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CLOSED FORM EXPRESSIONS FOR CURVED SURFACE AREA OF REVOLUTION OF HYPERBOLAS: A HYPERGEOMETRIC FUNCTION APPROACH

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Abstract. In this paper, we provide the exact expressions (not found in the literature) for the curved surface area of revolution (about the x-axis and y-axis) of horizontal and oblique hyperbolas. Closed-form expressions for the curved surface area are obtained in terms of Gauss function, Clausen function, and Appell's hypergeometric function of the first kind.

1. Introduction and preliminaries

For the sake of conciseness of the paper, we have used the following notations: $\mathbb{N} := \{1, 2, 3, \dots\}$; $\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \dots\}$; $\mathbb{Z} := \{0, \pm1, \pm2, \pm3, \dots\}$; $\mathbb{Z}^- := \{-1, -2, -3, \dots\}$; $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} = \{0, -1, -2, -3, \dots\}$ and $\mathbb{Z} := \mathbb{Z}_0^- \cup \mathbb{N}$, where the symbols $\mathbb{N}$ and $\mathbb{Z}$ are the set of natural numbers and the set of integers, respectively; the symbols $\mathbb{R}$ and $\mathbb{C}$ are the set of real numbers and the set of complex numbers, respectively.

The Pochhammer symbol $(\alpha)_p$, $(\alpha, p \in \mathbb{C})$ is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha + p)}{\Gamma(\alpha)}$$
$$= \begin{cases} 1, & (p = 0; \alpha \in \mathbb{C} \setminus \{0\}) \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & (p = n \in \mathbb{N}; \alpha \in \mathbb{C} \setminus \mathbb{Z}_0^-) \\ \frac{(-1)^k n!}{(n-k)!}, & (\alpha = -n; p = k; n, k \in \mathbb{N}_0; 0 \le k \le n) \\ 0, & (\alpha = -n; p = k; n, k \in \mathbb{N}_0; k > n) \\ \frac{(-1)^n}{(1-\alpha)_n}, & (p = -n; n \in \mathbb{N}; \alpha \in \mathbb{C} \setminus \mathbb{Z}). \end{cases}$$

It being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

If $a, p \in \mathbb{C}$ and $r = 0, 1, 2, 3, \dots$, then

$$a + pr = \frac{a\left(\frac{a+p}{p}\right)_r}{\left(\frac{a}{p}\right)_r}, \text{ such that each Pochhammer symbol is well defined.}$$

$$(\alpha)_{m+n} = (\alpha)_m(\alpha+m)_n = (\alpha)_n(\alpha+n)_m,$$

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$$\Gamma(z+1) = z\Gamma(z); \quad z \in \mathbb{C}\setminus\mathbb{Z}_0^-.$$

The generalized hypergeometric function of one variable ${}_pF_q$ is defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_p;\\ \beta_1,\ \beta_2,\ \ldots,\ \beta_q;\end{matrix}\ z\right] = {}_pF_q\left[\begin{matrix}(\alpha_p);\\ (\beta_q);\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(\alpha_j)_n}{\prod_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!},$$

where (α_p) is a set of parameters $\alpha_1, \alpha_2, \cdots, \alpha_p$ with similar interpretation for (β_q). By convention, the empty product is treated as unity and the empty sum is treated as zero, $p, q \in \mathbb{N}_0$.

Convergence conditions of ${}_pF_q$

(1) When $p \leq q$, then $|z| < \infty$;
(2) When $p = q+1$, then $|z| < 1$;
(3) When $p = q+1$ and $|z| = 1$, then $\Re(\omega) > 0$;
(4) When $p = q+1, |z| = 1$ and $z \neq 1$, then $-1 < \Re(\omega) \leq 0$,

where $\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$ and $\alpha_j \in \mathbb{C}$ $(j = 1, 2, 3, \ldots, p)$; $\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(j = 1, 2, 3, \ldots, q)$.

The Binomial expansion in terms of hypergeometric function can be written as

$$(1-z)^{-a} = {}_1F_0\left[\begin{matrix}a;\\ -;\end{matrix}\ z\right] = \sum_{n=0}^{\infty}\frac{(a)_n z^n}{n!},$$

where $a \in \mathbb{C}$ and $|z| < 1$.

