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ITERATIVE SYSTEM OF NABLA FRACTIONAL DIFFERENCE EQUATIONS WITH TWO-POINT BOUNDARY CONDITIONS

MAHAMMAD KHUDDUSH AND K. RAJENDRA PRASAD

Abstract. In this paper, we consider the nabla fractional order boundary value problem

$$\begin{aligned} \nabla^{ß-1}_{n_0}\left[\nabla z_j(t)\right] + \varphi(t) g_j(z_{j+1}(t)) = 0, \quad t \in \mathbb{N}^{n}_{n_0+2},\ 1 < ß < 2, \\ a z_j(n_0+1) - b\nabla z_j(n_0+1) = 0, \\ c z_j(n) + d\nabla z_j(n) = 0, \end{aligned}$$

where $j = 1, 2, \ldots, N$, $z_{N+1} = z_1$, $N \in \mathbb{N}$, $n_0, n \in \mathbb{R}$ with $n - n_0 \in \mathbb{N}$ and derive sufficient conditions for the existence of positive solutions by an application of Krasnoselskii's fixed point theorem on a Banach space. Later, we derive sufficient conditions for the existence of a unique solution by applying Rus's contraction mapping theorem in a metric space, where two metrics are employed.

1. Introduction

Fractional calculus is a generalization of classical integer order calculus and has been studied for more than three decades. Unlike integer order derivatives, the fractional derivative is a non local operator, which implies that the future states depend on the current state as well as the history of all the previous states. From this point of view, fractional differential equations provide a powerful tool for mathematical modeling of complex phenomena in science and engineering practices, see [5, 10, 12, 18, 30, 39] and references therein. In the qualitative theory of classical and fractional order differential equations, various theorems have been extensively deployed by researchers in establishing the existence, uniqueness and multiple solutions of boundary value problems, see [29,32–34,41] and the references therein.

Fractional difference equations have been of great interest recently. Díaz and Osler [13] introduced a fractional difference defined as an infinite series, a generalization of the binomial formula for the N^{th} order difference $\Delta^N f$. Gray and Zhang [22] developed a special case for one composition rule and Leibniz formula. They worked exclusively with the nabla operator. For the recent works in discrete fractional calculus, see [2–4, 19–21] and references therein.

On the other hand, differential equations with state-dependent delays have attracted a great deal of interest to the researchers since they widely arise from application models, such as population models [6], mechanical models [26], the

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dynamics of economical systems [7], position control [9], two-body problem of classical electrodynamics [14], etc. As a special type of state-dependent delay-differential equations, iterative differential equations have distinctive characteristics and have been investigated in recent years, e.g., equivariance [38], analyticity [40], convexity [36], monotonicity [15], smoothness [11]. In [17], Fečkan, Wang and Zhao established the maximal and minimal nondecreasing bounded solutions of the following iterative functional differential equations

$$z'(t) = g(t, z^{[1]}(t), z^{[2]}(t), \dots, z^{[n]}(t)),$$

where $z^{[j]}(t) := z(z^{[j-1]})(t)$ indicates the j-th iterate of z, where $j = 1, 2, \dots, n$, by the method of lower and upper solutions. Recently, Prasad, Khuddush and Leela [33] derived sufficient conditions for the existence and uniqueness of solutions by applying Schauder fixed point theorem and contraction mapping theorem in a Banach space to the following Caputo type fractional order boundary value problem

$$\begin{aligned} D^{\alpha} z(t) &= g(t, z(t), z^{[2]}(t)), \quad t \in [0, 1], \\ z(0) &= A, \; z(1) = B, \end{aligned}$$

where $1 < \alpha \le 2$ and $0 \le A \le B \le 1$. In [27], Jonnalagadda derived sufficient conditions for the existence of positive solutions for the following Riemann–Liouville nabla fractional order two-point boundary value problem,

$$\begin{aligned} \nabla^{\alpha}_{\rho(a)} z(t) + g(t, z(t)) &= 0, \quad t \in \mathbb{N}^{b}_{a+2}, \; 1 < \alpha < 2, \\ z(a) = 0, \; z(b) &= 0 \end{aligned}$$

where $\mathbb{N}^{n}_{n_0} := \{n_0, n_0+1, n_0+2, \dots, n\}$ by applying Krasnoselskii's fixed point theorem. Recently, Eralp and Topal [16] studied the following Caputo nabla fractional order two-point boundary value problem,

$$\begin{aligned} \nabla^{\mu}_{a^*} z(t) + g(t, z(t - (N-1))) &= 0, \quad t \in \mathbb{N}^{b-1}_{a+1}, \; 1 < \mu < 2, \\ \alpha z(a - j) - \beta \nabla z(a - (j-1)) &= 0, \quad j = 1, 2, \dots, N-1, \\ \gamma z(b) + \delta \nabla z(b) &= 0, \end{aligned}$$

and obtained the existence of positive solutions by using the Schauder and Banach fixed point theorems. Inspired by the above literature, in this paper, we derive sufficient conditions for the existence of positive solutions for the following iterative system of Riemann–Liouville nabla fractional order difference equations

$$\begin{aligned} \nabla^{ß-1}_{n_0} [\nabla z_j(t)] + \varphi(t) g_j(z_{j+1}(t)) &= 0, \quad t \in \mathbb{N}^{n}_{n_0+2}, \; 1 < ß < 2, \; j = 1, 2, \dots, N, \\ z_{N+1}(t) &= z_1(t), \quad t \in \mathbb{N}^{n}_{n_0+2}, \end{aligned} \tag{1.1}$$

satisfying two-point general boundary conditions

$$\begin{aligned} a z_j(n_0 + 1) - b \nabla z_j(n_0 + 1) &= 0, \quad j = 1, 2, \dots, N, \\ c z_j(n) + d \nabla z_j(n) &= 0, \quad j = 1, 2, \dots, N, \end{aligned} \tag{1.2}$$

where $N \in \mathbb{N}$, $n_0, n \in \mathbb{R}$ with $n - n_0 \in \mathbb{N}$, $\varphi(t) \in \ell^p(\mathbb{N}^{n}_{n_0+2})$, $g_j : [0, +\infty) \to [0, +\infty)$ is continuous for each $1 \le j \le n$, by an application of Krasnoselskii's fixed point theorem on a Banach space.

The rest of the paper is organized in the following fashion. In Section 2, we estimate bounds for the kernel and provide some lemmas which are needed in establishing our main results. In Section 3, we establish criteria for the existence of positive solutions for the boundary value problem (1.1)–(1.2) by applying Hölder's inequality and Krasnoselskii's cone fixed point theorem in a Banach space. Also, we derive sufficient conditions for the existence of a unique positive solution to the problem by an application of a fixed point theorem in a complete metric space. Finally, we provide an example to illustrate the main results of the paper.

