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EXISTENCE AND UNIQUENESS RESULTS FOR NONLINEAR FRACTIONAL LANGEVIN INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. This paper is devoted to the study of nonlinear fractional Langevin integro differential equations with boundary conditions. Some effective results concerning the existence and uniqueness are obtained by applying the Banach contraction mapping principle and the Schauder fixed point theorem. An example is presented illustrating the effectiveness of the theoretical results.

1. Introduction

Fractional differential equations have many applications in different problems and phenomenons from fields of science and engineering, see [1–17] and [19–24]. The Langevin equation is an important equation in mathematical physics used to model the phenomenon that exists in fluctuating environments such as Brownian motion. Paul Langevin first proposed the definition of the classical model of LE in terms of ordinary derivatives in 1908. LE is also unowned as a stochastic differential equation, because it controls the rapid movement of the dynamic systems microscopic variables. In recent years, several scholars have researched the solvability or existence and uniqueness of solutions to boundary value problems for Langevin fractional differential equations.

In [16], the authors discussed the existence and uniqueness of solutions to nonlinear Langevin equations involving two fractional orders with nonlocal integral and three-point boundary conditions

$$\begin{cases} {}^{C}D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x(t)=f(t,x(t)),\ t\in[0,1]\,,\ \lambda\in\mathbb{R}, \\ x(0)=x\left(1\right)=0,\ {}^{C}D^{\alpha}x\left(0\right)+{}^{C}D^{\alpha}x\left(1\right)=\gamma\int_0^{\mu}x\left(s\right)ds,\ \mu\in\left(0,1\right), \end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives of orders α and β, respectively, $0<\alpha<1$, $1<\beta\leq 2$, λ, $\gamma\in\mathbb{R}$ and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

In [3], Baghani and J. Nieto studied the existence results of solutions for an initial value problem of Langevin equation involving two fractional orders in different

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intervals as follows

$$\begin{cases} {}^{C}D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x(t)=f(t,x(t)),\ t\in[0,1]\,,\ \lambda\in\mathbb{R},\\ x(0)=x(1)=0,\ {}^{C}D^{2\alpha}x(1)+{}^{C}D^{\alpha}x\,(1)=0,\end{cases}$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the Caputo fractional derivatives of orders α and β, respectively, $0<\alpha\leq 1$, $1<\beta\leq 2$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

In [5], by using the Schaefer fixed point theorem and the contraction mapping principle, the authors obtained the existence and uniqueness of the solutions to the following Hadamard fractional Langevin equation of the form

$$\begin{cases} {}^{H}D^{\beta}\left({}^{H}D^{\alpha}+\lambda\right)x(t)=f(t,x(t),{}^{H}D^{\eta}x(t)),\ t\in[1,T]\,,\ \lambda\in\mathbb{R},\\ x(1)=0,\ x(T)+\frac{\lambda}{\Gamma(\alpha)}\int_1^T\left(\log\frac{T}{s}\right)^{\alpha-1}\frac{x(s)}{s}ds=0,\\ D^{\alpha}x(\zeta)+\lambda x(\zeta)=0,\ 1<\zeta\leq T,\end{cases}$$

where ${}^{H}D^{\beta}$, ${}^{H}D^{\alpha}$ and ${}^{H}D^{\eta}$ are the Hadamard fractional derivatives of orders β, α and η, respectively, $1<\beta<2$, $\beta-1<\alpha<1$, $0<\eta<\beta+\alpha-2$, and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

Inspired and motivated by the works mentioned above, in this paper, we study the existence and uniqueness of the solutions for the following nonlinear fractional Langevin integro-differential equation

$$\begin{cases} D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x(t)=f(t,x(t),I^{\gamma}x(t)),\ t\in(0,T)\,,\ \lambda\in\mathbb{R},\\ {}^{C}D^{\alpha}x(0)+\lambda x(0)={}^{C}D^{\alpha}\ x(T)=0,\ x(0)=a\int_0^T x(s)ds+b,\ a,b\in\mathbb{R},\end{cases}\tag{1.1}$$

where D^{β} and ${}^{C}D^{\alpha}$ are the Riemann–Liouville fractional derivative and Caputo fractional derivative of orders β and α, respectively, $1<\beta<2$, $0<\alpha<1$, I^{γ} is the Riemann–Liouville fractional integral of order $\gamma\in(0,1)$, and $f:[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a nonlinear continuous function.