A relation between inverse sine and logarithm functions

$$\sin^{-1}(i\,x) = i\sinh^{-1}(x) = i\log_e\left(x + \sqrt{(x^2+1)}\right); \quad x \in \mathbb{R}.$$

Some formulas recorded in the table of Prudnikov *et al.*

- [5, p. 468, entry 7.3.2(1)]

$${}_2F_1\left[\begin{matrix}-\frac{1}{2},\ -\frac{1}{2};\\ \frac{1}{2};\end{matrix}\ z\right] = \sqrt{(1-z)} + (\sqrt{z})\sin^{-1}(\sqrt{z}), \tag{1.1}$$

- [5, p. 513, entry 258]

$${}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{1}{2};\\ 2,\ 2;\end{matrix}\ z\right] = \frac{4}{z}\left[1 - \sqrt{(1-z)} + \log_e\left(\frac{1+\sqrt{(1-z)}}{2}\right)\right], \tag{1.2}$$

- [5, p. 519, entry 367]

$${}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2};\\ 2,\ 3;\end{matrix}\ z\right] = \frac{4}{z^2}\left[2\sqrt{(1-z)} - 2 + z - 2z\log_e\left(\frac{1+\sqrt{(1-z)}}{2}\right)\right], \tag{1.3}$$

which are valid for $|z| \leq 1$.

Appell's function of first kind F_1 [1, p. 73, eq. 1] (see also [2])

$$F_1\Big[a;b,c;d;x,y\Big] = \sum_{m,r=0}^{\infty} \frac{(a)_{m+r}(b)_m(c)_r}{(d)_{m+r}} \frac{x^m y^r}{m!r!},$$

where $\max\{|x|,|y|\} < 1$, $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$ and $a,b,c \in \mathbb{C}$.

Some basic integrals and reduction formula

$$\int \cosh^2(x)\,dx = C + \frac{x}{2} + \frac{\sinh(2x)}{4},$$

where C is the constant of the integration.

The following reduction formula is available in all textbooks of integral calculus

$$\int \sec^n(x)\,dx = C + \frac{\tan(x)\sec^{n-2}(x)}{(n-1)} + \frac{(n-2)}{(n-1)}\int \sec^{n-2}(x)\,dx, \tag{1.4}$$

where n is a positive integer greater than or equal to 2.

From reduction formula (1.4) we can write

$$\int \sec^{2r}(x)\,dx = C + \frac{\tan(x)\sec^{2r-2}(x)}{(2r-1)} + \frac{(2r-2)}{(2r-1)}\int \sec^{2r-2}(x)\,dx, \quad r \geq 1. \tag{1.5}$$

By successive applications of reduction formula (1.4) in the right hand side of eq. (1.5), we can find the same integral in a finite series form containing the Pochhammer symbol

$$\int \frac{dx}{\cos^{2r}(x)} = \qquad C + \frac{(r-1)!\tan(x)}{2\left(\frac{1}{2}\right)_r}\left(\sum_{p=0}^{r-1} \frac{\left(\frac{1}{2}\right)_p}{p!\cos^{2p}(x)}\right); \quad r \geq 1. \tag{1.6}$$

The integral (1.6) can be verified with the reduction formula (1.5) by taking $r = 1, 2, 3\ldots$

In equation (1.6), by replacing x with $(i\,x)$ and using the properties of hyperbolic functions, we get the closed form expression

$$\int \frac{dx}{\cosh^{2r}(x)} = C + \frac{(r-1)!\tanh(x)}{2\left(\frac{1}{2}\right)_r}\left(\sum_{p=0}^{r-1} \frac{\left(\frac{1}{2}\right)_p}{p!\cosh^{2p}(x)}\right); \quad r \geq 1. \tag{1.7}$$

Cauchy's double series identity $\big($[7, p. 100, eq. 2.1(2)], [6, p. 57, eq. 2]$\big)$

$$\sum_{r=0}^{\infty}\sum_{p=0}^{r} \Phi(r,p) \quad = \quad \sum_{r=0}^{\infty}\sum_{p=0}^{\infty} \Phi(r+p,p), \tag{1.8}$$

provided that the series involved are absolutely convergent.