2. Preliminaries, kernel and bounds

In this section, we present some definitions, lemmas and kernel for the homogeneous boundary value problem corresponding to the problem (1.1)–(1.2), which is useful in the later sections. Throughout the paper, we shall use the following notations. Denote the set of all positive integers, nonpositive integers, real numbers and positive real numbers by $\mathbb{N}$, $\mathbb{N}^-$, $\mathbb{R}$ and $\mathbb{R}^+$, respectively. The generalized rising function is defined by

$$t^{\overline{ß}} = \frac{\Gamma(t+ß)}{\Gamma(t)}$$

for $t \in \mathbb{R}\backslash\mathbb{N}^-$, $ß \in \mathbb{R}$ and $t + ß \in \mathbb{R}\backslash\mathbb{N}^-$. Also, if $t \in \mathbb{N}^-$ and $t + ß \in \mathbb{R}\backslash\mathbb{N}^-$, then we use the convention that $t^{\overline{ß}} = 0$.

For $ß \in \mathbb{R}\backslash\mathbb{N}^-$, define the $ß^{th}$ order nabla fractional Taylor monomial by

$$\mathcal{H}_ß(t, n_0) = \frac{(t-n_0)^{\overline{ß}}}{\Gamma(ß+1)},$$

provided the right hand side exists. Also, $\mathcal{H}_ß(t, n_0) = 0$ for $t \in \mathbb{N}_{n_0}$ and $ß \in \mathbb{N}^-\backslash\{0\}$. Also, we denote

$$\begin{aligned}
\mathbb{N}_{n_0} &= \{n_0, n_0+1, n_0+2, \dots\},\\
\lambda &= (b-a)c + ac\mathcal{H}_{ß-1}(n, n_0) + ad\mathcal{H}_{ß-2}(n, n_0) \neq 0,\\
\widehat{\aleph} &= \frac{1}{\lambda}\Big[ad\mathcal{H}_{ß-1}(n_0+1, n_0)\mathcal{H}_{ß-2}(n, n_0) + (b-a)d\mathcal{H}_{ß-2}(n, n_0)\Big],\\
\aleph^* &= \frac{1}{\lambda}\Big[ac(\mathcal{H}_{ß-1}(n, n_0))^2 + ad\mathcal{H}_{ß-1}(n, n_0) + (b-a)c\mathcal{H}_{ß-1}(n, n_0) + (b-a)d\Big],\\
\varkappa &= \frac{1}{\lambda}\Big[ac\mathcal{H}_{ß-1}(n, n_0)\mathcal{H}_ß(n, n_0) + ad\left[\mathcal{H}_{ß-1}(n, n_0)\right]^2\\
&\qquad + (b-a)c\mathcal{H}_ß(n, n_0) + (b-a)d\mathcal{H}_{ß-1}(n, n_0)\Big].
\end{aligned}$$

Definition 2.1. [21] The operator $\varrho : \mathbb{N}_{n_0+1} \to \mathbb{N}_{n_0}$ for the nabla transformation defined by $\varrho(t) = t - 1$ is called the *backward jump operator*.

Definition 2.2. [21] Let $z : N_{n_0+1} \to \mathbb{R}$ and $N \in \mathbb{N}$. The N^{th} order nabla sum of z based at n_0 is given by

$$\nabla_{n_0}^{-N} z(t) = \sum_{\tau=n_0+1}^{t} \mathcal{H}_{N-1}(t, \varrho(\tau))z(\tau),\ t \in \mathbb{N}_{n_0},$$

where, by convention, $\nabla_{n_0}^{-N} z(n_0) = 0$.

Definition 2.3. [21] Let $z : N_{n_0+1} \to \mathbb{R}$ and $ß \in \mathbb{R}^+$. The $ß^{th}$ order nabla sum of z based at n_0 is given by

$$\nabla_{n_0}^{-ß} z(t) = \sum_{\tau=n_0+1}^{t} \mathcal{H}_{ß-1}(t, \varrho(\tau)) z(\tau),\ t \in \mathbb{N}_{n_0},$$

where, by convention, $\nabla_{n_0}^{-ß} z(n_0) = 0$.

Definition 2.4. [21] Let $z : N_{n_0+1} \to \mathbb{R}$, $ß \in \mathbb{R}^+$ and choose $N \in \mathbb{N}_1$ such that $ß \in (N-1, N]$. The $ß^{th}$ order Riemann–Liouville nabla fractional difference of z is given by

$$\nabla_{n_0}^{ß} z(t) = \nabla^N \nabla_{n_0}^{-(N-ß)} z(t),\ t \in N_{n_0+N}.$$

Lemma 2.5. [25] *Let $ß > -1$ and $\tau \in \mathbb{N}_{n_0}$. Then, we have the following properties:*

(i) *If $t \in \mathbb{N}_{\varrho(\tau)}$, then $\mathcal{H}_{ß-1}(t, \varrho(\tau)) \geq 0$, and if $t \in \mathbb{N}_\tau$, then $\mathbb{N}_{\varrho(\tau)} > 0$.*
(ii) *If $t \in \mathbb{N}_{\varrho(\tau)}$ and $ß > 0$, then $\mathcal{H}_ß(t, \varrho(\tau))$ is a decreasing function of τ.*
(iii) *If $t \in \mathbb{N}_\tau$ and $-1 < ß < 0$, then $\mathcal{H}_ß(t, \varrho(\tau))$ is an increasing function of τ.*
(iv) *If $t \in \mathbb{N}_{\varrho(\tau)}$ and $ß \geq 0$, then $\mathcal{H}_ß(t, \varrho(\tau))$ is a nondecreasing function of t.*
(v) *If $t \in \mathbb{N}_\tau$ and $ß > 0$, then $\mathcal{H}_ß(t, \varrho(\tau))$ is an increasing function of t.*
(vi) *If $t \in \mathbb{N}_{\tau+1}$ and $-1 < ß < 0$, then $\mathcal{H}_ß(t, \varrho(\tau))$ is a decreasing function of τ.*

Definition 2.6. [24] Let $0 < p < \infty$ be a finite real number. A sequence of scalars $z = (z_k)_{k\in\mathbb{N}}$ is p-summable if $\sum_{k=1}^{\infty} |z_k|^p < \infty$. A sequence of scalars $z = (z_k)_{k\in\mathbb{N}}$ is bounded if $\sup_{k\in\mathbb{N}} |z_k| < \infty$.

