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# NECESSARY CONDITIONS FOR HYPONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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*Abstract.* In this paper, we present necessary conditions for the hyponormality of Toeplitz operator $T_\phi$ on the Bergman space $L^2_a(\mathbb{D})$ when the symbol $\phi$ is a polynomial in $z$ and $\overline{z}$.

## 1. Introduction

Let $T$ be a bounded linear operator acting on a complex Hilbert space $H$. The operator is said to be hyponormal if its self commutator $[T^*, T] = T^*T - TT^*$ is positive semi-definite, where $T^*$ denotes the adjoint of $T$. This is equivalent to saying that $\|Tu\| \geq \|T^*u\|$ for every $u$ in $H$.

Let $\mathbb{D}$ denote the open unit disc in the complex plane and $\mathrm{d}A$ the normalized area measure on $\mathbb{D}$. Let the space $L^2(\mathbb{D})$ be the Hilbert space of square integrable measurable functions with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\,\mathrm{d}A(z).$$

The Bergman space $L^2_a(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of analytic functions on $\mathbb{D}$, that is,

$$L^2_a(\mathbb{D}) = \{f : \int_{\mathbb{D}} |f(z)|^2\,\mathrm{d}A(z) < \infty,\ f \text{ is analytic on } \mathbb{D}\}.$$

Let $P$ denote the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ denote the space of bounded measurable functions on the unit disc $\mathbb{D}$. Let $\phi$ be a function in $L^\infty(\mathbb{D})$. The multiplication operator $M_\phi$, induced by the symbol $\phi$, is defined as $M_\phi f = \phi f$ for every $f \in L^2(\mathbb{D})$. The Toeplitz operators are the compressions of the multiplication operators to the subspace $L^2_a(\mathbb{D})$ and are defined as $T_\phi f = P(\phi f)$ for every $f \in L^2_a(\mathbb{D})$.

In [1, 2], C. Cowen completely characterised the hyponormality of Toeplitz operators (in the setting of Hardy spaces) by the properties of its symbol $\phi \in L^\infty(\mathbb{D})$. To prove this characterisation, he used dilation theorem due to D. Sarason [11] and the fact that space $(H^2)^\perp$ is the space of conjugate of functions in $zH^2$. But in the Bergman space setting, Sarason dilation theorem's equivalence is lacking. Additionally, the space $(L^2_a)^\perp$ is much larger than the space of conjugates of functions in $zL^2_a$.

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The characterisation of the hyponormal Toeplitz operators acting on the Bergman space turns out to be elusive. However, a substantial amount of work has been done by H. Sadraoui [10], I. S. Hwang [6,7], Čučković and R. Curto [3], A. Phukon [9], J. Lee [8], and A. Gupta and S. K. Singh [5] for the polynomial harmonic symbol $\phi$. Very recently, Matthew Fleeman and Constanze Liaw [4] studied the necessary condition on the coefficients of the non-harmonic polynomial $\phi$, under which $T_\phi$ is hyponormal. They showed that the Toeplitz operator $T_\phi$ with symbol $\phi(z) = a_{m,n} z^m \overline{z}^n$, $m \geq n$, $a_{m,n} \in \mathbb{C}$ is always hyponormal. However, this is not the case when $\phi$ is a two-term non harmonic polynomial. These results were further extended by Brian Simanek [12]. We shall consider $\phi$ the sum of the polynomial harmonic and non harmonic symbols and obtain the necessary conditions for the hyponormality.

## 2. The necessary conditions

In this section, we shall present some necessary conditions for the Toeplitz operartor acting on the Bergman space to be hyponormal, when the symbol $\phi$ is a polynomial in $z$ and $\overline{z}$. Some well known properties of the Toeplitz operators on the Bergman space are listed here.

**Proposition 2.1.** [10] *Let $f, g \in L^\infty(\mathbb{D})$, then*
(i) $T_{f+g} = T_f + T_g$;
(ii) $T_f^* = T_{\overline{f}}$;
(iii) $T_{\overline{f}} T_g = T_{\overline{f}g}$, *if $f$ or $g$ is analytic .*

**Proposition 2.2.** [6] *For any non negative integers $s$ and $t$, we have*

$$P(\overline{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \text{if } s \geq t \\ 0 & \text{if } s < t \end{cases},$$

*where $P$ is an orthogonal projection on $L_a^2(\mathbb{D})$.*

A result given in [4] will be helpful in deriving the necessary conditions.