In this paper, we have derived the exact expressions for the curved surface area of revolution (about the x-axis and y-axis) of horizontal and oblique hyperbolas. For the sake of convenience in the derivation of our expressions, we have considered the well known simple and standard equations of hyperbolas. Without any loss of generality, we are not considering the general conic equation of second degree in x and y in the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, to represent an oblique hyperbola under the conditions $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$ and $h^2 > ab$, (for a rectangular and oblique hyperbola $a + b = 0$ also), because our expressions of the curved surface area depend on the length of the semi-transverse axis, semi-conjugate axis, eccentricity, parameter and co-ordinates of the points lying on the arc of the hyperbolas. Analytical expressions for the curved surface area of revolution of an ellipse are given in [3].

Here, some exact expressions for the curved surface area of a three dimensional surface (obtained by revolving the arc of hyperbolas about the co-ordinate axes) are obtained using a hypergeometric function approach.

In this paper any values of parameters and arguments leading to the results which do not make sense are tacitly excluded.

2. Expressions for curved surface area (CSA) of revolution of horizontal hyperbola

Case 1: When an arc (lying in the first quadrant) between two arbitrary points $A(a\cosh(t_1), b\sinh(t_1))$ and $B(a\cosh(t_2), b\sinh(t_2))$ lying on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is revolved *about the x-axis (through the angle of* 2π*)*, then the closed form expression for the curved surface area of the generated three dimensional figure (like a frustum of a hyperboloid of two sheets) will be

$$\begin{aligned} \text{CSA} = 2\pi Eab\Bigg[\Bigg\{&\frac{\cosh^2(t_2)}{2} + \frac{\cosh(t_2)}{2E}\sqrt{(E^2\cosh^2(t_2) - 1)} + \\ &+ \frac{(1 - 2E^2\cosh^2(t_2))}{4E^2} - \frac{1}{2E^2}\log_e\left(E\cosh(t_2) + \sqrt{(E^2\cosh^2(t_2) - 1)}\right)\Bigg\} - \\ &- \Bigg\{\frac{\cosh^2(t_1)}{2} + \frac{\cosh(t_1)}{2E}\sqrt{(E^2\cosh^2(t_1) - 1)} + \frac{(1 - 2E^2\cosh^2(t_1))}{4E^2} - \\ &- \frac{1}{2E^2}\log_e\left(E\cosh(t_1) + \sqrt{(E^2\cosh^2(t_1) - 1)}\right)\Bigg\}\Bigg]. \qquad (2.1) \end{aligned}$$

Remark. When a vertical hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is revolved about the y-axis, then it will generate a surface like a frustum of a hyperboloid of two sheets and we will get the expressions of the curved surface area as given in equation (2.1).

Case 2: When an arc (lying in the first quadrant) between two arbitrary points $A(a\cosh(t_1), b\sinh(t_1))$ and $B(a\cosh(t_2), b\sinh(t_2))$ lying on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is revolved *about the y-axis (through the angle of* 2π*)*, then the closed form expression for the curved surface area of the generated three dimensional figure (like a frustum of a hyperboloid of one sheet) will be

$$\text{CSA} = 2\pi Ea^2\left[\left\{\frac{t_2(E^2-1)}{2E^2} + \frac{\sinh(2t_2)}{4} - \frac{\tanh(t_2)}{8E^4}F_1\left[1;1,\frac{1}{2};3;\frac{1}{E^2},\frac{1}{E^2\cosh^2(t_2)}\right]\right\}\right.$$

$$-\left\{\frac{t_1(E^2-1)}{2E^2}+\frac{\sinh(2t_1)}{4}-\frac{\tanh(t_1)}{8E^4}F_1\left[1;1,\frac{1}{2};3;\frac{1}{E^2},\frac{1}{E^2\cosh^2(t_1)}\right]\right\}\Bigg], \quad (2.2)$$

where $E=\frac{\sqrt{(a^2+b^2)}}{a}$ is the eccentricity of the hyperbola, a, b are the lengths of the semi-transverse and semi-conjugate axes, respectively.

Remark. When a vertical hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ is revolved about the x-axis, then it will generate a surface like a frustum of a hyperboloid of one sheet and we will get the expressions of the curved surface area as given in equation (2.2).