Definition 2.7. [24] If $0 < p < \infty$, then the space ℓ^p consists of all p-summable sequences of scalars. That is, a sequence $z = (z_k)_{k\in\mathbb{N}}$ belongs to ℓ^p if and only if

$$\|z\|_p = \|(z_k)_{k\in\mathbb{N}}\|_p = \left[\sum_{k=1}^{\infty} |z_k|^p\right]^{1/p} < \infty.$$

For $p = \infty$, the space ℓ^p consists of all bounded sequences of scalars. That is, a sequence $z = (z_k)_{k\in\mathbb{N}}$ belongs to ℓ^p if and only if

$$\|z\|_p = \|(z_k)_{k\in\mathbb{N}}\|_p = \sup_{k\in\mathbb{N}} |z_k| < \infty.$$

Remark 2.8. [24] By making appropriate changes in the preceeding definitions, we can consider spaces of sequences that are indexed by sets other than the natural numbers N. For example, if I is a countable index set, then we say that a sequence $z = (z_k)_{k\in I}$ is p-summable if and only if $\sum_{k=1}^{\infty} |z_k|^p < \infty$. For finite p, we let $\ell^p(I)$ be the space of all p-summable sequences indexed by I, and we define $\ell^\infty(I)$ to be the space of all bounded sequences indexed by I.

Lemma 2.9. [8] *Let $b \geq a$ and $f : N_{n_0+1} \to \mathbb{R}$. Then, the boundary value problem*

$$\begin{aligned} \nabla_{n_0}^{ß-1}\left[\nabla z_1(t)\right] + f(t) = 0, &\quad t \in \mathbb{N}_{n_0+2}^{n}, 1 < ß < 2,\\ a z_1(n_0+1) - b\nabla z_1(n_0+1) = 0,&\\ c z_1(n) + d\nabla z_1(n) = 0&\end{aligned}$$

has a unique solution

$$z_1(t) = \sum_{\tau=n_0+1}^{n} \aleph(t,\tau) f(\tau),$$

where

$$\aleph(t,\tau) = \begin{cases} \aleph_1(t,\tau), & t < \varrho(\tau), \\ \aleph_2(t,\tau), & \tau < t, \end{cases}$$

in which

$$\aleph_1(t,\tau) = \frac{1}{\lambda}\Big[ac\mathcal{H}_{\text{ß}-1}(t,n_0)\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau)) + ad\mathcal{H}_{\text{ß}-1}(t,n_0)\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau)) + (b-a)c\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau)) + (b-a)d\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau))\Big]$$

and

$$\aleph_2(t,\tau) = \aleph_1(t,\tau) - \mathcal{H}_{\text{ß}-1}(t,\varrho(\tau)).$$

Lemma 2.10. [28] *Let $b \geq a$. Then*

(i) *$\aleph_1(t,\tau)$ is an increasing function of t for all $(t,\tau) \in \mathbb{N}_{n_0}^n \times \mathbb{N}_{n_0+1}^n$ and $t \leq \varrho(\tau)$.*
(ii) *$\aleph_2(t,\tau)$ is a decreasing function of t for all $(t,\tau) \in \mathbb{N}_{n_0}^n \times \mathbb{N}_{n_0+1}^n$ and $\tau < t$.*

Lemma 2.11. *Suppose that $\widehat{\aleph} > \mathcal{H}_{\text{ß}-1}(n,n_0)$ and let $\eta = (\widehat{\aleph} - \mathcal{H}_{\text{ß}-1}(n,n_0))/\aleph^*$. Then, kernel function $\aleph(t,\tau)$ has the following properties:*

(i) *$\aleph(t,\tau) \geq 0$ for $(t,\tau) \in \mathbb{N}_{n_0}^n \times \mathbb{N}_{n_0+1}^n$.*
(ii) *$\aleph(t,\tau) \leq \aleph(\varrho(\tau),\tau)$ for $(t,\tau) \in \mathbb{N}_{n_0}^n \times \mathbb{N}_{n_0+1}^n$.*
(iii) *$\aleph(t,\tau) \geq \eta\aleph(\varrho(\tau),\tau)$ for $(t,\tau) \in \mathbb{N}_{n_0}^n \times \mathbb{N}_{n_0+1}^n$.*
(iv) *$\sum_{\tau=n_0+1}^{n} \aleph(t,\tau) \leq \varkappa$ for $t \in \mathbb{N}_{n_0}^n$.*

Proof. The results (i) and (iv) are established in [28]. To prove (ii), we let $t < \varrho(\tau)$, then, we have from (i) of Lemma 2.10 that

$$\aleph(t,\tau) = \aleph_1(t,\tau) \leq \aleph_1(\varrho(\tau),\tau).$$

Now, for $\tau < t$, we have from (ii) of Lemma 2.10 that

$$\aleph(t,\tau) = \aleph_2(t,\tau) \leq \aleph_2(\tau,\tau) \leq \aleph_2(\varrho(\tau),\tau) \text{ as } \varrho(\tau) < \tau.$$

From the above two inequalities, we obtain $\aleph(t,\tau) \leq \aleph(\varrho(\tau),\tau)$. Next, we prove (iii). We note from Lemma 2.5 that

$$\begin{gathered} \mathcal{H}_{\text{ß}-1}(n_0+1,n_0) \leq \mathcal{H}_{\text{ß}-1}(t,n_0) \ for \ t \in N_{n_0+1}, \\ 0 = \mathcal{H}_{\text{ß}-1}(n,n) \leq \mathcal{H}_{\text{ß}-1}(n,\varrho(\tau)) \leq \mathcal{H}_{\text{ß}-1}(n,n_0), \\ \mathcal{H}_{\text{ß}-2}(n,n_0) \leq \mathcal{H}_{\text{ß}-2}(n,\varrho(\tau)) \leq 1, \\ \mathcal{H}_{\text{ß}-1}(\varrho(\tau),n_0) \leq \mathcal{H}_{\text{ß}-1}(\tau,n_0) \leq \mathcal{H}_{\text{ß}-1}(n,n_0), \\ \mathcal{H}_{\text{ß}-1}(t,\varrho(\tau)) \leq \mathcal{H}_{\text{ß}-1}(t,n_0) \leq \mathcal{H}_{\text{ß}-1}(n,n_0). \end{gathered}$$

So,

$$\begin{aligned}\aleph_1(t,\tau)&=\frac{1}{\lambda}\Big[ac\mathcal{H}_{\text{ß}-1}(t,n_0)\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau))+ad\mathcal{H}_{\text{ß}-1}(t,n_0)\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau))\\&\qquad+(b-a)c\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau))+(b-a)d\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau))\Big]\\&\geq\frac{1}{\lambda}\Big[ad\mathcal{H}_{\text{ß}-1}(n_0+1,n_0)\mathcal{H}_{\text{ß}-2}(n,n_0)+(b-a)d\mathcal{H}_{\text{ß}-2}(n,n_0)\Big]=\widehat{\aleph}\end{aligned}$$

and

$$\begin{aligned}\aleph_1(\varrho(\tau),\tau)&=\frac{1}{\lambda}\Big[ac\mathcal{H}_{\text{ß}-1}(\varrho(\tau),n_0)\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau))+ad\mathcal{H}_{\text{ß}-1}(\varrho(\tau),n_0)\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau))\\&\qquad+(b-a)c\mathcal{H}_{\text{ß}-1}(n,\varrho(\tau))+(b-a)d\mathcal{H}_{\text{ß}-2}(n,\varrho(\tau))\Big]\\&\leq\frac{1}{\lambda}\Big[ac(\mathcal{H}_{\text{ß}-1}(n,n_0))^2+ad\mathcal{H}_{\text{ß}-1}(n,n_0)\\&\qquad+(b-a)c\mathcal{H}_{\text{ß}-1}(n,n_0)+(b-a)d\Big]=\aleph^*.\end{aligned}$$