**Proposition 2.3.** [4] *Let $H$ be a complex Hilbert space and $T$ and $S$ be the operators on $H$, then*

$$\begin{aligned}\langle [(T+S)^*, T+S]u, u\rangle = \langle Tu, Tu\rangle - \langle T^*u, T^*u\rangle + 2\mathrm{Re}[\langle Tu, Su\rangle - \langle T^*u, S^*u\rangle] \\ + \langle Su, Su\rangle - \langle S^*u, S^*u\rangle\end{aligned}$$

*for every $u$ in $H$.*

**Theorem 2.4.** *Let $\phi(z) = f(z) + g(z)$, where $f(z) = \alpha z^m + \beta z^m \overline{z}^n$ and $g(z) = \gamma \overline{z}^m + \delta \overline{z}^m z^n$, $m > n$. If $T_\phi$ is hyponormal, then*

(i) $$\frac{m^2}{(m+1)^2}(|\alpha|^2 - |\gamma|^2) \geq \frac{2m^3 - n^3 + m^2 - m^2 n - n^2}{(2m+1)(m+n+1)^2}(|\delta|^2 - |\beta|^2);$$

(ii) $$\frac{(2m-n+1)^2}{(2m+1)^2(3m-n+1)^2}|\beta|^2|\delta|^2 \leq$$
$$\left\{\frac{m^2}{(m+1)^2(2m+1)}(|\alpha|^2 - |\gamma|^2) + \frac{2m^3 - n^3 + m^2 - m^2 n - n^2}{(2m+1)^2(m+n+1)^2}(|\beta|^2 - |\delta|^2)\right\}$$

$$\Big\{\frac{m^2}{(4m-2n+1)(3m-2n+1)^2}(|\alpha|^2-|\gamma|^2) + \frac{4m^3+n^3-2mn^2-3m^2n+m^2-n^2}{(4m-2n+1)^2(3m-n+1)^2}(|\beta|^2-|\delta|^2)\Big\}.$$

*Proof.* Let $u = az^m + bz^{m+j}$, where $a$ and $b$ are complex numbers and $j$ is a non-negative integer. Then,

$$fu = \alpha a z^{2m} + \alpha b z^{2m+j} + \beta a z^{2m}\overline{z}^n + \beta b z^{2m+j}\overline{z}^n,$$
$$gu = \gamma a z^m\overline{z}^m + \gamma b z^{m+j}\overline{z}^m + \delta a z^{m+n}\overline{z}^m + \delta b z^{m+n+j}\overline{z}^m.$$

Taking $j = 2(m-n)$ and using Proposition 2.2, we get

$$Pfu = \alpha a z^{2m} + \alpha b z^{4m-2n} + \beta a\frac{2m-n+1}{2m+1}z^{2m-n} + \beta b\frac{4m-3n+1}{4m-2n+1}z^{4m-3n},$$
$$Pgu = \gamma a\frac{1}{m+1} + \gamma b\frac{2m-2n+1}{3m-2n+1}z^{2m-2n} + \delta a\frac{n+1}{m+n+1}z^n + \delta b\frac{2m-n+1}{3m-n+1}z^{2m-n}.$$

Using the definition of inner product on the Bergman space,

$$\langle Pfu, Pgu\rangle = a\overline{b}\beta\overline{\delta}\frac{2m-n+1}{(3m-n+1)(2m+1)},$$
$$\langle Pfu, Pfu\rangle = \frac{1}{2m+1}|\alpha a|^2 + \frac{1}{4m-2n+1}|\alpha b|^2 + \frac{2m-n+1}{(2m+1)^2}|\beta a|^2 + \frac{4m-3n+1}{(4m-2n+1)^2}|\beta b|^2,$$
$$\langle Pgu, Pgu\rangle = \frac{1}{(m+1)^2}|\gamma a|^2 + \frac{2m-2n+1}{(3m-2n+1)^2}|\gamma b|^2 + \frac{n+1}{(m+n+1)^2}|\delta a|^2 + \frac{2m-n+1}{(3m-n+1)^2}|\delta b|^2.$$

