NECESSARY CONDITIONS FOR HYPERSONORMALITY OF TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. In this paper, we present necessary conditions for the hyponormality of Toeplitz operator $T_\phi$ on the Bergman space $L^2_a(D)$ when the symbol $\phi$ is a polynomial in $z$ and $\overline{z}$.

1. Introduction

Let $T$ be a bounded linear operator acting on a complex Hilbert space $H$. The operator is said to be hyponormal if its self commutator $[T^*, T] = T^*T - TT^*$ is positive semi-definite, where $T^*$ denotes the adjoint of $T$. This is equivalent to saying that $\|Tu\| \geq \|T^*u\|$ for every $u$ in $H$.

Let $\mathbb{D}$ denote the open unit disc in the complex plane and $dA$ the normalized area measure on $\mathbb{D}$. Let the space $L^2(\mathbb{D})$ be the Hilbert space of square integrable measurable functions with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) g(\overline{z}) \, dA(z).$$

The Bergman space $L^2_a(\mathbb{D})$ is the subspace of $L^2(\mathbb{D})$ consisting of analytic functions on $\mathbb{D}$, that is,

$$L^2_a(\mathbb{D}) = \{ f : \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty, \, f \text{ is analytic on } \mathbb{D} \}.$$

Let $P$ denote the orthogonal projection of $L^2(\mathbb{D})$ onto $L^2_a(\mathbb{D})$. Let $L^\infty(\mathbb{D})$ denote the space of bounded measurable functions on the unit disc $\mathbb{D}$. Let $\phi$ be a function in $L^\infty(\mathbb{D})$. The multiplication operator $M_\phi$, induced by the symbol $\phi$, is defined as $M_\phi f = \phi f$ for every $f \in L^2(\mathbb{D})$. The Toeplitz operators are the compressions of the multiplication operators to the subspace $L^2_a(\mathbb{D})$ and are defined as $T_\phi f = P(\phi f)$ for every $f \in L^2_a(\mathbb{D})$.

In [1, 2], C. Cowen completely characterised the hyponormality of Toeplitz operators (in the setting of Hardy spaces) by the properties of its symbol $\phi \in L^\infty(\mathbb{D})$. To prove this characterisation, he used dilation theorem due to D. Sarason [11] and the fact that space $(H^2)\perp$ is the space of conjugate of functions in $zH^2$. But in the Bergman space setting, Sarason dilation theorem’s equivalence is lacking. Additionally, the space $(L^2_a)^\perp$ is much larger than the space of conjugates of functions in $zL^2_a$.


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The characterisation of the hyponormal Toeplitz operators acting on the Bergman space turns out to be elusive. However, a substantial amount of work has been done by H. Sadraoui [10], I.S. Hwang [6, 7], Čučković and R. Curto [3], A. Phukon [9], J. Lee [8], and A. Gupta and S.K. Singh [5] for the polynomial harmonic symbol \( \phi \). Very recently, Matthew Fleeman and Constanze Liaw [4] studied the necessary condition on the coefficients of the non-harmonic polynomial \( \phi \), under which \( T_\phi \) is hyponormal. They showed that the Toeplitz operator \( T_\phi \) with symbol \( \phi(z) = a_{m,n}z^m\overline{z}^n, m \geq n, a_{m,n} \in \mathbb{C} \) is always hyponormal. However, this is not the case when \( \phi \) is a two-term non-harmonic polynomial. These results were further extended by Brian Simanek [12]. We shall consider \( \phi \) the sum of the polynomial harmonic and non-harmonic symbols and obtain the necessary conditions for the hyponormality.

2. The necessary conditions

In this section, we shall present some necessary conditions for the Toeplitz operator acting on the Bergman space to be hyponormal, when the symbol \( \phi \) is a polynomial in \( z \) and \( \overline{z} \). Some well known properties of the Toeplitz operators on the Bergman space are listed here.

**Proposition 2.1.** [10] Let \( f, g \in L^\infty(\mathbb{D}) \), then

(i) \( T_{f+g} = T_f + T_g \);

(ii) \( T_f^* = T_f^\dagger \);

(iii) \( T_fT_g = T_{fg} \), if \( f \) or \( g \) is analytic.

**Proposition 2.2.** [6] For any non negative integers \( s \) and \( t \), we have

\[
P(\pi^t z^s) = \begin{cases} s-t+1 \frac{z^{s-t}}{s+1} & \text{if } s \geq t \\ 0 & \text{if } s < t \end{cases}
\]

where \( P \) is an orthogonal projection on \( L^2_a(\mathbb{D}) \).