Derivation of formulae (2.1) and (2.2)

Consider the equation of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \qquad \text{(Cartesian form)},$$

where $b^2=a^2(e^2-1)$ or $\frac{a^2+b^2}{a^2}=e^2$ or $\frac{a^2}{a^2+b^2}=\frac{1}{e^2}<1$, a and b are the semi-transverse and semi-conjugate axes of the hyperbola and e (>1) is called the eccentricity of the hyperbola.

Its parametric form is given by

$$x=a\cosh(t), \quad y=b\sinh(t).$$

Case 1: The curved surface area of revolution of any curve *about the x-axis* is given by

$$\text{CSA}=2\pi\int_{t_1}^{t_2} y\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}\,dt \quad (2.3)$$

Put $y=b\sinh(t)$, $\frac{dx}{dt}=a\sinh(t)$, $\frac{dy}{dt}=b\cosh(t)$, in equation (2.3) and integrate w. r. to t over the interval $[t_1,t_2]$ s.t. $0\le t_1<t_2<\infty$,

$$\text{CSA}=2\pi\int_{t_1}^{t_2} b\sinh(t)\sqrt{\left(a^2\sinh^2(t)+b^2\cosh^2(t)\right)}\,dt$$

$$=2b\pi\int_{t_1}^{t_2}\sinh(t)\sqrt{\left((a^2+b^2)\cosh^2(t)-a^2\right)}\,dt,$$

$$=2\pi b\sqrt{(a^2+b^2)}\int_{t_1}^{t_2}\sinh(t)\cosh(t)\sqrt{\left(1-\frac{1}{e^2\cosh^2(t)}\right)}\,dt,$$

$$=2\pi b\sqrt{(a^2+b^2)}\int_{t_1}^{t_2}\sinh(t)\cosh(t)\ {}_1F_0\left[\begin{matrix}-\frac{1}{2};\\ -;\end{matrix}\ \frac{1}{e^2\cosh^2(t)}\right]dt;$$

$$\left(\text{since } \frac{1}{e^2\cosh^2(t)}<1, \quad \forall t,\right)$$

$$= 2\pi b\sqrt{(a^2+b^2)}\int_{t_1}^{t_2}\sinh(t)\cosh(t)\left(\sum_{r=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}\cosh^{2r}(t)}\right)dt,$$

$$= 2\pi b\sqrt{(a^2+b^2)}\left[\int_{t_1}^{t_2}\sinh(t)\cosh(t)\,dt - \frac{1}{2e^2}\int_{t_1}^{t_2}\frac{\sinh(t)}{\cosh(t)}\,dt + \sum_{r=2}^{\infty}\frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}}\int_{t_1}^{t_2}\frac{\sinh(t)}{\cosh^{2r-1}(t)}\,dt\right].$$

On taking $\cosh(t) = u$ and $\sinh(t)\,dt = du$, we get

$$\text{CSA} = 2\pi eab\int_{u(t_1)}^{u(t_2)}\left[u - \frac{1}{2ue^2} + \sum_{r=2}^{\infty}\left[\frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}u^{2r-1}}\right]\right]du,$$

$$= 2\pi eab\left[\frac{u^2}{2} - \frac{1}{2e^2}\log_e(u) + \sum_{r=2}^{\infty}\left[\frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}(-2r+2)u^{2r-2}}\right]\right]_{u(t_1)}^{u(t_2)}.$$

Now, putting $u = \cosh(t)$, we get

$$\text{CSA} = 2\pi eab\left[\frac{\cosh^2(t)}{2} - \frac{1}{2e^2}\log_e(\cosh(t)) + \sum_{r=2}^{\infty}\frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}(-2r+2)\cosh^{2r-2}(t)}\right]_{t_1}^{t_2}$$

$$= 2\pi eab\left[\frac{\cosh^2(t)}{2} - \frac{1}{2e^2}\log_e(\cosh(t)) + \frac{\left(-\frac{1}{2}\right)_2}{(-2)\left(1\right)_2 e^4\cosh^2(t)}\sum_{r=0}^{\infty}\frac{\left(\frac{3}{2}\right)_r(1)_r(1)_r}{(3)_r(2)_r r!e^{2r}\cosh^{2r}(t)}\right]_{t_1}^{t_2}$$