The rest of the paper is divided into four sections. In Section 2, some notations, definitions of fractional calculus and fixed point theorems are presented. In Section 3, some useful results concerning the existence and uniqueness of nonlinear fractional differential equations are obtained. In Section 4, an example is presented illustrating the effectiveness of the theoretical results.

2. Preliminaries

Some definitions, notations and results of the fractional calculus are introduced throughout this section, which will be utilized in this paper.

Let $J=[0,T]$. Denote by $\mathcal{C}=C(J)$ the Banach space of all continuous functions defined on J endowed with the norm

$$\|x\|=\sup\left\{|x(t)|:t\in J\right\}.$$

$AC(J)$ is the space of absolutely continuous valued functions from J into $\mathbb{R}$, and set

$$AC^{m}(J)=\left\{x:J\to\mathbb{R}:x,x',x'',\ldots,x^{m-1}\in\mathcal{C}\text{ and }x^{m-1}\in AC(J)\right\}.$$

Now, we are giving out some fractional calculus results and properties.

Definition 2.1 ([10])**.** The fractional integral of order $\alpha > 0$ of a function $h : J \to \mathbb{R}$ is defined by

$$I^{\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

provided the integral exists.

Definition 2.2 ([10])**.** *The Caputo fractional derivative of order $\alpha > 0$ of* function $h : J \to \mathbb{R}$ is defined by

$$^{C}D^{\alpha}h(t) = D^{\alpha}\left[h(t) - \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} t^{k}\right],$$

where

$$m = [\alpha] + 1 \text{ for } \alpha \notin \mathbb{N}_0,\ m = \alpha \text{ for } \alpha \in \mathbb{N}_0, \tag{2.1}$$

and D^{α}_{0+} is a fractional derivative in Riemann–Liouville sense of order α given by

$$D^{\alpha}h(t) = D^{m}I^{m-\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-s)^{m-\alpha-1} h(s)ds.$$

The Caputo fractional derivative $^{C}D^{\alpha}_{0+}$ exists for h belonging to $AC^{m}(J)$. In this case, we have

$$^{C}D^{\alpha}h(t) = I^{m-\alpha}h^{(m)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{m-\alpha-1} h^{(m)}(s)ds.$$

Note that, when $\alpha = m$, we get $^{C}D^{\alpha}h(t) = h^{(m)}(t)$.

Lemma 2.3 ([10])**.** *Let $\alpha > 0$ and m be given by (2.1). If $h \in AC^{m}(J,\mathbb{R})$, then*

$$\left(I^{\alpha C}D^{\alpha}h\right)(t) = h(t) - \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} t^{k},$$

where $h^{(k)}$ is the usual derivative of h of order k.

Lemma 2.4 ([10])**.** *If $\alpha > 0$ and m is given by (2.1), then the Caputo fractional differential equation $^{C}D^{\alpha}h(t) = 0$ has a general solution*

$$h(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{m-1}t^{m-1},$$

where $a_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$. Further, the Riemann–Liouville fractional differential equation

$$D^{\alpha}h(t) = 0$$

has a general solution

$$h(t) = a_1 t^{\alpha-1} + a_2 t^{\alpha-2} + a_3 t^{\alpha-3} + ... + a_m t^{\alpha-m},\ a_i \in \mathbb{R},\ i = 1, 2, ..., m.$$

Lemma 2.5 ([10])**.** *For any $\alpha, \beta \in [0,\infty)$ and $\mu > -1$,*

$$\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\beta-1} s^{\alpha-1}ds = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\alpha+\beta-1}.$$

Lemma 2.6 (Banach fixed point theorem [18])**.** *Any contraction mapping on a complete nonempty metric space has a unique fixed point.*

Lemma 2.7 (Schauder fixed point theorem [18])**.** *Let Ω be a nonempty bounded closed convex subset of a Banach space S and $\Phi : \Omega \to \Omega$ be a continuous compact mapping. Then, it has a fixed point in Ω.*

To obtain our results, we need the following lemma.