A result given in [4] will be helpful in deriving the necessary conditions.

**Proposition 2.3.** [4] Let \( H \) be a complex Hilbert space and \( T \) and \( S \) be the operators on \( H \), then

\[
\langle (T + S)^\dagger T + S \rangle u, u \rangle = \langle Tu, Tu \rangle - \langle T^*u, T^*u \rangle + 2\text{Re}[\langle Tu, Su \rangle - \langle T^*u, S^*u \rangle] + \langle Su, Su \rangle - \langle S^*u, S^*u \rangle
\]

for every \( u \) in \( H \).

**Theorem 2.4.** Let \( \phi(z) = f(z) + g(z) \), where \( f(z) = \alpha z^m + \beta z^m\overline{z}^n \) and \( g(z) = \gamma z^m + \delta z^m\overline{z}^n \), \( m > n \). If \( T_\phi \) is hyponormal, then

(i) \[
\frac{m^2}{(m+1)^2}(|\alpha|^2 - |\gamma|^2)^2 \geq \frac{2m^3 - n^3 + m^2 - m^2n - n^2}{(2m+1)(m+n+1)^2}(|\delta|^2 - |\beta|^2);
\]

(ii) \[
\frac{(2m-n+1)^2}{(2m+1)^2(3m-n+1)^2} |\beta|^2 |\delta|^2 \leq \left\{ \frac{m^2}{(m+1)^2(2m+1)}(|\alpha|^2 - |\gamma|^2) + \frac{2m^3 - n^3 + m^2 - m^2n - n^2}{(2m+1)^2(2m+n+1)^2}(|\beta|^2 - |\delta|^2) \right\}
\]
\[
\left\{ \frac{m^2}{(4m - 2n + 1)(3m - 2n + 1)^2}(|\alpha|^2 - |\gamma|^2) + \frac{4m^3 + n^3 - 2mn^2 - 3m^2n + m^2 - n^2}{(4m - 2n + 1)^2(3m - n + 1)^2}(|\beta|^2 - |\delta|^2) \right\}.
\]

**Proof.** Let \( u = az^m + bz^{m+j} \), where \( a \) and \( b \) are complex numbers and \( j \) is a non-negative integer. Then,

\[
\begin{align*}
fu &= \alpha az^{2m} + \alpha bz^{2m+j} + \beta az^{m+2m} + \beta bz^{2m+j}z^n, \\
gu &= \gamma az^{m+2m} \gamma z^{m+2m} + \gamma bz^{m+j} \gamma z^m + \delta az^{m+n} \gamma z^m + \delta bz^{m+n+j} \gamma z^m.
\end{align*}
\]

Taking \( j = 2(m-n) \) and using Proposition 2.2, we get

\[
\begin{align*}
Pfu &= \alpha az^{2m} + \alpha bz^{4m-2n} + \beta az^{2m-2n} + \beta bz^{4m-3n}, \\
Pgu &= \gamma a \frac{1}{m+1} + \gamma b \frac{2m-2n+1}{3m-2n+1} z^{2m-2n} + \delta a \frac{n+1}{m+n+1} z^n \\
&\quad + \delta b \frac{2m-2n+1}{3m-2n+1} z^{2m-n}.
\end{align*}
\]

Using the definition of inner product on the Bergman space,

\[
\begin{align*}
\langle Pfu, Pgu \rangle &= \frac{2m-n+1}{(3m-n+1)(2m+1)}, \\
\langle Pfu, Pfu \rangle &= \frac{1}{2m+1} |\alpha a|^2 + \frac{1}{4m-2n+1} |\alpha b|^2 \\
&\quad + \frac{2m-n+1}{(2m+1)^2} |\beta a|^2 + \frac{4m-3n+1}{(4m-2n+1)^2} |\beta b|^2, \\
\langle Pgu, Pgu \rangle &= \frac{1}{(m+1)^2} |\gamma a|^2 + \frac{2m-2n+1}{(3m-2n+1)^2} |\gamma b|^2 \\
&\quad + \frac{n+1}{(m+n+1)^2} |\delta a|^2 + \frac{2m-n+1}{(3m-n+1)^2} |\delta b|^2.
\end{align*}
\]

