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THIRD ORDER DIFFERENTIAL SUBORDINATION ASSOCIATED WITH JANOWSKI FUNCTIONS

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Abstract. Using the admissibility condition, we obtain certain third order differential subordination results for an analytic function p with $p(0) = 1$ belonging to the class of Janowski functions, $\mathcal{P}[A, B]$. As an application, certain second order differential inequalities involving special functions are obtained.

1. Introduction and preliminaries

Let $\mathcal{H}[a, n]$ denote the class of analytic functions defined in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$, where n is a positive integer and $a \in \mathbb{C}$. We denote $\mathcal{H}_1 := \mathcal{H}[1, 1]$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}[0, 1]$ consisting of all analytic functions f normalized by the condition $f(0) = f'(0) - 1 = 0$. Let $\mathcal{S}$ be a subclass of $\mathcal{A}$ containing univalent functions. Let the functions f and g be analytic in $\mathbb{D}$, then we say that f is subordinate to g in $\mathbb{D}$ (written $f \prec g$) if there exists a function $w(z)$ analytic in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. In particular, if the function g is univalent in $\mathbb{D}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Let us denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathbb{D}} \backslash E(q)$, where $E(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\}$, such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D}\backslash E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$. The class $\mathcal{P}$ consists of Carathéodory functions $\mathcal{P} : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ that map the unit disc $\mathbb{D}$ into the region on the right half of the plane. For $-1 \leq B < A \leq 1$, $\mathcal{P}[A, B]$ denotes the class of analytic functions $p \in \mathcal{P}$, such that

$$p(z) \prec \frac{1 + Az}{1 + Bz}$$

and the functions in $\mathcal{P}[A, B]$ are called Janowski functions introduced in [7]. Special functions play an important role in Geometric Function Theory and its related fields. The confluent (Kummer) hypergeometric function $\Phi(a, c; z)$ is given by

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a + n)}{\Gamma(c + n)} \frac{z^n}{n!},$$

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where $a, c \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$ and $(\lambda)_n$ denotes the Pochhammer symbol given by $(\lambda)_0 = 1$, $(\lambda)_n = \lambda(\lambda+1)_{n-1}$. The function $\Phi \in \mathcal{H}[1,1]$ is a solution of the differential equation

$$z\Phi''(a,c;z) + (c-z)\Phi'(a,c;z) - a\Phi(a,c;z) = 0.$$

Note that $\Phi(a,c;z)$ satisfies

$$c\Phi'(a;c;z) = a\Phi(a+1;c+1;z) \quad \text{and} \quad \Phi(a;a;z) = e^z. \tag{1.1}$$

Next, we consider the generalized Bessel function

$$u_p(z) = u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(\kappa)_n} \frac{z^n}{n!}, \tag{1.2}$$

where $p, b, c \in \mathbb{C}$ and $\kappa = p + (b+1)/2 \neq 0, -1, -2, \dots$. We note that the function $u_p(z)$ is analytic in $\mathbb{D}$ and satisfies the differential equation

$$4z^2 u_p''(z) + 4\kappa z u_p'(z) + czu_p(z) = 0. \tag{1.3}$$

The normalized Lommel function can be expressed as

$$h_{\mu,\nu}(z) = z + \sum_{n\geq 1} \frac{(-1/4)^n}{((\mu-\nu+3)/2)_n((\mu+\nu+3)/2)_n} z^{n+1},$$

which satisfies the following second order differential equation

$$z^2 h''_{\mu,\nu}(z) + \mu z h'_{\mu,\nu}(z) + \frac{1}{4}\big((\mu-1)^2 - \nu^2 + z\big)h_{\mu,\nu}(z) = \frac{1}{4}((\mu+1)^2 - \nu^2)z. \tag{1.4}$$

We denote the normalized form of the generalized struve function by

$$u_{\nu,b,c}(z) = \sum_{n\geq 0} \frac{(-c/4)^n}{(3/2)_n(\kappa)_n} z^n,$$

where $\kappa = \nu + (b+2)/2 \neq 0, -1, -2, \dots$ and denote $u_{\nu,b,c}$ simply by u_ν. We note that the function u_ν satisfies

$$4z^2 u_\nu''(z) + 2(2\kappa+1)zu_\nu'(z) + (cz + 2(\kappa-1))u_\nu(z) - 2(\kappa-1) = 0.$$

There is an extensive literature in geometric function theory that deals with analytic and geometric properties of the above special functions. For more details, see [11,13,16–18]. Relations between the generalized Bessel function and Janowski class were obtained in [8]. In [1], the authors obtained various sufficient conditions on the parameters of the confluent hypergeometric function and established a certain second order subordination relation with the Janowski function.