$$= 2\pi eab\left[\frac{\cosh^2(t)}{2} - \frac{1}{2e^2}\log_e(\cosh(t)) + \frac{1}{16e^4\cosh^2(t)}\;{}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2};\\ 2,\ 3;\end{matrix}\ \frac{1}{e^2\cosh^2(t)}\right]\right]_{t_1}^{t_2},$$

or we have

$$\text{CSA} = 2\pi eab\left[\left\{\frac{\cosh^2(t_2)}{2} - \frac{1}{2e^2}\log_e(\cosh(t_2)) + \frac{1}{16e^4\cosh^2(t_2)}\times{}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2};\\ 2,\ 3;\end{matrix}\ \frac{1}{e^2\cosh^2(t_2)}\right]\right\} - \left\{\frac{\cosh^2(t_1)}{2} - \frac{1}{2e^2}\log_e(\cosh(t_1)) + \frac{1}{16e^4\cosh^2(t_1)}\;{}_3F_2\left[\begin{matrix}1,\ 1,\ \frac{3}{2};\\ 2,\ 3;\end{matrix}\ \frac{1}{e^2\cosh^2(t_1)}\right]\right\}\right] \tag{2.4}$$

Now, using the result (1.3) in eq. (2.4) and after simplification, we get the desired formula (2.1). Where the eccentricity e is replaced with E.

Case 2: The curved surface area of revolution of any curve *about the y-axis* is given by

$$\text{CSA} = 2\pi \int_{t_1}^{t_2} x\sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt. \tag{2.5}$$

Put $x = a\cosh(t)$, $\frac{dx}{dt} = a\sinh(t)$, $\frac{dy}{dt} = b\cosh(t)$ in equation (2.5) and integrate w. r. to t over the interval $[t_1, t_2]$ s.t. $0 \leq t_1 < t_2 < \infty$,

$$\begin{aligned}
\text{CSA} &= 2\pi \int_{t_1}^{t_2} a\cosh(t)\sqrt{\left(a^2\sinh^2(t) + b^2\cosh^2(t)\right)}\, dt \\
&= 2a\pi \int_{t_1}^{t_2} \cosh(t)\sqrt{\left((a^2+b^2)\cosh^2(t) - a^2\right)}\, dt, \\
&= 2\pi a\sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh^2(t)\sqrt{\left(1 - \frac{1}{e^2\cosh^2(t)}\right)}\, dt, \\
&= 2\pi a\sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh^2(t)\ {}_1F_0\left[\begin{matrix} -\frac{1}{2}; \\ -; \end{matrix}\ \frac{1}{e^2\cosh^2(t)}\right] dt; \\
&\qquad\qquad \left(\text{since}\quad \frac{1}{e^2\cosh^2(t)} < 1, \quad \forall \quad t\right), \\
&= 2\pi a\sqrt{(a^2+b^2)} \int_{t_1}^{t_2} \cosh^2(t)\left(\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}\cosh^(2r)t}\right) dt, \\
&= 2\pi a\sqrt{(a^2+b^2)}\left[\int_{t_1}^{t_2} \cosh^2(t)\, dt - \frac{1}{2e^2}\int_{t_1}^{t_2} dt + \sum_{r=2}^{\infty} \frac{\left(-\frac{1}{2}\right)_r}{r!e^{2r}} \int_{t_1}^{t_2} \frac{1}{\cosh^{2r-2}(t)}\, dt\right].
\end{aligned}$$

On integrating (using eq. (1.7)), we get

$$\begin{aligned}
\text{CSA} &= 2\pi e a^2\left[\frac{t}{2} + \frac{\sinh 2t}{4} - \frac{t}{2e^2} + \sum_{r=2}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (r-1)!\tanh(t)}{2r!e^{2r}\left(\frac{1}{2}\right)_r} \sum_{p=0}^{r-1} \frac{\left(\frac{1}{2}\right)_p}{p!\cosh^{2p}(t)}\right]_{t_1}^{t_2} \\
&= 2\pi e a^2\left[\frac{t}{2e^2}(e^2-1) + \frac{\sinh 2t}{4} - \frac{\tanh(t)}{8e^4} \sum_{r=0}^{\infty}\sum_{p=0}^{r} \frac{r!\left(\frac{1}{2}\right)_p}{(3)_r e^{2r} p!\cosh^{2p}(t)}\right]_{t_1}^{t_2}.
\end{aligned} \tag{2.6}$$