Similarly, computing for $P\overline{f}u$ and $P\overline{g}u$, we have

$$\langle P\overline{f}u, P\overline{g}u\rangle = \overline{a}b\overline{\beta}\delta\frac{2m-n+1}{(3m-n+1)(2m+1)},$$
$$\langle P\overline{f}u, P\overline{f}u\rangle = \frac{1}{(m+1)^2}|\alpha a|^2 + \frac{2m-2n+1}{(3m-2n+1)^2}|\alpha b|^2 + \frac{n+1}{(m+n+1)^2}|\beta a|^2 + \frac{2m-n+1}{(3m-n+1)^2}|\beta b|^2,$$
$$\langle P\overline{g}u, P\overline{g}u\rangle = \frac{1}{2m+1}|\gamma a|^2 + \frac{1}{4m-2n+1}|\gamma b|^2 + \frac{2m-n+1}{(2m+1)^2}|\delta a|^2 + \frac{4m-3n+1}{(4m-2n+1)^2}|\delta b|^2.$$

Since $T_\phi$ is hyponormal, we have $\langle({T_\phi}^*T_\phi - T_\phi{T_\phi}^*)u, u\rangle \geq 0$. Using Propositions 2.1 and 2.3, we get

$$\|T_fu\|^2 - \|T_f^*u\|^2 + 2Re[\langle T_fu, T_gu\rangle - \langle T_f^*u, T_g^*u\rangle] + \|T_gu\|^2 - \|T_g^*u\|^2 \geq 0.$$

Substituting for $T_f u, T_g u, T_f^* u$ and $T_g^* u$ (and using Proposition 2.1), we have

$$\left[|a|^2\Big(\frac{1}{2m+1}-\frac{1}{(m+1)^2}\Big)+|b|^2\Big(\frac{1}{4m-2n+1}-\frac{2m-2n+1}{(3m-2n+1)^2}\Big)\right](|\alpha|^2-|\gamma|^2)$$
$$+2\mathrm{Re}(a\bar{b}\beta\bar{\delta})\Big[\frac{2m-n+1}{(2m+1)(3m-n+1)}\Big]+\left[|a|^2\Big(\frac{2m-n+1}{(2m+1)^2}-\frac{n+1}{(m+n+1)^2}\Big)\right.$$
$$\left.+|b|^2\Big(\frac{4m-3n+1}{(4m-2n+1)^2}-\frac{2m-n+1}{(3m-n+1)^2}\Big)\right](|\beta|^2-|\delta|^2)\geq\ 0.$$

Further, using the inequality $\mathrm{Re}(xy)\leq|x||y|$, we get

$$|a|^2\left[\Big(\frac{m^2}{(2m{+}1)(m{+}1)^2}\Big)(|\alpha|^2{-}|\gamma|^2){+}\Big(\frac{2m^3{-}n^3{+}m^2{-}m^2n{-}n^2}{(2m{+}1)^2(m{+}n{+}1)^2}\Big)(|\beta|^2{-}|\delta|^2)\right]$$
$$+2|a||b|\Big[\frac{2m-n+1}{(2m+1)(3m-n+1)}\Big]|\beta\delta|+|b|^2\left[\Big(\frac{m^2}{(4m-2n+1)(3m-2n+1)^2}\Big)\right.$$
$$\left.(|\alpha|^2-|\gamma|^2)+\Big(\frac{4m^3-2mn^2-3m^2n+m^2+n^3-n^2}{(4m-2n+1)^2(3m-n+1)^2}\Big)(|\beta|^2-|\delta|^2)\right]\geq 0. \quad (2.1)$$

The following cases arise:

Case (i) Let $b=0$, then from inequality (2.1), it follows that

$$\Big(\frac{m^2}{(2m+1)(m+1)^2}\Big)(|\alpha|^2-|\gamma|^2)\geq\Big(\frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2}\Big)(|\delta|^2-|\beta|^2). \quad (2.2)$$

Case (ii) Let $b\neq 0$, then again using inequality (2.1), we have

$$\Big|\frac{a}{b}\Big|^2\left[\Big(\frac{m^2}{(2m+1)(m+1)^2}\Big)(|\alpha|^2{-}|\gamma|^2){+}\Big(\frac{2m^3-n^3{+}m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2}\Big)(|\beta|^2{-}|\delta|^2)\right]$$
$$+2\Big|\frac{a}{b}\Big|\Big[\frac{2m-n+1}{(2m+1)(3m-n+1)}\Big]|\beta\delta|+\Big(\frac{m^2}{(4m-2n+1)(3m-2n+1)^2}\Big)(|\alpha|^2{-}|\gamma|^2)$$
$$+\Big(\frac{4m^3-2mn^2-3m^2n+m^2+n^3-n^2}{(4m-2n+1)^2(3m-n+1)^2}\Big)(|\beta|^2-|\delta|^2)\geq 0, \quad (2.3)$$