The theory of differential subordination provides techniques to reduce differential subordination problems into verifying a simple algebraic condition called admissible condition. Antonino and Miller [2] obtained some general results of third order differential inequalities and subordination for a large class. It should be remarked in passing that only a few articles [2,14] deal with a very narrow class of third order differential subordination.

We exploit the third order differential subordination theory to get several sufficient conditions for functions satisfying several subordinations to be Janowski functions. As an application, we obtain various sufficient conditions on parameters involved in a certain special function and establish their subordination relation with the Janowski function.

We will now recall some definitions and a Lemma due to Antonino-Miller [2], which are required in our investigations.

Definition 1.1. [2, p. 440, Def. 1] Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ and satisfies the following third-order differential subordination:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.5}$$

then $p(z)$ is called a solution of the differential subordination (1.5). Furthermore, a given univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.5) or, more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant.

Definition 1.2. [2, p. 449, Def. 3] Let Ω be a set in $\mathbb{C}$. Also, let $q \in \mathcal{Q}$ and $n \geq 2$. The class $\Psi[\Omega, q]$ consists of functions $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$, which satisfy the following admissibilty conditions:

$$\psi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(\zeta),\ \ s = k\zeta q'(\zeta),\ \ \mathrm{Re}\left\{\frac{t}{s} + 1\right\} \geq k\,\mathrm{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\}$$

and

$$\mathrm{Re}\left\{\frac{u}{s}\right\} \geq \mathrm{Re}\left\{\frac{\zeta^2 q'''(\zeta)}{q'(\zeta)}\right\},$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$ and $k \geq n$.

Lemma 1.3. [2, p.449, Theorem 1] *Let $p \in \mathcal{H}[a, n]$ with $n \geq 2$ and $q \in \mathcal{Q}(a)$ satisfying the following conditions:*

$$\mathrm{Re}\left\{\frac{\zeta q''(z)}{q'(z)}\right\} \geq 0 \qquad \text{and} \qquad \left|\frac{zp'(z)}{q'(z)}\right| \leq k,$$

where $z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\backslash E(q)$ and $k \geq n$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \subset \Omega,$$

then $p(z) \prec q(z)$.

2. Third order subordination associated with Janowski function

We now describe the class of admissible function $\Psi_n[\Omega, q]$, where $q : \mathbb{D} \to \mathbb{C}$ is given by

$$q(z) = \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < 0 < A \leq 1.$$

We note that q is univalent in $\overline{\mathbb{D}}$ with $q(\mathbb{D}) = \Delta$ and $q(0) = 1$, where

$$\Delta = \left\{w \in \mathbb{C} : \left|\frac{w - 1}{A - Bw}\right| < 1\right\}.$$

For $\zeta = e^{i\theta}$ and $0 < \theta < 2\pi$ such that $(\cos\theta + B) \geq 0$, we have

$$q(\zeta) = \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}, \quad q'(\zeta) = \frac{A-B}{(1+Be^{i\theta})^2}, \quad q''(\zeta) = \frac{-2B(A-B)}{(1+Be^{i\theta})^3},$$

and

$$q'''(\zeta) = \frac{6B^2(A-B)}{(1+Be^{i\theta})^4},$$

and a simple calculation yields

$$\operatorname{Re}\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right\} = \frac{(1-B^2)m}{1+B^2+2B\cos\theta},$$

$$\operatorname{Re}\left\{\frac{\zeta q'''(\zeta)}{q'(\zeta)}\right\} = \frac{6m^2B^2(B^2+2B\cos\theta+\cos 2\theta)}{(1+B^2+2B\cos\theta)^2}.$$

Thus, we get the following admissibility condition

$$\psi(r,s,t,u:z) \notin \Omega$$

whenever

$$\left.\begin{aligned} &r = \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}, \qquad s = \frac{m(A-B)e^{i\theta}}{(1+Be^{i\theta})^2},\\ &\operatorname{Re}\left\{\frac{t}{s}+1\right\} \geq \frac{1-B^2}{1+B^2-2B\cos\theta},\\ &\operatorname{Re}\left\{\frac{u}{s}\right\} \geq \frac{6m^2B^2(B^2+2B\cos\theta+\cos 2\theta)}{(1+B^2+2B\cos\theta)^2}, \end{aligned}\right\} \tag{2.1}$$

where $0 < \theta < 2\pi$, such that $B + \cos\theta \geq 0$ and $m \geq n \geq 1$.