Now, using double series identity (1.8) in equation (2.6), we get

CSA

$$= 2\pi e a^2 \left[\frac{t}{2e^2}(e^2-1) + \frac{\sinh 2t}{4} - \frac{\tanh(t)}{8e^4} \sum_{r=0}^{\infty}\sum_{p=0}^{\infty} \frac{(1)_{r+p}\left(\frac{1}{2}\right)_p (1)_r}{(3)_{r+p} e^{2r+2p} p! r! \cosh^{2p}(t)} \right]_{t_1}^{t_2},$$

$$= 2\pi e a^2 \left[\frac{t}{2e^2}(e^2-1) + \frac{\sinh(2t)}{4} - \frac{\tanh(t)}{8e^4}\; F_1\left[1;1,\frac{1}{2};3;\frac{1}{e^2},\frac{1}{e^2\cosh^2(t)}\right] \right]_{t_1}^{t_2}.$$

After simplification, we get the required formula (2.2).

Remark. To avoid the confusion between eccentricity (denoted by e) and logarithm to the base e, we are writing eccentricity e as E in the main expressions of the curved surface area, which will be useful in the numerical computation.

3. Expressions for the curved surface area (CSA) of revolution (through the angle of 2π) of an oblique hyperbola

When an arc (lying in the first quadrant) between two arbitrary points $A(c\, t_1, \frac{c}{t_1})$ and $B(c\, t_2, \frac{c}{t_2})$ lying on the rectangular hyperbola $xy = c^2$ is revolved *about the x-axis* (where c is a parameter), then the closed form expressions for the curved surface area will be

When $0 < t_1 < t_2 < \infty$ and $c > 0$, then

$$\text{CSA (about the } x\text{-axis)}$$
$$= \pi c^2 \left[\frac{\sqrt{(1+t_1^4)}}{t_1^2} - \frac{\sqrt{(1+t_2^4)}}{t_2^2} + \log_e \left\{ \frac{t_2^2 + \sqrt{(1+t_2^4)}}{t_1^2 + \sqrt{(1+t_1^4)}} \right\} \right]. \tag{3.1}$$

Derivations of the formula (3.1)

Consider the equation of a rectangular and oblique hyperbola

$$xy = c^2 \qquad \text{(Cartesian form)}.$$

Its parametric form is given by

$$x = ct, \quad y = \frac{c}{t}.$$

Putting $\frac{dx}{dt} = c$ and $\frac{dy}{dt} = -\frac{c}{t^2}$ in (2.3), we get

$$\text{CSA} = 2\pi \int_{t_1}^{t_2} \frac{c}{t}\sqrt{\left(c^2 + \frac{c^2}{t^4}\right)}\, dt = 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t}\sqrt{\left(1 + \frac{1}{t^4}\right)}\, dt.$$

Since no standard formula for the indefinite integral of $\sqrt{\left(1+\frac{1}{t^4}\right)}$ w. r. to t is available in the literature of integral calculus and other mathematical tables. Therefore, we shall apply a hypergeometric approach to evaluate the closed form expressions for the curved surface area (CSA) of revolution (through the angle of 2π) of a hyperbola *about the x-axis.* Three cases arise based on the position of points A and B.

Case 3: When $0 < t_1 < t_2 < 1$ and $c > 0$, then

$$\text{CSA} = 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t^3}\sqrt{(1+t^4)}\, dt, \qquad (\text{since } t^4 < 1 \text{ here}),$$
$$= 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t^3}\ {}_1F_0\left[\begin{matrix}-\frac{1}{2};\\ -;\end{matrix}\ -t^4\right] dt,$$

$$\text{CSA} = 2\pi c^2 \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{t_2} t^{4r-3}\, dt,$$
$$= 2\pi c^2 \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!(4r-2)} \left(t_2^{4r-2} - t_1^{4r-2}\right),$$
$$= \frac{\pi c^2}{t_1^2}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},\ -\frac{1}{2};\\ \frac{1}{2};\end{matrix}\ -t_1^4\right] - \frac{\pi c^2}{t_2^2}\ {}_2F_1\left[\begin{matrix}-\frac{1}{2},\ -\frac{1}{2};\\ \frac{1}{2};\end{matrix}\ -t_2^4\right]. \tag{3.2}$$

Both ${}_2F_1$ series are convergent, since $t_1 < 1$ and $t_2 < 1$. Therefore, our result (3.2) is also convergent. Using the result (1.1) in the right hand side of equation (3.2), we get the desired result (3.1).