Therefore, for $t<\varrho(\tau)$, we have

$$\frac{\aleph(t,\tau)}{\aleph(\varrho(\tau),\tau)}=\frac{\aleph_1(t,\tau)}{\aleph_1(\varrho(\tau),\tau)}\geq\frac{\widehat{\aleph}}{\aleph^*}\geq\frac{\widehat{\aleph}-\mathcal{H}_{\text{ß}-1}(n,n_0)}{\aleph^*}$$

and, for $\tau<t$, we get from (i) that

$$\frac{\aleph(t,\tau)}{\aleph(\varrho(\tau),\tau)}=\frac{\aleph_2(t,\tau)}{\aleph_1(\varrho(\tau),\tau)}=\frac{\aleph_1(t,\tau)-\mathcal{H}_{\text{ß}-1}(t,\varrho(\tau))}{\aleph_1(\varrho(\tau),\tau)}\geq\frac{\widehat{\aleph}-\mathcal{H}_{\text{ß}-1}(n,n_0)}{\aleph^*}.$$

This completes the proof. □

We note that an N-tuple $(z_1,z_2,\ldots,z_N)$ is a solution of (1.1)–(1.2) if and only if

$$\begin{aligned}z_1(t)=&\sum_{\tau_1=n_0+1}^{n}\aleph(t,\tau_1)\phi(\tau_1)g_1\Bigg[\sum_{\tau_2=n_0+1}^{n}\aleph(\tau_1,\tau_2)\phi(\tau_2)g_2\Bigg[\sum_{\tau_3=n_0+1}^{n}\aleph(\tau_2,\tau_3)\phi(\tau_3)\\&\vdots\\&\times g_{N-2}\Bigg[\sum_{\tau_{N-1}=n_0+1}^{n}\aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1})\\&\times g_{N-1}\Bigg[\sum_{\tau_N=n_0+1}^{n}\aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\Bigg]\Bigg]\end{aligned}$$

and

$$\begin{aligned}z_j(t)&=\sum_{\tau=n_0+1}^{n}\aleph(t,\tau)\phi(\tau)g_j(z_{j+1}(\tau)),\quad j=2,3,\ldots,N,\\z_{N+1}(t)&=z_1(t),\quad t\in\mathbb{N}_{n_0}^{n}.\end{aligned}$$

Denote the set of all real-valued functions z defined on $\mathbb{N}_{n_0}^{n}$ by $\mathcal{B}$. Then, $\mathcal{B}$ is a Banach space equipped with the norm $\|z\|=\max_{t\in\mathbb{N}_{n_0}^{n}}|z(t)|$. Next, we take the cone $\mathcal{D} \subset \mathcal{B}$ defined by

$$\mathcal{D} = \Big\{ z \in \mathcal{B} : z(t) \geq 0 \text{ for } t \in \mathbb{N}_{n_0}^{n} \text{ and } \min_{t \in \mathbb{N}_{n_0+1}^{n-1}} z(t) \geq \eta \|z\| \Big\}.$$

For any $z_1 \in \mathcal{D}$, define an operator $\mathscr{F} : \mathcal{D} \to \mathcal{B}$ by

$$\begin{aligned}(\mathscr{F} z_1)(t) = \sum_{\tau_1=n_0+1}^{n} \aleph(t,\tau_1)\phi(\tau_1) g_1 \Bigg[\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1,\tau_2)\phi(\tau_2) g_2 \Bigg[\sum_{\tau_3=n_0+1}^{n} &\aleph(\tau_2,\tau_3) \\ &\vdots \\ \times g_{N-2} \Bigg[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1}) & \\ \times g_{N-1} \Bigg[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) \Bigg] \Bigg]. &\end{aligned}$$

Lemma 2.12. *The operator $\mathscr{F}$ is self mapping and completely continuous on $\mathcal{D}$.*

Proof. Since $g_j(z_{j+1}(\tau))$ is nonnegative for $\tau \in \mathbb{N}_{n_0}^{n}$, $z_1 \in \mathcal{D}$ and $\aleph(t,\tau)$ is nonnegative for all $(t,\tau) \in \mathbb{N}_{n_0}^{n} \times \mathbb{N}_{n_0+1}^{n}$, it follows that $\mathscr{F}(z_1(t)) \geq 0$ for all $t \in \mathbb{N}_{n_0}^{n}$, $z_1 \in \mathcal{D}$. Now, by Lemma 2.11, we have

$$\begin{aligned}&\min_{t \in \mathbb{N}_{n_0+1}^{n-1}} (\mathscr{F} z_1)(t) \\ &= \min_{t \in \mathbb{N}_{n_0+1}^{n-1}} \Bigg\{ \sum_{\tau_1=n_0+1}^{n} \aleph(t,\tau_1)\phi(\tau_1) g_1 \Bigg[\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1,\tau_2)\phi(\tau_2) g_2 \Bigg[\sum_{\tau_3=n_0+1}^{n} \aleph(\tau_2,\tau_3) \\ &\qquad \vdots \\ &\quad \times g_{N-2} \Bigg[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1}) \\ &\quad \times g_{N-1} \Bigg[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) \Bigg] \Bigg] \Bigg\} \\ &= \eta \sum_{\tau_1=n_0+1}^{n} \aleph(\varrho(\tau_1),\tau_1)\phi(\tau_1) g_1 \Bigg[\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1,\tau_2)\phi(\tau_2) g_2 \Bigg[\sum_{\tau_3=n_0+1}^{n} \aleph(\tau_2,\tau_3) \\ &\qquad \vdots \\ &\quad \times g_{N-2} \Bigg[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1}) \\ &\quad \times g_{N-1} \Bigg[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) \Bigg] \Bigg]\end{aligned}$$

$$\geq \eta \max_{t\in\mathbb{N}_{n_0}^n}\Bigg\{\sum_{\tau_1=n_0+1}^{n}\aleph(t,\tau_1)\phi(\tau_1)g_1\Bigg[\sum_{\tau_2=n_0+1}^{n}\aleph(\tau_1,\tau_2)\phi(\tau_2)g_2\Bigg[\sum_{\tau_3=n_0+1}^{n}\aleph(\tau_2,\tau_3)$$

$$\vdots$$

$$\times g_{N-2}\Bigg[\sum_{\tau_{N-1}=n_0+1}^{n}\aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1})$$

$$\times g_{N-1}\Bigg[\sum_{\tau_N=n_0+1}^{n}\aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\Bigg]\Bigg] \geq \eta \max_{t\in\mathbb{N}_{n_0}^n}|\mathscr{F}z_1(t)|.$$

Thus, $\mathscr{F}(\mathcal{D}) \subset \mathcal{D}$. Therefore, the operator $\mathscr{F}$ is completely continuous by standard methods and by the Arzelà–Ascoli theorem. □

3. Existence of positive solutions

By an application of the following theorems, we derive the sufficient conditions for the existence of positive solutions to the problem (1.1)–(1.2).