When

$$q(z) = \left(\frac{1+Az}{1+Bz}\right),$$

the following Theorem is a special case of Lemma 1.3:

Theorem 2.1. *Let Ω be a subset of $\mathbb{C}$ and $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ satisfz the admissibility condition $\psi(r,s,t,u;z) \notin \Omega$ for all $z \in \mathbb{D}$, where r,s,t,u are given by* (2.1)*. If $p \in \mathcal{H}[1,n]$, with $n \in \mathbb{N}$, satisfies*

$$|zp'(z)||1+B\zeta|^2 \leq n|A-B| \quad (-1 \leq B < 0 < A \leq 1,\ z \in \mathbb{D},\ \zeta \in \partial\mathbb{D}\backslash E(q))$$

and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega$, then $p \in \mathcal{P}[A,B]$.

Now, using Theorem 2.1, we obtain the following third order differential subordination result for the Janowski function.

Theorem 2.2. *Let $-1 \leq B < 0 < A \leq 1$ and C, D be non negative real numbers. Suppose that the functions $E, F, G : \mathbb{D} \to \mathbb{C}$ satisfy the conditions*

(i) $\operatorname{Re}\{E(z)\} \geq D$,
(ii) $6CB^2(A-B)(1-|B|)^2 + (2B)D(A-B)(1-B)^3 + (A-B)(1-B)^4 \operatorname{Re}(E(z)) - (|A+B|+2)(1-B)^4|F(z)| \geq 2(1-B)^4(1+|B|)|G(z)|$.

If $p \in \mathcal{H}[1,n]$ *satisfy* $|zp'(z)||(1+B\zeta)|^2 \leq n|A-B|$, *where* $\zeta \in \partial\mathbb{D}\backslash\{E(q) \subset \{-1\}\}$ *and*

$$Cz^3p'''(z) + Dz^2p''(z) + E(z)zp'(z) + F(z)p(z) + G(z) = 0, \tag{2.2}$$

then $p \in \mathcal{P}[A,B]$.

Proof. Let $\Omega = \{0\}$ and define a function $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ by $\psi(r,s,t,u;z) = Cu + Dt + E(z)s + F(z)r + G(z)$. Now, by (2.2), we observe that

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

for all $z \in \mathbb{D}$. Consider

$$\begin{aligned}
&|\psi(r,s,t,u;z)| = |Cu + Dt + E(z)s + F(z)r + G(z)| \\
&= |s|\left|C\frac{u}{s} + D\left(\frac{t}{s}+1\right) + E(z) + F(z)\frac{r}{s} + \frac{G(z)}{s} - D\right| \\
&\geq |s|\left\{C\operatorname{Re}\left(\frac{u}{s}\right) + D\operatorname{Re}\left(\frac{t}{s}+1\right) + \operatorname{Re}(E(z))\right. \\
&\quad \left. + \operatorname{Re}\left(F(z)\frac{r}{s}\right) + \operatorname{Re}\left(\frac{G(z)}{s}\right) - D\right\} \\
&\geq \left(\frac{m(A-B)\sqrt{1+B^4+4B^2+4B\cos\theta(1+B^2)+2B^2\cos 2\theta}}{(1+2B\cos\theta+B^2)^2}\right) \\
&\quad \times \left(6Cm^2B^2\frac{(B^2+2B\cos\theta+\cos 2\theta)}{(1+B^2+2B\cos\theta)^2} + D\left(\frac{(1-B^2)m}{1+B^2+2B\cos\theta}\right) + \operatorname{Re}(E(z))\right. \\
&\quad + \frac{[(A+B)+(AB+1)\cos\theta]\operatorname{Re}(F(z))}{m(A-B)} + \frac{(AB-1)\sin\theta\operatorname{Im}(F(z))}{m(A-B)} \\
&\quad \left. + \frac{[(1+B^2)\cos\theta+2B]\operatorname{Re}(G(z))}{m(A-B)} - \frac{(B^2-1)\sin\theta\operatorname{Im}(G(z))}{m(A-B)} - D\right) \\
&> (A-B)\frac{(1-|B|)^2}{(1-B)^4} \\
&\quad \times \left(6CB^2\frac{(1-|B|)^2}{(1-B)^4} + \frac{D(1+B)}{(1-B)} + \operatorname{Re}(E(z)) - \left(\frac{|A+B|+AB+1}{(A-B)}\right)|F(z)|\right. \\
&\quad \left. + \frac{(AB-1)}{(A-B)}|F(z)| - \frac{(1+B^2+2|B|)}{(A-B)}|G(z)| - \frac{(B^2-1)}{(A-B)}|G(z)| - D\right) \\
&= \frac{(1-|B|)^2}{(1-B)^8}\left(6CB^2(1-|B|)^2(A-B) + D(1+B)(1-B)^3(A-B)\right. \\
&\quad + (A-B)(1-B)^4\operatorname{Re}(E(z)) \\
&\quad \left. - (|A+B|+2)|F(z)| - 2(1+|B|)(1-B)^4|G(z)|\right) \geq 0,
\end{aligned}$$