Case 4: When $0 < t_1 < 1 < t_2 < \infty$ (or $t_1 < 1$ and $\frac{1}{t_2} < 1$) and $c > 0$, then

$$\text{CSA} = 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t}\sqrt{\left(1+\frac{1}{t^4}\right)}\, dt$$
$$= 2\pi c^2 \int_{t_1}^{1} \frac{1}{t}\sqrt{\left(1+\frac{1}{t^4}\right)}\, dt + 2\pi c^2 \int_{1}^{t_2} \frac{1}{t}\sqrt{\left(1+\frac{1}{t^4}\right)}\, dt$$
$$= 2\pi c^2 \int_{t_1}^{1} \frac{1}{t^3}\ {}_1F_0\left[\begin{matrix}-\frac{1}{2};\\ -;\end{matrix}\ -t^4\right] dt + 2\pi c^2 \int_{1}^{t_2} \frac{1}{t}\ {}_1F_0\left[\begin{matrix}-\frac{1}{2};\\ -;\end{matrix}\ -\frac{1}{t^4}\right] dt$$
$$= 2\pi c^2 \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{1} t^{4r-3}\, dt + 2\pi c^2 \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{1}^{t_2} t^{-4m-1}\, dt$$
$$= 2\pi c^2 \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{1} t^{4r-3}\, dt$$
$$+ 2\pi c^2 \Big[\int_{1}^{t_2} \frac{1}{t}\, dt + \sum_{m=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{1}^{t_2} t^{-4m-1}\, dt\Big]$$
(replacing m with $m+1$)

$$= 2\pi c^2 \sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!} \int_{t_1}^{1} t^{4r-3}\, dt +$$

$$+ 2\pi c^2 \Big[\int_{1}^{t_2} \frac{1}{t}\, dt + \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{m+1} (-1)^{m+1}}{(m+1)!} \int_{1}^{t_2} t^{-4m-5}\, dt \Big]$$

$$= 2\pi c^2 \Big[\sum_{r=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_r (-1)^r}{r!(4r-2)} \left(t^{4r-2}\right) \Big]_{t_1}^{1} + 2\pi c^2 \log_e(t)_1^{t_2}$$

$$+ 2\pi c^2 \Big[\sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{m+1} (-1)^{m+1}}{(m+1)!(-4m-4)} \left(t^{-4m-4}\right) \Big]_{1}^{t_2},$$

$$\text{CSA} = 2\pi c^2 \log_e(t_2) + \frac{\pi c^2}{t_1^2}\, {}_2F_1\left[\begin{matrix} -\frac{1}{2},\ -\frac{1}{2}; \\ \frac{1}{2}; \end{matrix} -t_1^4\right] - \pi c^2\, {}_2F_1\left[\begin{matrix} -\frac{1}{2},\ -\frac{1}{2}; \\ \frac{1}{2}; \end{matrix} -1\right] -$$

$$- \frac{\pi c^2}{4\, t_2^4}\, {}_3F_2\left[\begin{matrix} \frac{1}{2},\ 1,\ 1; \\ 2,\ 2; \end{matrix} -\frac{1}{t_2^4}\right] + \frac{\pi c^2}{4}\, {}_3F_2\left[\begin{matrix} \frac{1}{2},\ 1,\ 1; \\ 2,\ 2; \end{matrix} -1\right]. \tag{3.3}$$

Both ${}_2F_1$ and ${}_3F_2$ series are convergent, since $t_1 < 1$ and $\frac{1}{t_2} < 1$. Therefore, our result (3.3) is also convergent. Using the results (1.1) and (1.2) in the right hand side of equation (3.3), we get the desired result (3.1).