Lemma 2.8. *For any $h \in C(J)$, the problem*

$$\begin{cases} D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x(t)=h(t),\ t\in(0,T),\ \lambda\in\mathbb{R}, \\ {}^{C}D^{\alpha}x(0)+\lambda x(0)={}^{C}D^{\alpha}x(T)=0,\ x(0)=a\int_0^T x(s)ds+b,\ a,b\in\mathbb{R} \end{cases} \tag{2.2}$$

is equivalent to the integral equation

$$\begin{aligned} x(t) &= I^{\alpha+\beta}h(t)-\lambda I^{\alpha}x(t)-\frac{\Gamma(\beta)t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(I^{\beta}h(T)-\lambda x(T)\right) \\ &\quad + a\int_0^T x(s)ds+b \\ &= \frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)ds-\frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds \\ &\quad -\frac{\Gamma(\beta)t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(\frac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}h(s)ds-\lambda x(T)\right) \\ &\quad + a\int_0^T x(s)ds+b. \end{aligned} \tag{2.3}$$

Proof. Taking the integrator operator I^{β} to the first equation of (2.2), and from Lemma 2.4, we get

$$ {}^{C}D^{\alpha}x(t)+\lambda x(t)=I^{\beta}h(t)+a_1 t^{\beta-1}+a_2 t^{\beta-2}. \tag{2.4}$$

According to conditions ${}^{C}D^{\alpha}x(0)+\lambda x(0)={}^{C}D^{\alpha}x(T)=0$, it yields

$$a_1=\frac{1}{T^{\beta-1}}\left(\lambda x(T)-I^{\beta}h(T)\right),\ a_2=0.$$

Replacing a_1 and a_2 with their values in (2.4), we get

$${}^{C}D^{\alpha}x(t)=I^{\beta}h(t)-\lambda x(t)+\frac{t^{\beta-1}}{T^{\beta-1}}\left(\lambda x(T)-I^{\beta}h(T)\right).$$

Taking the integrator operator I^{α} again to the above equation and using Lemmas 2.4 and 2.5, we obtain

$$x(t)=I^{\alpha+\beta}h(t)-\lambda I^{\alpha}x(t)-\frac{\Gamma(\beta)t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}\left(I^{\beta}h(T)-\lambda x(T)\right)+a_3. \tag{2.5}$$

Using the integral condition, we find

$$a_3=a\int_0^T x(s)ds+b.$$

Substituting the value of a_3 into (2.5), we obtain the integral equation (2.3).

Conversely, putting $t=0$ in (2.3), we have

$$x(0)=a\int_0^T x(s)ds+b.$$

Taking ${}^{C}D^{\alpha}$ to (2.3), we get

$$ {}^{C}D^{\alpha}x(t)+\lambda x(t)=I^{\beta}h\left(t\right)+\frac{t^{\beta-1}}{T^{\beta-1}}\left(\lambda x(T)-I^{\beta}h(T)\right). \tag{2.6} $$

Then, for $t=0$ and $t=T$, we have

$$ {}^{C}D^{\alpha}x(0)+\lambda x(0)={}^{C}D^{\alpha}x(T)=0. $$

Taking D^{β} to (2.6), we obtain

$$ D^{\beta}\left({}^{C}D^{\alpha}+\lambda\right)x(t)=h\left(t\right),\ t\in(0,T). $$

This completes the proof. □

3. Main results

In the following we employ fixed point theorems to prove the existence and uniqueness results for problem (1.1).

For obtaining our results, we need the following hypotheses:

(H1) There exist constants $l_1,l_2>0$ such that

$$ \left|f(t,x_1,y_1)-f(t,x_2,y_2)\right|\leq l_1\left|x_1-x_2\right|+l_2\left|y_1-y_2\right|, $$

for any $t\in J$ and each $x_i,y_i\in\mathbb{R}$, $i=1,2$.