which is a quadratic polynomial in $|a/b|$ and takes only non negative values. We know that, if a quadratic polynomial $f(x)=a_2x^2+a_1x+a_0$ (for $a_2,a_1,a_0$ real and $a_2\geq 0$) takes only non negative values for all $x$, then it cannot have non distinct real roots. Thus, its discriminant is non positive. Consequently, from (2.2) and (2.3), it follows that

$$\frac{(2m-n+1)^2}{(2m+1)^2(3m-n+1)^2}|\beta|^2|\delta|^2$$
$$\leq\Big\{\frac{m^2}{(m+1)^2(2m+1)}(|\alpha|^2-|\gamma|^2)+\frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2}(|\beta|^2-|\delta|^2)\Big\}$$
$$\Big\{\frac{m^2}{(4m-2n+1)(3m-2n+1)^2}(|\alpha|^2-|\gamma|^2)$$

$$+\frac{4m^3+n^3-2mn^2-3m^2n+m^2-n^2}{(4m-2n+1)^2(3m-n+1)^2}(|\beta|^2-|\delta|^2)\Big\}$$

as required. □

**Remark 2.5.** The following example shows that the conditions in the theorem are only necessary but not sufficient:

Let $\phi(z) = 2z^3 + 2z^3\overline{z} + \overline{z}^3 + 3\overline{z}^3 z$, then the conditions of Theorem 2.4 are satisfied. Using Proposition 2.2, we get that

$$T_\phi(z) = 2z^4 + \frac{8}{5}z^3 \quad \text{and} \quad T_\phi{}^*(z) = z^4 + \frac{12}{5}z^3.$$

Therefore, we have $\|T_\phi(z)\|^2 = \frac{36}{25}$ and $\|T_\phi{}^*(z)\|^2 = \frac{41}{25}$. Thus, it follows that $\|T_\phi(z)\| \le \|T_\phi{}^*(z)\|$, showing $T_\phi$ is not hyponormal.

**Corollary 2.6.** *Let $\phi(z) = f(z)+g(z)$, where $f(z) = \alpha z^m+\beta z^m\overline{z}^{m-1}$* and $g(z)$ $= \gamma\overline{z}^m + \delta\overline{z}^m z^{m-1}$; *$m > 1$. If $T_\phi$ is hyponormal, then*

(i) $$\frac{m^2}{(m+1)^2}(|\alpha|^2-|\gamma|^2) \ge \frac{4m-1}{4m(2m+1)}(|\delta|^2-|\beta|^2).$$

(ii) $$\frac{(m+2)^2}{4(m+1)^2(2m+1)^2}|\beta|^2|\delta|^2 \le$$
$$\Big\{\frac{m^2}{(m+1)^2(2m+1)}(|\alpha|^2-|\gamma|^2)+\frac{4m-1}{4m(2m+1)^2}(|\beta|^2-|\delta|^2)\Big\}$$
$$\Big\{\frac{m^2}{(2m+3)(m+3)^2}(|\alpha|^2-|\gamma|^2)+\frac{4m^2+3m-2}{4(m+1)^2(2m+3)^2}(|\beta|^2-|\delta|^2)\Big\}.$$

**Theorem 2.7.** *Let $\phi(z) = f(z) + g(z)$, where*

$$f(z) = \sum_{i=1}^{k}\alpha_i z^{m+i}\overline{z}^m \quad \textit{and} \quad g(z) = \sum_{i=1}^{k}\beta_i\overline{z}^{m+i}z^m.$$

*If $T_\phi$ is hyponormal, then*

(i) $$\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}(|\alpha_i|^2-|\beta_i|^2) \ge 0.$$

(ii) $$\left(\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}(|\alpha_i|^2-|\beta_i|^2)\right)\left(\sum_{i=1}^{k}\frac{i+2}{(m+i+2)^2}(|\alpha_i|^2-|\beta_i|^2)\right.$$
$$\left.+\frac{1}{(m+2)^2}(|\beta_1|^2)-|\alpha_1|^2)\right) \ge \left|\sum_{i=2}^{k}\frac{(\alpha_i\overline{\alpha_{i-1}}-\overline{\beta_i}\beta_{i-1})(i+1)}{(m+i+1)^2}\right|^2.$$