Case 5: When $1 < t_1 < t_2 < \infty$ $\left(\text{or } \frac{1}{t_2} < \frac{1}{t_1} < 1\right)$ and $c > 0$, then

$$\text{CSA} = 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t} \sqrt{\left(1 + \frac{1}{t^4}\right)}\, dt$$

$$= 2\pi c^2 \int_{t_1}^{t_2} \frac{1}{t}\, {}_1F_0\left[\begin{matrix} -\frac{1}{2}; \\ -; \end{matrix} -\frac{1}{t^4}\right] dt,$$

$$= 2\pi c^2 \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{t_1}^{t_2} t^{-4m-1}\, dt,$$

$$= 2\pi c^2 \Big[\int_{t_1}^{t_2} \frac{1}{t}\, dt + \sum_{m=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_m (-1)^m}{m!} \int_{t_1}^{t_2} t^{-4m-1}\, dt \Big]$$

(replacing m with $m+1$)

$$= 2\pi c^2 \Big[\int_{t_1}^{t_2} \frac{1}{t}\, dt + \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_{m+1} (-1)^{m+1}}{(m+1)!} \int_{t_1}^{t_2} t^{-4m-5}\, dt \Big] \tag{3.4}$$

or

$$\text{CSA} = \frac{\pi c^2}{4\, t_1^4}\, {}_3F_2\left[\begin{matrix} \frac{1}{2},\ 1,\ 1; \\ 2,\ 2; \end{matrix} -\frac{1}{t_1^4}\right] - \frac{\pi c^2}{4\, t_2^4}\, {}_3F_2\left[\begin{matrix} \frac{1}{2},\ 1,\ 1; \\ 2,\ 2; \end{matrix} -\frac{1}{t_2^4}\right] + 2\pi c^2 \log_e\left(\frac{t_2}{t_1}\right).$$

Both ${}_3F_2$ series are convergent, since $\frac{1}{t_1} < 1$ and $\frac{1}{t_2} < 1$. Therefore, our result (3.4) is also convergent. Using the result (1.2) in the right hand side of equation (3.4), we get the desired result (3.1).

4. Numerical examples

(1) When $b = a = 1$ (then $E = \sqrt{2}$), $t_1 = 0$ and $t_2 = \log_e(\sqrt{2} + \sqrt{3})$ in equation (2.1), we have

$$\text{CSA (about the } x\text{-axis)} = \pi\big(-1 + \sqrt{15} - \frac{1}{\sqrt{2}} \log_e[\sqrt{12} + \sqrt{10} - \sqrt{6} - \sqrt{5}]\big)$$
$$\approx (2.404092803...)\pi \approx 7.552680289 \text{ square unit.}$$

(2) When $b = a = 1$ (then $E = \sqrt{2}$), $t_1 = 0$ and $t_2 = 1$ in equation (2.2), we have

$$\text{CSA (about } y\text{-axis)} \approx 2\pi\sqrt{2}\Big\{ (1.156715102) - $$
$$- (0.023799817)\ F_1\Big[1;\ 1,\ 0.5;\ 3;\ 0.5,\ 0.209987171\Big]\Big\}.$$

Now, using a computer program to calculate the sum of double series corresponding to the above-mentioned Appell's function F_1, we get

$$\text{CSA (about the } y\text{-axis)} \approx 10.00767142 \text{ square unit.}$$

(3) When $b = a = 1$ (then $E = \sqrt{2}$), $t_1 = 0$ and $t_2 = \log_e(\sqrt{2} + \sqrt{3})$ in equation (2.2), we have

$$\text{CSA (about the } y\text{-axis)} \approx \frac{\pi}{\sqrt{2}}\Big\{ \log_e\Big\{\sqrt{2} + \sqrt{3}\Big\} + 2\sqrt{6} - $$
$$- \frac{1}{4\sqrt{6}}\ F_1\Big[1;\ 1,\ 0.5;\ 3;\ 0.5,\ 0.166666667\Big]\Big\}$$
$$\approx 13.141539977 \text{ square unit.}$$

(4) When $c = 1$, $t_1 = 1$ and $t_2 = 2$ in equation (3.1), we have

$$\text{CSA (about } x\text{-axis)} = \pi\Big[\sqrt{2} - \frac{\sqrt{17}}{4} + \log_e\Big\{\frac{4 + \sqrt{17}}{1 + \sqrt{2}}\Big\}\Big]$$
$$\approx 5.016420116 \text{ square unit.}$$

Remark

We have also derived some formulae [4] for arc-length between two arbitrary points lying on different hyperbolas.

We conclude our present investigation by observing that the solutions to such problems can be obtained in an analogous manner. All the results obtained are believed to be new and exact.

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