(H2) There exists a function $\Psi\in L^1\left(J,\mathbb{R}^{+}\right)$ such that

$$ \left|f(t,x,y)\right|\leq\Psi(t),\ \forall(t,x,y)\in J\times\mathbb{R}\times\mathbb{R}. $$

3.1. Existence and uniqueness results via Banach's fixed point theorem

Theorem 3.1. *Let* (H1) *hold. If*

$$ \begin{aligned} \theta=&\left(\frac{T^{\alpha+\beta}}{\Gamma\left(\alpha+\beta+1\right)}+\frac{T^{\beta+\alpha}}{\beta\Gamma\left(\beta+\alpha\right)}\right)\left(l_1+l_2\frac{T^{\eta}}{\Gamma\left(\eta+1\right)}\right)\\ &+\left|\lambda\right|\left(\frac{T^{\alpha}}{\Gamma\left(\alpha+1\right)}+\frac{\Gamma\left(\beta\right)T^{\alpha}}{\Gamma\left(\beta+\alpha\right)}\right)+\left|a\right|T<1, \end{aligned} \tag{3.1} $$

then problem (1.1) *has at least one solution.*

Proof. We convert problem (1.1) into a fixed point problem by defining the mapping $\Phi:\mathcal{C}\to\mathcal{C}$ as

$$ \begin{aligned} &(\Phi x)(t)\\ &=\frac{1}{\Gamma\left(\alpha+\beta\right)}\int_0^t(t-s)^{\alpha+\beta-1}f(s,x(s),I^{\gamma}x(s))\,ds-\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^t(t-s)^{\alpha-1}x(s)\,ds\\ &-\frac{\Gamma\left(\beta\right)t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma\left(\beta+\alpha\right)}\left(\frac{1}{\Gamma\left(\beta\right)}\int_0^T(T-s)^{\beta-1}f(s,x(s),I^{\gamma}x(s))\,ds-\lambda x(T)\right)\\ &+a\int_0^T x(s)\,ds+b. \end{aligned} $$

Obviously, the fixed points of the mapping Φ are solutions to problem (1.1). By (H1), for each $x,y\in\mathcal{C}$ and $t\in J$, we get

$$ \left|(\Phi x)(t)-(\Phi y)(t)\right| $$

$$\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s,x(s),I^\eta x(s)) - f(s,y(s),I^\eta y(s))| \, ds$$
$$+ \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)-y(s)| \, ds + \frac{\Gamma(\beta) t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}$$
$$\times \left(\frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} |f(s,x(s),I^\eta x(s)) - f(s,y(s),I^\eta y(s))| \, ds \right.$$
$$\left. + |\lambda| |x(T)-y(T)| \right) + |a| \int_0^T |x(s)-y(s)| \, ds$$
$$\leq \left(\left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{T^{\beta+\alpha}}{\beta\Gamma(\beta+\alpha)} \right) \left(l_1 + l_2 \frac{T^\eta}{\Gamma(\eta+1)} \right) \right.$$
$$\left. + |\lambda| \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta) T^\alpha}{\Gamma(\beta+\alpha)} \right) + |a| T \right) \|x-y\| .$$

Thus,

$$\|\Phi x - \Phi y\| \leq \theta \|x-y\| .$$

From (3.1), Φ is a contraction. As a result of Banach's fixed point theorem, Φ has a unique fixed point which is the unique solution to problem (1.1) on J. This finishes the proof. □

3.2. Existence results via Schauder's fixed point theorem.

For the sake of convenience, we put

$$\Lambda_1 = \frac{\Psi^* T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{\Psi^* T^{\alpha+\beta}}{\beta\Gamma(\alpha+\beta)} + |b| ,$$

where $\Psi^* = \sup\{\Psi(t) : t \in J\}$.