in view of hypothesis (ii) of Theorem 2.2. This proves that $|\psi(r,s,t,u;z)| \neq 0$ and hence $\psi(r,s,t,u;z) \notin \Omega$. Therefore, by Theorem 2.1,

$$p(z) \prec \left(\frac{1+Az}{1+Bz}\right),$$

for all $z \in \mathbb{D}$. □

Now, we deduce the following result from Theorem 2.2.

Corollary 2.3. *Let* $-1 \leq B < 0 < A \leq 1$ *and* D *be a non negative real number. Suppose that the functions* $E, F, G : \mathbb{D} \to \mathbb{C}$ *satisfy the conditions* $\operatorname{Re}(E(z)) \geq D$ *and*

$$\begin{aligned} 2B(A-B)D + (A-B)(1-B)\operatorname{Re}(E(z)) \\ - (|A+B|+2)(1-B)\big|F(z)\big| > 2(1+|B|)(1-B)|G(z)|. \end{aligned} \tag{2.3}$$

If $p \in \mathcal{H}[1, n]$ *and*

$$Dz^2p''(z) + E(z)zp'(z) + F(z)p(z) + G(z) = 0,$$

then $p \in \mathcal{P}[A, B]$.

Proof. By taking $C = 0$ in Theorem 2.2, then we can define the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, by $\psi(r, s, t; z) = Dt + E(z)s + F(z)r + G(z)$. Let $\Omega := \{0\}$. Now, we observe that

$$\begin{aligned} &|\psi(r,s,t;z)| = |Dt + E(z)s + F(z)r + G(z)| \\ &\geq |s| \left\{ D\operatorname{Re}\left(\frac{t}{s}+1\right) + \operatorname{Re}(E(z)) + \operatorname{Re}\left(F(z)\frac{r}{s}\right) + \operatorname{Re}\left(\frac{G(z)}{s}\right) - D \right\} \\ &\geq \left(\frac{m(A-B)\sqrt{1+B^4+4B^2+4B\cos\theta(1+B^2)+2B^2\cos 2\theta}}{(1+2B\cos\theta+B^2)^2} \right) \\ &\quad \times \left\{ D\left(\frac{(1-B^2)m}{1+B^2+2B\cos\theta} \right) + \operatorname{Re}(E(z)) + \frac{[(A+B)+(AB+1)\cos\theta]}{m(A-B)} \right. \\ &\quad \times \operatorname{Re}(F(z)) + \frac{(AB-1)\sin\theta \operatorname{Im} F(z)}{m(A-B)} + \frac{[(1+B^2)\cos\theta+2B]\operatorname{Re}(G(z))}{m(A-B)} \\ &\quad \left. - \frac{(B^2-1)\sin\theta \operatorname{Im} G(z)}{m(A-B)} - D \right\} \\ &> \frac{(A-B)(1-|B|)^2}{(1-B)^4} \left\{ \frac{D(1-B^2)}{(1-B)^2} + \operatorname{Re}(E(z)) - \frac{(|A+B|+AB+1)}{(A-B)}|F(z)| \right. \\ &\quad \left. + \frac{(AB-1)}{(A-B)}|F(z)| - \frac{(1+B^2+2|B|)}{(A-B)}|G(z)| - \frac{(B^2-1)}{(A-B)}|G(z)| - D \right\} \\ &= \frac{(1-|B|)^2}{(1-B)^5} \left\{ D(2B)(A-B) + (A-B)(1-B)\operatorname{Re}(E(z)) \right. \\ &\quad \left. - (|A+B|+2)(1-B)|F(z)| - 2(1+|B|)(1-B)|G(z)| \right\} \geq 0, \end{aligned}$$