Theorem 3.1. (Krasnoselskii's [23]) *Let $\mathcal{D}$ be a cone in a Banach space $\mathcal{B}$ and E_1, E_2 are open sets with $0 \in E_1, \overline{E}_1 \subset E_2$. Let $\mathscr{F} : \mathcal{D}\cap(\overline{E}_2\backslash E_1) \to \mathcal{D}$ be a completely continuous operator such that*

(a) $\|\mathscr{F}z\| \leq \|z\|$, $z \in \mathcal{D}\cap\partial E_1$, *and* $\|\mathscr{F}z\| \geq \|z\|$, $z \in \mathcal{D}\cap\partial E_2$, *or*
(b) $\|\mathscr{F}z\| \geq \|z\|$, $z \in \mathcal{D}\cap\partial E_1$, *and* $\|\mathscr{F}z\| \leq \|z\|$, $z \in \mathcal{D}\cap\partial E_2$.

Then, $\mathscr{F}$ has a fixed point in $\mathcal{D}\cap(\overline{E}_2\backslash E_1)$.

Theorem 3.2. (Hölder's inequality [31]) *Fix $1 \leq p \leq \infty$ and let q be the dual index to p, i.e., $1/p + 1/q = 1$. Let $z = (z_k)_{k=1}^n \in \ell^p$ and $\vartheta = (\vartheta_k)_{k=1}^n \in \ell^q$, then the sequence $z\vartheta = (z\vartheta_k)_{k=1}^n \in \ell^1$, and*

$$\|z\vartheta\|_1 \leq \|z\|_p\|\vartheta\|_q.$$

Moreover, if $z \in \ell^1$ and $\vartheta \in \ell^\infty$, then $z\vartheta = (z\vartheta_k)_{k=1}^n \in \ell^1$, and

$$\|z\vartheta\|_1 \leq \|z\|_1\|\vartheta\|_\infty.$$

We consider the follwoing three possible cases for $\varphi \in \ell^p(N_{n_0}^n) : p > 1$, $p = 1$ and $p = \infty$. The case $p > 1$ is treated in the following theorem.

Theorem 3.3. *Let φ be bounded below by a positive real number $\widehat{\varphi}$ and there exist positive numbers R, r such that $\eta r < r < \theta r < R$, where*

$$\theta > \left[\eta\widehat{\varphi}\sum_{\tau_N=n_0+1}^{n}\aleph(\varrho(\tau_N),\tau_N)\right]^{-1}$$

and g_j, $j = 1, 2, \ldots, N$, satisfies the following conditions

($\mathcal{J}_1$) $g_j(z) \leq M_1R$ *for* $0 \leq z \leq R$, *where*

$$M_1 < \frac{1}{\|\aleph\|_q\|\varphi\|_p},$$

($\mathcal{J}_2$) $g_j(z) \geq \theta r$ *for* $\eta r \leq z \leq r$.

Then, the problem (1.1)–(1.2) *has at least one positive solution.*

Proof. Consider the open subsets E_1 and E_2 of $\mathcal{B}$ defined by

$$E_1 = \{z \in \mathcal{B} : \|z\| < R\},\ E_2 = \{z \in \mathcal{B} : \|z\| < r\}.$$

Let $z_1 \in \mathcal{D} \cap \partial E_1$. Then, $z_1(\tau) \leq R = \|z_1\|$ for all $\tau \in N^n_{n_0+1}$. By $(\mathcal{J}_1)$ and $\tau_{N-1} \in N^n_{n_0+1}$, we have

$$\begin{aligned}\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N)) &\leq \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\\ &\leq M_1 R \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\phi(\tau_N).\end{aligned}$$

There exists a $q > 1$ such that $1/p + 1/q = 1$. By the first part of Theorem 3.2 and for $\aleph := \aleph(\varrho(\tau_N),\tau_N)$, we get

$$\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N)) \leq M_1 R\|\aleph\|_q\|\varphi\|_p \leq R.$$

It follows in similar manner for $\tau_{N-2} \in N^n_{n_0+1}$,

$$\begin{aligned}\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1})g_{N-1}&\left[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\right]\\ &\leq \sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\varrho(\tau_{N-1}),\tau_{N-1})\phi(\tau_{N-1})g_{N-1}(R)\\ &\leq M_1 R \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_{N-1}),\tau_{N-1})\phi(\tau_{N-1})\\ &\leq M_1 R\|\aleph\|_q\|\varphi\|_p \leq R.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\mathscr{F}z_1)(t) = \sum_{\tau_1=n_0+1}^{n} \aleph(t,\tau_1)\phi(\tau_1)g_1&\left[\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1,\tau_2)\phi(\tau_2)g_2\left[\sum_{\tau_3=n_0+1}^{n} \aleph(\tau_2,\tau_3)\right.\right.\\ &\vdots\\ \times g_{N-2}&\left[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1})\right.\\ \times g_{N-1}&\left.\left.\left[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\right]\right]\right] \leq R, \quad t \in \mathbb{N}^n_{n_0}.\end{aligned}$$

Since $R = \|z_1\|$ for $z_1 \in \mathcal{D} \cap \partial E_1$, we get

$$\|\mathscr{F}z_1\| \leq \|z_1\|. \tag{3.1}$$

Let $\tau \in \mathbb{N}_{n_0+1}^{n}$. Then, $r = \|z_1\| \geq z_1(\tau) \geq \min_{\tau \in \mathbb{N}_{n_0+1}^{n-1}} z_1(t) \geq \eta \|z_1\| \geq \theta r$. By $(\mathcal{J}_2)$ and for $\tau_{N-1} \in \mathbb{N}_{n_0+1}^{n}$, we have

$$\begin{aligned}\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N)) &\geq \eta \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\\ &\geq \eta\theta r \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\phi(\tau_N)\\ &\geq \eta\theta r\widehat{\varphi} \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N) \geq r.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$\begin{aligned}(\mathscr{F} z_1)(t) = &\sum_{\tau_1=n_0+1}^{n} \aleph(t,\tau_1)\phi(\tau_1)g_1\Bigg[\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1,\tau_2)\phi(\tau_2)g_2\Bigg[\sum_{\tau_3=n_0+1}^{n} \aleph(\tau_2,\tau_3)\\ &\vdots\\ &\times g_{N-2}\Bigg[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1})\\ &\times g_{N-1}\Bigg[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N)g_N(z_1(\tau_N))\Bigg]\Bigg] \geq r, \quad t \in \mathbb{N}_{n_0}^{n}.\end{aligned}$$