Similarly, computing for \( P_f^*u \) and \( P_g^*u \), we have

\[
\begin{align*}
\langle P_f^*u, P_g^*u \rangle &= \frac{2m-n+1}{(3m-n+1)(2m+1)}, \\
\langle P_f^*u, P_f^*u \rangle &= \frac{1}{(m+1)^2} |\alpha a|^2 + \frac{2m-2n+1}{(3m-2n+1)^2} |\alpha b|^2 \\
&\quad + \frac{n+1}{(m+n+1)^2} |\beta a|^2 + \frac{2m-n+1}{(3m-n+1)^2} |\beta b|^2, \\
\langle P_g^*u, P_g^*u \rangle &= \frac{1}{2m+1} |\gamma a|^2 + \frac{1}{4m-2n+1} |\gamma b|^2 \\
&\quad + \frac{2m-n+1}{(2m+1)^2} |\delta a|^2 + \frac{4m-3n+1}{(4m-2n+1)^2} |\delta b|^2.
\end{align*}
\]

Since \( T_\phi \) is hyponormal, we have \( \langle (T_\phi^*T_\phi - T_\phi T_\phi^*)u, u \rangle \geq 0 \). Using Propositions 2.1 and 2.3, we get

\[
\|T_f^*u\|^2 - \|T_f^*u\|^2 + 2Re\left[\langle T_f^*u, T_g^*u \rangle - \langle T_f^*u, T_g^*u \rangle \right] + \|T_g^*u\|^2 - \|T_g^*u\|^2 \geq 0.
\]
Substituting for $T_f u, T_g u, T_f^* u$ and $T_g^* u$ (and using Proposition 2.1), we have

$$
\left[|a|^2 \left( \frac{1}{2m+1} - \frac{1}{(m+1)^2} \right) + |b|^2 \left( \frac{1}{4m-2n+1} - \frac{2m-2n+1}{(3m-2n+1)^2} \right) \right] (|\alpha|^2 - |\gamma|^2)
+ 2 \text{Re}(\bar{a}b\beta\delta) \left[ \frac{2m-n+1}{(2m+1)(3m-n+1)} \right] + \left[ |a|^2 \left( \frac{2m-n+1}{(2m+1)^2} - \frac{n+1}{(m+n+1)^2} \right)
+ |b|^2 \left( \frac{4m-3n+1}{(4m-2n+1)^2} - \frac{2m-n+1}{(3m-n+1)^2} \right) \right] (|\beta|^2 - |\delta|^2) \geq 0.
$$

Further, using the inequality $\text{Re}(xy) \leq |x||y|$, we get

$$
|a|^2 \left[ \left( \frac{m^2}{(2m+1)(m+1)^2} \right) (|\alpha|^2 - |\gamma|^2) + \left( \frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2} \right) (|\beta|^2 - |\delta|^2) \right]
+ 2|a||b| \left[ \frac{2m-n+1}{(2m+1)(3m-n+1)} \right] |\beta\delta| + |b|^2 \left[ \left( \frac{m^2}{(4m-2n+1)(3m-2n+1)^2} \right) \right] (|\alpha|^2 - |\gamma|^2)
+ \left( \frac{4m^3-2mn^2-3m^2n+m^2+n^3-n^2}{(4m-2n+1)^2(3m-n+1)^2} \right) (|\beta|^2 - |\delta|^2) \geq 0. \quad (2.1)
$$

The following cases arise:

Case (i) Let $b = 0$, then from inequality (2.1), it follows that

$$
\left( \frac{m^2}{(2m+1)(m+1)^2} \right) (|\alpha|^2 - |\gamma|^2) \geq \left( \frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2} \right) (|\beta|^2 - |\delta|^2).
$$

Case (ii) Let $b \neq 0$, then again using inequality (2.1), we have

$$
\frac{|a|^2}{|b|^2} \left[ \left( \frac{m^2}{(2m+1)(m+1)^2} \right) (|\alpha|^2 - |\gamma|^2) + \left( \frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2} \right) (|\beta|^2 - |\delta|^2) \right]
+ 2 \left[ \frac{2m-n+1}{(2m+1)(3m-n+1)} \right] |\beta\delta| + \left( \frac{2m^3-n^3+m^2-m^2n-n^2}{(4m-2n+1)(3m-2n+1)^2} \right) (|\alpha|^2 - |\gamma|^2)
+ \left( \frac{4m^3-2mn^2-3m^2n+m^2+n^3-n^2}{(4m-2n+1)^2(3m-n+1)^2} \right) (|\beta|^2 - |\delta|^2) \geq 0,
$$

which is a quadratic polynomial in $|a/b|$ and takes only non-negative values. We know that, if a quadratic polynomial $f(x) = a_2x^2 + a_1x + a_0$ (for $a_2, a_1, a_0$ real and $a_2 \geq 0$) takes only non-negative values for all $x$, then it cannot have non distinct real roots. Thus, its discriminant is non-positive. Consequently, from (2.2) and (2.3), it follows that