*Proof.* Let $u = a + bz$, where $a$ and $b$ are complex numbers. Then,

$$T_\phi(u) = P\phi(a+bz)$$
$$= P\Big[a\Big(\sum_{i=1}^{k}\alpha_i z^{m+i}\overline{z}^m+\sum_{i=1}^{k}\beta_i\overline{z}^{m+i}z^m\Big)+b\Big(\sum_{i=1}^{k}\alpha_i z^{m+i+1}\overline{z}^m+\sum_{i=1}^{k}\beta_i\overline{z}^{m+i}z^{m+1}\Big)\Big]$$

$$= a\sum_{i=1}^{k}\alpha_i\frac{i+1}{m+i+1}z^i + b\Big(\sum_{i=1}^{k}\alpha_i\frac{i+2}{m+i+2}z^{i+1} + \frac{\beta_1}{m+2}\Big)$$

(using Proposition 2.2). Therefore,

$$\begin{aligned}\langle T_\phi^* T_\phi u, u\rangle = |a|^2\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}|\alpha_i|^2 + 2\mathrm{Re}\Big[a\bar{b}\sum_{i=2}^{k}\alpha_i\overline{\alpha_{i-1}}\frac{i+1}{(m+i+1)^2}\Big] \\ + |b|^2\Big(\sum_{i=1}^{k}\frac{i+2}{(m+i+2)^2}|\alpha_i|^2 + \frac{|\beta_1|^2}{(m+2)^2}\Big).\end{aligned} \tag{2.4}$$

Using Proposition 2.1, we have

$$\begin{aligned}&T_\phi^*(u) = P\overline{\phi}(a+bz)\\ &= P\Big[a\Big(\sum_{i=1}^{k}\overline{\alpha_i z^{m+i}}z^m + \sum_{i=1}^{k}\overline{\beta_i}z^{m+i}\overline{z^m}\Big) + b\Big(\sum_{i=1}^{k}\overline{\alpha_i z^{m+i}}z^{m+1} + \sum_{i=1}^{k}\overline{\beta_i}z^{m+i+1}\overline{z^m}\Big)\Big]\\ &= a\sum_{i=1}^{k}\overline{\beta_i}\frac{i+1}{m+i+1}z^i + b\Big(\sum_{i=1}^{k}\overline{\beta_i}\frac{i+2}{m+i+2}z^{i+1} + \frac{\overline{\alpha_1}}{m+2}\Big).\end{aligned}$$

Again, using the definition of inner product, we get

$$\begin{aligned}\langle T_\phi T_\phi^* u, u\rangle = |a|^2\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}|\beta_i|^2 + 2\mathrm{Re}\Big[a\bar{b}\sum_{i=2}^{k}\overline{\beta_i}\beta_{i-1}\frac{i+1}{(m+i+1)^2}\Big] \\ + |b|^2\Big(\sum_{i=1}^{k}\frac{i+2}{(m+i+2)^2}|\beta_i|^2 + \frac{|\alpha_1|^2}{(m+2)^2}\Big).\end{aligned} \tag{2.5}$$

Since $T_\phi$ is hyponormal, from (2.4) and (2.5) it follows that

$$\begin{aligned}\langle (T_\phi^* T_\phi - T_\phi T_\phi^*)u, u\rangle &= |a|^2\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}(|\alpha_i|^2 - |\beta_i|^2)\\ &+ 2\mathrm{Re}\Big[a\bar{b}\sum_{i=2}^{k}(\alpha_i\overline{\alpha_{i-1}} - \overline{\beta_i}\beta_{i-1})\frac{i+1}{(m+i+1)^2}\Big]\\ &+ |b|^2\Big(\sum_{i=1}^{k}\frac{i+2}{(m+i+2)^2}(|\alpha_i|^2 - |\beta_i|^2) + \frac{|\beta_1|^2 - |\alpha_1|^2}{(m+2)^2}\Big) \geq 0.\end{aligned}$$

Using Theorem 1.5 [9], we have

(i) If $b = 0$, then $\displaystyle\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}(|\alpha_i|^2 - |\beta_i|^2) \geq 0.$

(ii) If $b \neq 0$, then

$$\Big(\sum_{i=1}^{k}\frac{i+1}{(m+i+1)^2}(|\alpha_i|^2 - |\beta_i|^2)\Big)\Big(\sum_{i=1}^{k}\frac{i+2}{(m+i+2)^2}(|\alpha_i|^2 - |\beta_i|^2)$$

$$+\frac{1}{(m+2)^2}(|\beta_1|^2)-|\alpha_1|^2)\Bigg)\geq\left|\sum_{i=2}^{k}\frac{(\alpha_i\overline{\alpha_{i-1}}-\overline{\beta_i}\beta_{i-1})(i+1)}{(m+i+1)^2}\right|^2$$

as required. □

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