in view of (2.3). Therefore $|\psi(r, s, t; z)| > 0$. Now by making use of Theorem 2.1, we have

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

□

3. Applications to special functions

Now, using Theorem 2.2, we derive a sufficient condition for the confluent hypergeometric function $\Phi(a;c;z)$ to be in the Janowski class.

Corollary 3.1. *Let* $-1 \le B < 0 < A \le 1$*. If* $a, c \in \mathbb{C}$, $|a| > 1$, $c \neq 0, -1, -2, \dots$ *and*

$$\operatorname{Re}(c) \ge \frac{(|A+B|+2)|a+1|}{(A-B)} - \frac{2B}{(1-B)}, \tag{3.1}$$

then

$$\Phi(a;c;z) \prec \frac{1+Az}{1+Bz}.$$

Proof. Let $p(z) = \frac{c}{a}\Phi'(a;c;z)$, then p satisfies the second order differential equation

$$z^2p''(z) + (c - z + 1)zp'(z) - (a+1)zp(z) = 0. \tag{3.2}$$

By taking $C = 0, D = 1, E(z) = c - z + 1, F(z) = -(a+1)z, G(z) = 0$, in Theorem 2.2, the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ can be defined as $\psi(r, s, t; z) = t + (c - z + 1)s - (a+1)zr$. Suppose that $\Omega = \{0\}$, then in respect of (3.2) $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$.

Now, we consider

$$\begin{aligned}
&|\psi(r, s, t; z)| = |t + (c - z + 1)s - (a+1)zr| \\
&\ge |s| \left\{ \operatorname{Re}\left(\frac{t}{s} + 1\right) + \operatorname{Re}(c - z + 1) - \operatorname{Re}\left((a+1)z\frac{r}{s}\right) - 1 \right\} \\
&\ge \left(\frac{m(A-B)\sqrt{1 + B^4 + 4B^2 + 4B\cos\theta(1+B^2) + 2B^2\cos 2\theta}}{(1 + 2B\cos\theta + B^2)^2} \right) \\
&\quad \times \left(\frac{(1-B^2)m}{1 + B^2 + 2B\cos\theta} + \operatorname{Re}(c - z + 1) \right. \\
&\quad \left. - \frac{[(A+B) + (AB+1)\cos\theta]\operatorname{Re}\left((a+1)z\frac{r}{s}\right)}{m(A-B)} - \frac{(AB-1)\sin\theta\operatorname{Im}((a+1)z)}{m(A-B)} \right) \\
&\ge \frac{(1-|B|)^2}{(1-B)^6} \Big\{ 2B(A-B)(1-B) + (A-B)(1-B)^2\operatorname{Re}(c - z + 1) \\
&\quad - (|A+B|+2)(1-B)^2|(a+1)z| \Big\} \\
&> \frac{(1-|B|)^2}{(1-B)^5} \Big\{ 2B(A-B) + (A-B)(1-B)\operatorname{Re}(c) \\
&\quad - (|A+B|+2)(1-B)|a+1| \Big\} \ge 0,
\end{aligned}$$

in view of (3.1). Thus,

$$\frac{c}{a}\Phi'(a;c;z) \prec \frac{1+Az}{1+Bz}.$$

Now, using relation (1.1), we have

$$(a-1)\Phi(a;c;z) \quad = (c-1)\Phi'(a-1;c-1;z).$$

Since $|a|>1$, our assertion that

$$\Phi(a;c;z) \prec \frac{1+Az}{1+Bz}$$

follows. □

Sufficient conditions for the Bessel function of the first kind $u_p(z)$ given by (1.2) to belong to the class of Janowski functions are as follows:

Corollary 3.2. *Let* $-1 \le B < 0 < A \le 1$ *and if* $b, p, c \in \mathbb{R}$ *and*

$$\kappa = p + \left(\frac{b+1}{2}\right),$$

$\kappa \neq 0, -1, -2, \dots,$ *satisfy the conditions,* $\kappa \ge 1$ *and*

$$8B(A-B) + 4(A-B)(1-B)\kappa - (|A+B|+2)(1-B)|c| \ge 0, \tag{3.3}$$

then $u_p(z) \in \mathcal{P}[A,B]$.