$$
\frac{(2m-n+1)^2}{(2m+1)^2(3m-n+1)^2} |\beta|^2 |\delta|^2
\leq \left\{ \frac{m^2}{(m+1)^2(2m+1)} (|\alpha|^2 - |\gamma|^2) + \frac{2m^3-n^3+m^2-m^2n-n^2}{(2m+1)^2(m+n+1)^2} (|\beta|^2 - |\delta|^2) \right\}
\leq \left\{ \frac{m^2}{(4m-2n+1)(3m-2n+1)^2} (|\alpha|^2 - |\gamma|^2) \right\}
$$
Therefore, we have \( \|s\| = \frac{4m^3 + n^2 - 2mn^2 - 3m^2n + m^2 - n^2}{(4m - 2n + 1)(3m - n + 1)^2} (|\beta|^2 - |\delta|^2) \) as required. \( \square \)

**Remark 2.5.** The following example shows that the conditions in the theorem are only necessary but not sufficient:

Let \( \phi(z) = 2z^3 + 2z^3\overline{z} + \overline{z}^3 + 3\overline{z}^3z \), then the conditions of Theorem 2.4 are satisfied. Using Proposition 2.2, we get that \( T \) is hyponormal, then

\[
\sum_{k=1}^{\infty} \frac{i + 1}{(m + i + 1)^2} (|\alpha_i|^2 - |\beta_i|^2) \geq 0.
\]

Using Proposition 2.2, we get that

\[
T_{\phi}(z) = 2z^4 + \frac{8}{5}z^3 \quad \text{and} \quad T_{\phi}^*(z) = z^4 + \frac{12}{5}z^3.
\]

Therefore, we have \( \|T_{\phi}(z)\|^2 = \frac{36}{25} \) and \( \|T_{\phi}^*(z)\|^2 = \frac{41}{25} \). Thus, it follows that \( \|T_{\phi}(z)\| \leq \|T_{\phi}^*(z)\| \), showing \( T_{\phi} \) is not hyponormal.

**Corollary 2.6.** Let \( \phi(z) = f(z) + g(z) \), where \( f(z) = \alpha z^m + \beta z^m \overline{z}^m - 1 \) and \( g(z) = \gamma \overline{z}^m + \delta \overline{z}^m z^m - 1 \); \( m > 1 \). If \( T_{\phi} \) is hyponormal, then

\[
\begin{align*}
(i) \quad & \frac{m^2}{(m + 1)^2} (|\alpha|^2 - |\gamma|^2) \geq \frac{4m - 1}{4m(2m + 1)} (|\beta|^2 - |\delta|^2), \\
(ii) \quad & \frac{(m + 2)^2}{4(m + 1)^2 (2m + 1)^2} (|\beta|^2 |\delta|^2) \leq \\
& \left\{ \frac{m^2}{(m + 1)^2 (2m + 1)} (|\alpha|^2 - |\gamma|^2) + \frac{4m - 1}{4m(2m + 1)^2} (|\beta|^2 - |\delta|^2) \right\} \\
& \left\{ \frac{m^2}{(2m + 3)(m + 3)^2} (|\alpha|^2 - |\gamma|^2) + \frac{4m^2 + 3m - 2}{4(m + 1)^2 (2m + 3)^2} (|\beta|^2 - |\delta|^2) \right\}.
\end{align*}
\]

**Theorem 2.7.** Let \( \phi(z) = f(z) + g(z) \), where

\[
f(z) = \sum_{i=1}^{k} \alpha_i z^{m+i} \overline{z}^m \quad \text{and} \quad g(z) = \sum_{i=1}^{k} \beta_i \overline{z}^{m+i} z^m.
\]

If \( T_{\phi} \) is hyponormal, then

\[
(i) \quad \sum_{i=1}^{k} \frac{i + 1}{(m + i + 1)^2} (|\alpha_i|^2 - |\beta_i|^2) \geq 0.
\]

\[
(ii) \quad \left( \sum_{i=1}^{k} \frac{i + 1}{(m + i + 1)^2} (|\alpha_i|^2 - |\beta_i|^2) \right) \left( \sum_{i=1}^{k} \frac{i + 2}{(m + i + 2)^2} (|\alpha_i|^2 - |\beta_i|^2) \right) \]

\[
+ \frac{1}{(m + 2)^2} (|\beta_1|^2 - |\alpha_1|^2) \geq \left( \sum_{i=2}^{k} \frac{\alpha_i \overline{\alpha}_{i-1} - \beta_i \overline{\beta}_{i-1} (i + 1)}{m + i + 1} \right)^2.
\]