Thus, if $z_1 \in \mathcal{D} \cap \partial E_2$, then

$$\|\mathscr{F} z_1\| \geq \|z_1\|. \tag{3.2}$$

It is evident that $0 \in E_2 \subset \overline{E}_2 \subset E_1$. From (3.1), (3.2), it follows from Theorem 3.1 that the operator $\mathscr{F}$ has a fixed point $z_1 \in \mathcal{D} \cap (\overline{E}_1 \backslash E_2)$ such that $z_1^{(\tau)} \geq 0$ on $N_{n_0+2}^{n}$. Next, setting $z_{N+1} = z_1$, we obtain a positive solution $(z_1, z_2, \ldots, z_\ell)$ of (1.1)–(1.2) given iteratively by

$$\begin{aligned}z_j(t) &= \sum_{\tau=n_0+1}^{n} \aleph(t,\tau)\phi(\tau)g_j(z_{j+1}(\tau)), \quad j = 2, 3, \ldots, N,\\ z_{N+1}(t) &= z_1(t), \quad t \in \mathbb{N}_{n_0}^{n}.\end{aligned}$$

The proof is completed. □

For $p = 1$, we have the following theorem.

Theorem 3.4. *Let φ be bounded below by a positive real number $\widehat{\varphi}$ and there exist positive numbers R, r such that $\eta r < r < \theta r < R$, where*

$$\theta > \left[\eta\widehat{\varphi} \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\right]^{-1}$$

and g_j, $j = 1, 2, \ldots, N$, satisfies $(\mathcal{J}_2)$ and

($\mathcal{J}_3$) *$g_j(z) \leq M_2 R$ for $0 \leq z \leq R$, where*

$$M_2 < \frac{1}{\|\aleph\|_\infty \|\varphi\|_1}.$$

Then, the problem (1.1)–(1.2) *has at least one positive solution.*

Proof. Consider the open subsets E_1 and E_2 of $\mathcal{B}$ defined by

$$E_1 = \{z \in \mathcal{B} : \|z\| < R\}, \ E_2 = \{z \in \mathcal{B} : \|z\| < r\}.$$

Let $z_1 \in \mathcal{D} \cap \partial E_1$. Then, $z_1(\tau) \leq R = \|z_1\|$ for all $\tau \in N^n_{n_0+1}$. By ($\mathcal{J}_3$) and $\tau_{N-1} \in N^n_{n_0+1}$, we have

$$\begin{aligned} \sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1}, \tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) &\leq \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N), \tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) \\ &\leq M_2 R \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N), \tau_N)\phi(\tau_N) \\ &\leq M_2 R \|\aleph\|_\infty \|\varphi\|_1 \leq R. \end{aligned}$$

It follows in similar manner for $\tau_{N-2} \in N^n_{n_0+1}$,

$$\begin{aligned} &\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2}, \tau_{N-1})\phi(\tau_{N-1}) g_{N-1}\left[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1}, \tau_N)\phi(\tau_N) g_N(z_1(\tau_N))\right] \\ &\leq \sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\varrho(\tau_{N-1}), \tau_{N-1})\phi(\tau_{N-1}) g_{N-1}(R) \\ &\leq M_2 R \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_{N-1}), \tau_{N-1})\phi(\tau_{N-1}) \leq M_2 R \|\aleph\|_\infty \|\varphi\|_1 \leq R. \end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned} (\mathscr{F} z_1)(t) = \sum_{\tau_1=n_0+1}^{n} \aleph(t, \tau_1)\phi(\tau_1) g_1\Bigg[&\sum_{\tau_2=n_0+1}^{n} \aleph(\tau_1, \tau_2)\phi(\tau_2) g_2\Bigg[\sum_{\tau_3=n_0+1}^{n} \aleph(\tau_2, \tau_3) \\ &\vdots \\ &\times g_{N-2}\Bigg[\sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2}, \tau_{N-1})\phi(\tau_{N-1}) \\ &\times g_{N-1}\Bigg[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1}, \tau_N)\phi(\tau_N) g_N(z_1(\tau_N))\Bigg]\Bigg] \leq R, \quad t \in \mathbb{N}^n_{n_0}. \end{aligned}$$

Since $R = \|z_1\|$ for $z_1 \in \mathcal{D} \cap \partial E_1$, we get

$$\|\mathscr{F} z_1\| \leq \|z_1\|.$$

The rest of the proof is similar to the proof of Theorem 3.3. This completes the proof. □

Theorem 3.5. *Let φ be bounded below by a positive real number $\widehat{\varphi}$ and there exist positive real numbers R, r such that $\eta r < r < \theta r < R$, where*

$$\theta > \left[\eta\widehat{\varphi} \sum_{\tau_N=n_0+1}^{n} \aleph(\varrho(\tau_N),\tau_N)\right]^{-1}$$

and g_j, $j = 1, 2, \ldots, N$, satisfies $(\mathcal{J}_2)$ and

$(\mathcal{J}_4)$ $g_j(z) \le M_3 R$ *for* $0 \le z \le R$, *where*

$$M_3 < \frac{1}{\|\aleph\|_1\|\varphi\|_\infty}.$$

Then, the problem (1.1)–(1.2) *has at least one positive solution.*

Proof. Proof of the present theorem is similar to the proofs of Therems 3.3 and 3.4. So, we omit the deltails here. □

4. Uniqueness and stability analysis

In this section, we derive sufficient conditions for the existence of a unique solution of the boundary value problem (1.1)–(1.2), where we employ two metrics under Rus's theorem (see [1,37] for more details). In this regard, let $z, \vartheta \in \mathcal{B}$ and consider the following two metrics on $\mathcal{B}$:

$$d_1(z,\vartheta) = \max_{t\in\mathbb{N}_{n_0+1}^{n}} |z(t) - \vartheta(t)|, \tag{4.1}$$

$$d_2(z,\vartheta) = \left[\sum_{t=n_0+1}^{n} |z(t) - \vartheta(t)|^p\right]^{\frac{1}{p}}, \ p > 1. \tag{4.2}$$

For d_1 in (4.1), the pair $(\mathcal{B}, d_1)$ forms a complete metric space. For d_2 in (4.2), the pair $(\mathcal{B}, d_2)$ forms a metric space. The relationship between the two metrics on $\mathcal{B}$ is given by

$$d_2(z,\vartheta) \le (n - n_0)^{1/p} d_1(z,\vartheta) \text{ for all } z,\vartheta \in \mathcal{B}. \tag{4.3}$$

Theorem 4.1. (Rus [35]) *Let $\mathcal{B}$ be a nonempty set and let d_1 and d_2 be two metrics on $\mathcal{B}$ such that $(\mathcal{B}, d_1)$ forms a complete metric space. If the mapping $\mathscr{F} : \mathcal{B} \to \mathcal{B}$ is continuous with respect to d_1 on $\mathcal{B}$ and*

$$d_1(\mathscr{F}z, \mathscr{F}\vartheta) \le \alpha d_2(z,\vartheta), \tag{4.4}$$

for some $\alpha > 0$ and for all $z, \vartheta \in \mathcal{B}$,

$$d_2(\mathscr{F}z, \mathscr{F}\vartheta) \le \beta d_2(z,\vartheta),$$

for some $0 < \beta < 1$ for all $z, \vartheta \in \mathcal{B}$, then there is a unique $z^* \in \mathcal{B}$ such that $\mathscr{F}z^* = z^*$.