Proof. Let $p(z) = u_p(z),\ C = 0,\ D = 4,\ E(z) = 4\kappa,\ F(z) = cz,\ G(z) = 0$ in Theorem 2.2, then $p(z)$ satisfies the differential equation (1.3). Now, by defining $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by $\psi(r,s,t;z) = 4t + 4\kappa s + czr$ and letting $\Omega := \{0\}$, then in view of (1.3), $\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega$. We now consider

$$\begin{aligned}
&|\psi(r,s,t;z)| = |4t + 4\kappa s + czr| \\
&= |s||4\frac{t}{s} + 4\kappa + cz\frac{r}{s}| \ge |s|\left\{4\operatorname{Re}\left(\frac{t}{s}+1\right) + \operatorname{Re}(4\kappa) + \operatorname{Re}\left(cz\frac{r}{s}\right) - 4\right\} \\
&> \frac{(A-B)(1-|B|)^2}{(1-B)^4} \\
&\quad \times \left\{\frac{4(1+B)}{(1-B)} + 4\kappa - \frac{|A+B| + (AB+1)}{(A-B)}|cz| + \frac{(AB-1)}{(A-B)}|cz| - 4\right\} \\
&\ge \frac{(1-|B|)^2}{(1-B)^6} \\
&\quad \times \left\{8B(1-B)(A-B) + 4(A-B)(1-B)^2\kappa - (|A+B|+2)(1-B)^2|cz|\right\} \\
&> \frac{(1-|B|^2)}{(1-B)^5}\left\{8B(A-B) + 4(A-B)(1-B)\kappa - (|A+B|+2)(1-B)|c|\right\} \ge 0
\end{aligned}$$

in respect to (3.3). Therefore, $|\psi(r,s,t;z)| > 0$ and now, by making use of Theorem 2.1, we conclude that

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

□

Remark 3.3. When $A = 1$ and $B = -1$, Corollary 3.2 reduces to [4, Theorem 2.2, p. 29].

Corollary 3.4. *Let* $-1 \leq B < 0 < A \leq 1$ *and* $\mu, \nu \in \mathbb{C}$ *such that* $\mu \pm \nu$ *are non negative odd integers with* $\mathrm{Re}(\mu) \geq 2$ *and satisfy the following condition*

$$\begin{aligned} &4(A-B)\Big((1-B)\,\mathrm{Re}(\mu)+2\Big) \\ &\geq (1-B)\Big((|A+B|+2)+(|A+B|-2B+4)\times|(\mu+1)^2-\nu^2|\Big), \end{aligned} \tag{3.4}$$

then

$$\frac{h_{\mu\nu}(z)}{z} \prec \frac{1+Az}{1+Bz}.$$

Proof. When

$$p(z)=\frac{h_{\mu\nu}(z)}{z}, \quad C=0, \quad D=1, \quad E(z)=(\mu+2),$$

$$F(z)=\frac{(\mu+1)^2-\nu^2+z}{4}, \quad G(z)=-\frac{(\mu+1)^2-\nu^2}{4}$$

in Theorem 2.2, and in view of (1.4), the function p satisfies the second-order differential equation

$$z^2p''(z)+(\mu+2)zp'(z)+\frac{1}{4}p(z)((\mu+1)^2-\nu^2+z)-\frac{1}{4}((\mu+1)^2-\nu^2)=0.$$