**Proof.** Let \( u = a + bz \), where \( a \) and \( b \) are complex numbers. Then,

\[
T_{\phi}(u) = P\phi(a + bz)
\]

\[
= P \left[ a \left( \sum_{i=1}^{k} \alpha_i z^{m+i} \overline{z}^m + \sum_{i=1}^{k} \beta_i \overline{z}^{m+i} z^m \right) + b \left( \sum_{i=1}^{k} \alpha_i z^{m+i} \overline{z}^m + \sum_{i=1}^{k} \beta_i z^{m+i} z^m \right) \right]
\]

as required.
Using Theorem 1.5 [9], we have

\[
\langle T_\phi^* T_\phi u, u \rangle = |a|^2 \sum_{i=1}^{k} \frac{i+1}{(m+i+1)^2} |\alpha_i|^2 + 2\text{Re} \left[ a \sum_{i=2}^{k} \alpha_i \alpha_{i-1} \frac{i+1}{(m+i+1)^2} \right] + |b|^2 \left( \sum_{i=1}^{k} \frac{i+2}{(m+i+2)^2} |\alpha_i|^2 + \frac{|\beta_1|^2}{(m+2)^2} \right).
\]

(2.4)

Using Proposition 2.1, we have

\[
T_\phi^* (u) = P \tilde{\phi}(a + bz)
\]

\[
= P \left[ a \left( \sum_{i=1}^{k} \alpha_i z^{m+i} + \sum_{i=1}^{k} \beta_i z^{m+i} \right) \right] + b \left( \sum_{i=1}^{k} \beta_i z^{m+i} \right). 
\]

\[
= a \sum_{i=1}^{k} \frac{\beta_i}{m+i+1} z^i + b \left( \sum_{i=1}^{k} \beta_i z^{m+i+1} + \frac{\alpha_1}{m+2} \right).
\]

Again, using the definition of inner product, we get

\[
\langle T_\phi^* T_\phi u, u \rangle = |a|^2 \sum_{i=1}^{k} \frac{i+1}{(m+i+1)^2} |\beta_i|^2 + 2\text{Re} \left[ a \sum_{i=2}^{k} \beta_i \beta_{i-1} \frac{i+1}{(m+i+1)^2} \right] + |b|^2 \left( \sum_{i=1}^{k} \frac{i+2}{(m+i+2)^2} |\beta_i|^2 + \frac{|\alpha_1|^2}{(m+2)^2} \right).
\]

(2.5)

Since \( T_\phi \) is hyponormal, from (2.4) and (2.5) it follows that

\[
\langle (T_\phi^* T_\phi - T_\phi T_\phi^*) u, u \rangle = |a|^2 \sum_{i=1}^{k} \frac{i+1}{(m+i+1)^2} (|\alpha_i|^2 - |\beta_i|^2) 
+ 2\text{Re} \left[ a \sum_{i=2}^{k} \frac{\alpha_i \alpha_{i-1} - \bar{\beta}_i \beta_{i-1}}{(m+i+1)^2} \right] 
+ |b|^2 \left( \sum_{i=1}^{k} \frac{i+2}{(m+i+2)^2} (|\alpha_i|^2 - |\beta_i|^2) + \frac{|\beta_1|^2 - |\alpha_1|^2}{(m+2)^2} \right) \geq 0.
\]

Using Theorem 1.5 [9], we have

(i) If \( b = 0 \), then \( \sum_{i=1}^{k} \frac{i+1}{(m+i+1)^2} (|\alpha_i|^2 - |\beta_i|^2) \geq 0. \)

(ii) If \( b \neq 0 \), then

\[
\left( \sum_{i=1}^{k} \frac{i+1}{(m+i+1)^2} (|\alpha_i|^2 - |\beta_i|^2) \right) \left( \sum_{i=1}^{k} \frac{i+2}{(m+i+2)^2} (|\alpha_i|^2 - |\beta_i|^2) \right) \geq 0.
\]
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\[ + \frac{1}{(m + 2)^2} (|\beta_1|^2 - |\alpha_1|^2) \geq \left| \sum_{i=2}^{k} \frac{(\alpha_i\alpha_{i-1} - \overline{\beta_i}\beta_{i-1})(i + 1)}{(m + i + 1)^2} \right|^2 \]

as required.

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REFERENCES


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