Theorem 4.2. *Let φ be bounded above by φ^* and there is some $K > 0$ such that*

$$|g_j(z) - g_j(\vartheta)| \le K|z - \vartheta| \quad \textit{for} \quad z,\vartheta \in \mathcal{B}.$$

Also, assume that there are constants $p > 1$ and $q > 1$ such that $1/p + 1/p = 1$ with

$$(n - n_0)^{1/p}(\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q < 1, \tag{4.5}$$

then the boundary value problem (1.1)–(1.2) *has a unique solution.*

Proof. Let $z_1, \vartheta_1 \in \mathcal{B}$ be any two solutions of (1.1)–(1.2) and $\tau_{N-1} \in N_{n_0+1}^{n}$. Then, by Hölder's inequality 3.2, we have

$$\begin{aligned}
&\Big| \sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) - \sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(\vartheta_1(\tau_N)) \Big| \\
&\qquad \le \sum_{\tau_N=n_0+1}^{n} |\aleph(\tau_{N-1},\tau_N)|\,|\phi(\tau_N)|\,|g_N(z_1(\tau_N)) - g_N(\vartheta_1(\tau_N))| \\
&\qquad \le \varphi^* K \sum_{\tau_N=n_0+1}^{n} |\aleph(\varrho(\tau_N),\tau_N)|\,|z_1(\tau_N) - \vartheta_1(\tau_N)| \\
&\qquad \le \varphi^* K \left[\sum_{\tau_N=n_0+1}^{n} |\aleph(\varrho(\tau_N),\tau_N)|^q \right]^{\frac{1}{q}} \left[\sum_{\tau_N=n_0+2}^{n} |z_1(\tau_N) - \vartheta_1(\tau_N)|^p \right]^{\frac{1}{p}} \\
&\qquad \le \varphi^* K \|\aleph\|_q d_2(z_1,\vartheta_1).
\end{aligned}$$

Similarly, for $\tau_{N-2} \in N_{n_0+1}^{n}$, we have

$$\begin{aligned}
&\Bigg| \sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1}) g_{N-1}\left[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) \right] \\
&- \sum_{\tau_{N-1}=n_0+1}^{n} \aleph(\tau_{N-2},\tau_{N-1})\phi(\tau_{N-1}) g_{N-1}\left[\sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(\vartheta_1(\tau_N)) \right] \Bigg| \\
&\le \sum_{\tau_{N-1}=n_0+1}^{n} |\aleph(\tau_{N-2},\tau_{N-1})||\phi(\tau_{N-1})| \\
&\times \Big| \sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(z_1(\tau_N)) - \sum_{\tau_N=n_0+1}^{n} \aleph(\tau_{N-1},\tau_N)\phi(\tau_N) g_N(\vartheta_1(\tau_N)) \Big| \\
&\le \varphi^* \sum_{\tau_{N-1}=n_0+1}^{n} |\aleph(\tau_{N-2},\tau_{N-1})|[\varphi^* K \|\aleph\|_q d_2(z_1,\vartheta_1)] \le (\varphi^*)^2 \varkappa K \|\aleph\|_q d_2(z_1,\vartheta_1).
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$|(\mathscr{F} z_1)(t) - (\mathscr{F}\vartheta_1)(t)| \le (\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q d_2(z_1,\vartheta_1). \tag{4.6}$$

Thus, defining

$$\alpha = (\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q,$$

we see that

$$d(\mathscr{F} z_1, \mathscr{F}\vartheta_1) \le \alpha d_2(z_1,\vartheta_1), \tag{4.7}$$

for some $\alpha > 0$ for all $z_1, \vartheta_1 \in \mathcal{B}$, and so the inequality (4.4) of Theorem 4.1 holds. Now, for all $z_1, \vartheta_1 \in \mathcal{B}$, we may apply (4.3) to (4.7) to obtain

$$d_1(\mathscr{F} z_1, \mathscr{F}\vartheta_1) \le \alpha d_2(z_1,\vartheta_1) \le \alpha(n-n_0)^{1/p} d_1(z_1,\vartheta_1).$$

Thus, given any ß > 0, we can choose $\delta = ß/[\alpha(n-n_0)^{1/p}]$ so that $d_1(\mathscr{F} z_1, \mathscr{F}\vartheta_1) <$ ß whenever $d(z_1,\vartheta_1) < \delta$. Hence, $\mathscr{F}$ is continuous on $\mathcal{B}$ with respect to the metric

d_1. Finally, we show that $\mathscr{F}$ is contractive on $\mathcal{B}$ with respect to the metric d_2. From (4.6), for each $z_1, \vartheta_1 \in \mathcal{B}$, consider

$$\begin{aligned}&\left[\sum_{t=n_0+1}^{n} |(\mathscr{F}z_1)(t) - (\mathscr{F}\vartheta_2)(t)|^p\right]^{\frac{1}{p}}\\ &\leq \left[\sum_{t=n_0+1}^{n} \left|(\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q d_2(z_1,\vartheta_1)\right|^p\right]^{\frac{1}{p}}\\ &\leq (n-n_0)^{1/p} (\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q d_2(z_1,\vartheta_1).\end{aligned}$$

That is,

$$d_2(\mathscr{F}z_1, \mathscr{F}\vartheta_1) \leq (n-n_0)^{1/p} (\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q d_2(z_1,\vartheta_1).$$

From the assumption (4.5), we have

$$d_2(\mathscr{F}z_1, \mathscr{F}\vartheta_1) \leq \beta d_2(z_1,\vartheta_1)$$

for some $\beta < 1$ and all $z_1, \vartheta_1 \in \mathcal{B}$. Thus, by Theorem 4.1, the operator $\mathscr{F}$ has a unique fixed point in $\mathcal{B}$. Therefore, the boundary value problem (1.1)–(1.2) has a unique solution. □

5. Examples

In this section, we present two examples to check the validity of our main results.