We now define the function $\psi(r,s,t;z):\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ as

$$\psi(r,s,t;z)=t+(\mu+2)s+\frac{1}{4}((\mu+1)^2-\nu^2+z)r-\frac{1}{4}((\mu+1)^2-\nu^2),$$

and suppose that $\Omega:=\{0\}$, then $\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega$, for all $z\in\mathbb{D}$. We note that

$$\begin{aligned} &|\psi(r,s,t;z)| = \left|t+(\mu+2)s+\frac{1}{4}((\mu+1)^2-\nu^2+z)r-\frac{1}{4}((\mu+1)^2-\nu^2)\right| \\ &\geq |s|\Bigg\{\mathrm{Re}\left(\frac{t}{s}+1\right)+\mathrm{Re}(\mu+2)+\mathrm{Re}\left(\frac{1}{4}((\mu+1)^2-\nu^2+z)\frac{r}{s}\right) \\ &\quad+\mathrm{Re}\left(\frac{-\frac{1}{4}((\mu+1)^2-\nu^2)}{s}\right)-1\Bigg\} \\ &\geq \frac{m(A-B)\sqrt{1+B^4+4B^2+4B\cos\theta(1+B^2)+2B^2\cos 2\theta}}{(1+2B\cos\theta+B^2)^2} \\ &\quad\Bigg\{\frac{(1-B^2)m}{1+B^2+2B\cos\theta}+\mathrm{Re}(\mu+2) \\ &\quad+\frac{[(A+B)+(AB+1)\cos\theta]\,\mathrm{Re}\big(\frac{1}{4}((\mu+1)^2-\nu^2+z)\big)}{m(A-B)} \\ &\quad+\frac{(AB-1)\sin\theta\,\mathrm{Im}\big(\frac{1}{4}((\mu+1)^2-\nu^2+z)\big)}{m(A-B)} \\ &\quad+\frac{[(1+B^2)\cos\theta+2B]\,\mathrm{Re}\big(-\frac{1}{4}((\mu+1)^2-\nu^2)\big)}{m(A-B)} \\ &\quad-\frac{(B^2-1)\sin\theta\,\mathrm{Im}\big(-\frac{1}{4}((\mu+1)^2-\nu^2)\big)}{m(A-B)}-1\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&> \frac{(A-B)(1-|B|)^2}{(1-B)^4}\bigg\{\frac{(1+B)}{(1-B)} + \operatorname{Re}(\mu+2)\\
&\quad - \frac{|A+B|+AB+1}{(A-B)}\left|\frac{1}{4}((\mu+1)^2-\nu^2+z)\right|\\
&\quad + \frac{(AB-1)}{(A-B)}\left|\frac{1}{4}((\mu+1)^2-\nu^2+z)\right| - \frac{(1-B)^2}{(A-B)}\left|-\frac{1}{4}((\mu+1)^2-\nu^2)\right|\\
&\quad + \frac{(B^2-1)}{(A-B)}\left|-\frac{1}{4}((\mu+1)^2-\nu^2)\right| - 1\bigg\}\\
&> \frac{(1-|B|)^2}{(1-B)^5}\bigg\{(1+B)(A-B) + (A-B)(1-B)\operatorname{Re}(\mu) - \frac{(|A+B|+2)(1-B)}{4}\\
&\quad - \left|\frac{(\mu+1)^2-\nu^2}{4}\right|(1-B)\{|A+B|+2+2(1-B)\}\bigg\}\\
&= \frac{(1-|B|)^2}{4(1-B)^5}\bigg\{8(A-B) + 4(A-B)(1-B)\operatorname{Re}(\mu) - (|A+B|+2)(1-B)\\
&\quad - (|A+B|-2B+4)(1-B)\left|(\mu+1)^2-\nu^2\right|\bigg\} \geq 0,
\end{aligned}
$$

in view of (3.4). Therefore, $\psi(r,s,t;z) \notin \Omega$ and, by using Theorem 2.1, we conclude that

$$\frac{h_{\mu\nu}(z)}{z} \prec \frac{1+Az}{1+Bz}.$$

□

By taking $C = 0, D = 4,\ E(z) = 2(2\kappa+1),\ F(z) = cz + 2(\kappa-1)$ and $G(z) = 2(\kappa-1)$, in Theorem 2.2, we have the following:

Corollary 3.5. *Let $-1 \leq B < 0 < A \leq 1$. If the parameter $\kappa, c \in \mathbb{C}$, such that $\kappa \neq 0, -1, -2, \dots$, satisfy the condition*

$$
\begin{aligned}
&4(A-B)(1-B)\operatorname{Re}(\kappa) - 2|\kappa-1|(1-B)(|A+B|-2B+4)\\
&\geq (|A+B|+2)(1-B)|c| - 2(A-B)(3B+1),
\end{aligned}
$$

then

$$u_v(z) \prec \frac{1+Az}{1+Bz}.$$

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