Theorem 3.2. *Suppose that the hypotheses* (H1) *and* (H2) *are satisfied. If*

$$\omega = |\lambda| \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta) T^\alpha}{\Gamma(\beta+\alpha)} \right) + |a| T < 1,$$

then problem (1.1) *has at least one solution on* J.

Proof. We consider a nonempty closed bounded convex subset

$$\Omega = \{x \in \mathcal{C} : \|x\| \leq M\} \text{ of } \mathcal{C},$$

where M is chosen so that

$$M \geq \frac{\Lambda_1}{1-\omega}.$$

Notice that the continuity of the mapping Φ follows from the continuity of the function f. Now, we need to show that the mapping Φ is compact by applying the well-known Arzelà–Ascoli theorem. So, we will show that $\Phi(\Omega) \subset \Omega$ and $\Phi(\Omega)$ is a uniformly bounded and equicontinuous set. For $x \in \Omega$, we have

$$|(\Phi x)(t)| \leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} |f(s,x(s),I^\eta x(s))| \, ds$$
$$+ \frac{|\lambda|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| \, ds + \frac{\Gamma(\beta) t^{\beta+\alpha-1}}{T^{\beta-1}\Gamma(\beta+\alpha)}$$

$$
\begin{aligned}
&\times \left(\frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \left| f(s, x(s), I^\eta x(s)) \right| ds + |\lambda| \left| x(T) \right| \right) \\
&+ |a| \int_0^T |x(s)| \, ds + |b| \\
&\leq \frac{\Psi^* T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + |\lambda| M \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(\beta) T^\alpha}{\Gamma(\alpha+\beta)} \right) \\
&+ \frac{\Psi^* T^{\alpha+\beta}}{\beta \Gamma(\alpha+\beta)} + |a| TM + |b| \\
&\leq M
\end{aligned}
$$

and, consequently,

$$
\|\Phi x\| \leq M,
$$

which means that $\Phi(\Omega) \subset \Omega$ and the set $\Phi(\Omega)$ is uniformly bounded. Next, we will prove that $\Phi(\Omega)$ is an equicontinuous set. For $t_1, t_2 \in J$ such that $t_1 < t_2$ and for $x \in \Omega$, we get

$$
\begin{aligned}
&|(\Phi x)(t_2) - (\Phi x)(t_1)| \\
&\leq \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_1} \left((t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1} \right) \left| f(s, x(s), I^\eta x(s)) \right| ds \\
&+ \frac{1}{\Gamma(\alpha+\beta)} \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} \left| f(s, x(s), I^\eta x(s)) \right| ds \\
&+ \frac{|\lambda|}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right) |x(s)| \, ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |x(s)| \, ds \right) \\
&+ \frac{\Gamma(\beta) \left(t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1} \right)}{T^{\beta-1} \Gamma(\beta+\alpha)} \\
&\times \left(\frac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \left| f(s, x(s), I^\eta x(s)) \right| ds + |\lambda| \left| x(T) \right| \right) \\
&\leq \frac{\Psi^*}{\Gamma(\alpha+\beta)} \left(\int_0^{t_1} \left((t_2-s)^{\alpha+\beta-1} - (t_1-s)^{\alpha+\beta-1} \right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha+\beta-1} ds \right) \\
&+ \frac{|\lambda| M}{\Gamma(\alpha)} \left(\int_0^{t_1} \left((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \\
&+ \frac{\Gamma(\beta) \left(t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1} \right)}{T^{\beta-1} \Gamma(\beta+\alpha)} \left(\frac{\Psi^* T^\beta}{\Gamma(\beta+1)} + |\lambda| M \right) \\
&\leq \frac{\Psi^*}{\Gamma(\alpha+\beta+1)} \left(t_2^{\alpha+\beta} - t_1^{\alpha+\beta} \right) + \frac{2 |\lambda| M}{\Gamma(\alpha+1)} (t_2 - t_1)^\alpha \\
&+ \frac{\Gamma(\beta) \left(t_2^{\beta+\alpha-1} - t_1^{\beta+\alpha-1} \right)}{T^{\beta-1} \Gamma(\beta+\alpha)} \left(\frac{\Psi^* T^\beta}{\Gamma(\beta+1)} + |\lambda| M \right).
\end{aligned}
$$

As $t_1 \to t_2$, we see that the right hand side of the above inequality tends to zero and the convergence is independent of x in Ω, which means $\Phi(\Omega)$ is equicontinuous.

