity on X such that each member of $\mathcal{F}$ is δ_μ-continuous.*

Proof. Define a binary relation β_μ on $\mathcal{P}(X)$ as follows:

$$(A, B) \in \beta_\mu \text{ if and only if } (f(A), f(B)) \in \delta_{\mu_f} \text{ for each } f \in \mathcal{F}.$$

It is to show that β_μ is a proximity base. Let $A \cap B \neq \emptyset$. Then, $f(A) \cap f(B) \neq \emptyset$ for each $f \in \mathcal{F}$. Therefore, $(f(A), f(B)) \in \delta_{\mu_f}$ for each $f \in \mathcal{F}$. So, $(A, B) \in \beta_\mu$. Thus, axiom (i) follows. For axiom (ii), let $(A, B) \notin \beta_\mu$. Then, there exists some $f \in \mathcal{F}$ such that $(f(A), f(B)) \notin \delta_{\mu_f}$. Therefore, there exists some $E_f \subset Y_f$ such that $(f(A), E_f) \notin \delta_{\mu_f}$ and $(Y_f \backslash E_f, f(B)) \notin \delta_{\mu_f}$. Let $E = f^{-1}(E_f)$. Then, $f(E) \subset E_f$ and $f(X \backslash E) \subset Y_f \backslash E_f$. Therefore, $(A, E) \notin \beta_\mu$ and $(X \backslash E, B) \notin \beta_\mu$. Thus, β_μ is a μ-proximity base for X. Hence, $\delta_\mu(\beta_\mu)$, the μ-proximity generated by β_μ, is the required μ-proximity on X. □

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