Example 5.1. Consider the following boundary value problem:

$$\begin{aligned}\nabla_1^{1/2}\left[\nabla z_j(t)\right] + \varphi(t) g_j(z_{j+1}(t)) = 0, \quad t \in \mathbb{N}_3^{100},\ j=1,2,\\ z_3(t) = z_1(t), \quad t \in \mathbb{N}_3^{100},\end{aligned} \tag{5.1}$$

satisfying two-point general boundary conditions

$$\begin{aligned}z_j(1) - 20\nabla z_j(1) = 0, \quad j=1,2,\\ \frac{1}{900} z_j(100) + 4\nabla z_j(100) = 0, \quad j=1,2,\end{aligned} \tag{5.2}$$

where $\varphi(t) = \frac{\pi}{12(t+1)^4}$ and

$$g_j(z) = \begin{cases} 0.2\times 10^{10}, & z \in (10^4, +\infty),\\ \frac{6.3\times 10^8 - 0.2\times 10^{10}}{10^2 - 10^4}(z - 10^4) + 0.2\times 10^{10}, & z \in [10^2, 10^4],\\ 6.3\times 10^8, & z \in (0.0132\times 10^2, 10^2),\\ \frac{6.3\times 10^8 - 0.2\times 10^{10}}{0.0132\times 10^2 - 0.012\times 10^2}(z - 0.012\times 10^2) + 0.2\times 10^{10}, & \\ & z \in (0.012\times 10^2, 0.0132\times 10^2],\\ 0.2\times 10^{10}, & z \in [0, 0.012\times 10^2],\end{cases}$$

Here, $n_0 = 0$, $n = 100$, $a = 1$, $b = 20$, $c = \frac{1}{900}$, $d = 4$. Let $p = q = 2$, $M_1 = 0.3$, $\theta = 6.2\times 10^6$. By using Maple, we calculated $\lambda = 0.2601595441$, $\widehat{\aleph} = 17.41443310$, $\aleph^* = 466.8589700$, $\widehat{\varphi} = 2.515840659\times 10^{-9}$,

$$\sum_{\tau=2}^{100} \aleph(\varrho(\tau),\tau) = 4879.295601, \quad \eta = \frac{\widehat{\aleph} - \mathcal{H}_{\text{ß}-1}(n, n_0)}{\aleph^*} = 0.01316187049,$$

$$\|\aleph\|_2 = \left[\sum_{\tau=2}^{100} |\aleph(\varrho(\tau),\tau)|^2\right]^{1/2} = 723.2219121,$$
$$\|\varphi\|_2 = \left[\sum_{\tau=2}^{100} |\varphi(\tau)|^2\right]^{1/2} = 0.003424534117.$$

Then,

$$M_1 < \frac{1}{\|\aleph\|_2\|\varphi\|_2} = 0.4037633796,$$
$$\theta > \left[\eta\widehat{\varphi}\sum_{\tau=2}^{100} \aleph(\varrho(\tau),\tau)\right]^{-1} = 6.189308432 \times 10^6.$$

Taking $R = 10^{10}$, $r = 10^2$,

$$\eta r = 0.01316187 \times 10^2 < 10^2 = r < \theta r = 6.2 \times 10^8 < 10^{10} = R,$$

and g_j satisfies the following growth conditions:

$$g_j(z) \le M_1 R = 0.3 \times 10^{10}, \quad z \in [0, 10^{10}]$$
$$g_j(z) \ge \theta r = 6.2 \times 10^8, \qquad z \in [0.0132 \times 10^2, 10^2].$$

All the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (5.1)–(5.2) has at least one positive solution.

Example 5.2. Consider the following boundary value problem:

$$\begin{aligned} \nabla_1^{1/2}\left[\nabla z_j(t)\right] + \varphi(t) g_j(z_{j+1}(t)) = 0, \quad t \in \mathbb{N}_3^{100}, \ j = 1, 2, \\ z_3(t) = z_1(t), \quad t \in \mathbb{N}_3^{100}, \end{aligned} \tag{5.3}$$

satisfying two-point general boundary conditions

$$\begin{aligned} z_j(1) - 20\nabla z_j(1) = 0, \quad j = 1, 2, \\ \frac{1}{900} z_j(100) + 4\nabla z_j(100) = 0, \quad j = 1, 2, \end{aligned} \tag{5.4}$$

where $n_0 = 0$, $n = 100$, $a = 1$, $b = 20$, $c = \frac{1}{900}$, $d = 4$, $\varphi(t) = \frac{\pi}{6}(t - 94)^2$, $g_j(z) = 10^{-11} \sin z$ for $j = 1, 2$. Then, it is clear that $N = 2$, $\varphi(t) \le 6\pi = \varphi^*$ and

$$|g_j(z) - g_j(\vartheta)| \le 10^{-11}|z - \vartheta| = K|z - \vartheta|$$

for $z, \vartheta \in \mathcal{B}$. Letting $p = q = \frac{1}{2}$. Then,

$$\|\aleph\|_2 = \left[\sum_{\tau=2}^{100} |\aleph(\varrho(\tau),\tau)|^2\right]^{1/2} = 723.2219121$$

and

$$\begin{aligned} \varkappa = {} & \frac{1}{\lambda}\Big[ac\mathcal{H}_{\text{ß}-1}(n,n_0)\mathcal{H}_{\text{ß}}(n,n_0) + ad\left[\mathcal{H}_{\text{ß}-1}(n,n_0)\right]^2 \\ & \qquad + (b-a)c\mathcal{H}_{\text{ß}}(n,n_0) + (b-a)d\mathcal{H}_{\text{ß}-1}(n,n_0)\Big] \\ = {} & 5342.553597. \end{aligned}$$

From the above values, we get

$$(n-n_0)^{1/p}(\varphi^*)^N \varkappa^{N-1} K^{N-1} \|\aleph\|_q = 0.1365967313 < 1.$$

Thus, all the conditions of Theorem 4.2 are satisfied. Therefore, by Theorem 4.2, the problem (5.3)–(5.4) has a unique solution.

6. Conclusion

In this paper, we developed a theory to study the existence and uniqueness of positive solutions of a certain type of a nabla fractional order difference equation by applying Krasnoselskii's cone fixed point theorem in a Banach space and Rus's fixed point theorem in a complete metric space, respectively. To tackle the growth conditions $(\mathcal{J}_i)$, $i = 1, 2, 3, 4$, we chose a particular form of a nonlinear term g_j in Example 5.1 and in Example 5.2 for simplicity. We feel that there are many other nice forms for g_j satisfying growth conditions. In the future, we will study the existence of multiple positive solutions of (1.1)–(1.2) and also study the problem with both Δ operators and mixed Δ and ∇ operators.

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Mahammad Khuddush, Department of Mathematics, Dr. Lankapalli Bullayya College, Visakhapatnam, 530013, India
e-mail: khuddush89@gmail.com

K. Rajendra Prasad, Department of Applied Mathematics, College of Science and Technology, Andhra University, Visakhapatnam, 530003, India
e-mail: rajendra92@rediffmail.com