The Arzelà–Ascoli theorem implies that Φ is compact. Thus, by the Schauder fixed point theorem, we prove that Φ has at least one fixed point $x \in \Omega$ which is a solution to problem (1.1) on J. □

4. Example

As an application of our results, we take the following fractional Langevin equation

$$\begin{cases} D^{\frac{3}{2}}\left({}^{C}D^{\frac{1}{2}}+\frac{1}{4}\right)x(t)=f(t,x(t),I^{\frac{1}{3}}x(t)),\ t\in(0,1), \\ {}^{C}D^{\frac{1}{2}}x(0)+\frac{1}{4}x(0)={}^{C}D^{\frac{1}{2}}x(1)=0,\ x(0)=\frac{1}{10}\int_0^1 x(s)ds+2. \end{cases} \tag{4.1}$$

Here, $\alpha=\frac{1}{2}$, $\beta=\frac{3}{2}$, $\eta=\frac{1}{3}$, $\lambda=\frac{1}{4}$, $a=\frac{1}{10}$ and $b=2$. Set

$$f(t,x(t),I^{\frac{1}{3}}x(t))=\frac{\sin(t)}{\exp(t^2)+7}\left(\frac{|x(t)|}{|x(t)|+1}+\frac{\left|I^{\frac{1}{3}}x(t)\right|}{1+\left|I^{\frac{1}{3}}x(t)\right|}\right).$$

For $x_i, y_i \in \mathbb{R}$, $i=1,2$, we have

$$\begin{aligned} &|f(t,x_1,x_2)-f(t,y_1,y_2)| \\ &=\left|\frac{\sin(t)}{\exp(t^2)+7}\left(\left(\frac{|x_1|}{|x_1|+1}-\frac{|y_1|}{|y_1|+1}\right)+\left(\frac{|x_2|}{|x_2|+1}-\frac{|y_2|}{|y_2|+1}\right)\right)\right| \\ &\le \frac{1}{\exp(t^2)+7}\left(\frac{|x_1-y_1|}{(1+|x|)(1+|y|)}+\frac{|x_2-y_2|}{(1+|x_2|)(1+|y_2|)}\right) \\ &\le \frac{1}{8}\left(|x_1-y_1|+|x_2-y_2|\right); \end{aligned}$$

thus, the assumption (H1) is satisfied with $l_1=l_2=\frac{1}{8}$. We will check that condition (3.1) is satisfied. Indeed,

$$\begin{aligned} \theta &= \left(\frac{T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{T^{\beta+\alpha}}{\beta\Gamma(\beta+\alpha)}\right)\left(l_1+l_2\frac{T^{\eta}}{\Gamma(\eta+1)}\right) \\ &+|\lambda|\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{\Gamma(\beta)T^{\alpha}}{\Gamma(\beta+\alpha)}\right)+|a|T \\ &=\left(\frac{1}{\Gamma(3)}+\frac{2}{3\Gamma(2)}\right)\left(\frac{1}{8}+\frac{1}{8}\frac{1}{\Gamma\left(\frac{1}{3}+1\right)}\right)+\frac{1}{4}\left(\frac{1}{\Gamma\left(\frac{3}{2}\right)}+\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}\right)+\frac{1}{10} \\ &\simeq 0.913<1. \end{aligned}$$

Then, by Theorem 3.1, problem (4.1) has a unique solution on $[0,1]$. Also, we have

$$f(t,x,y)\le\frac{2}{\exp(t^2)+7},\ \forall(t,x,y)\in J\times\mathbb{R}\times\mathbb{R}.$$

Hence, condition (H2) holds with $\Psi(t)=\frac{2}{\exp(t^2)+7}$. It follows from Theorem 3.2 that problem (4.1) has at least one solution on $[0,1]$.

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