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ON THE CATEGORY OF ORDERED PRE-TOPOLOGICAL SPACES

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Abstract. A pre-topological space equipped with an order is called an ordered pre-topological space. These spaces form the objects of a category which will be denoted by **OVPT**. The arrows of this category are certain increasing maps called V-continuous. Essentially, we will prove that the subcategory of ordered pre-topological spaces of type T_0, $\mathbf{OVPT}_0$, is reflective in **OVPT**. We introduce and study some new separation axioms and characterize the class of morphisms orthogonal to the objects of $\mathbf{OVPT}_0$.

1. Introduction

Categorical topologists are interested in developing different reflections in the category **TOP** of topological spaces with continuous maps as arrows. For more information on reflective subcategories we cite [4, 5, 8, 19]. Especially, concerning the category TOP, interesting results can be found in [6, 9, 10, 20]. Herrlich and Strecker described in their paper [13] some methods of constructing reflections related to separation axioms like T_0-, T_1- and T_2-reflection in **TOP**.

The study of T_0-reflection was generalized to other categories. As examples we can cite the category of the ordered topological space **ORDTOP** in [14, 15] (ordered topological spaces are presented by Nachbin in [18]), the category of the generalized topology **GenTOP** in [17], and the category of the pretopological spaces **PreTOP** in [1].

Here, we will consider this concept in the category of the ordered pre-topological space of type V denoted **OVPT**.

In Section 2, we recall some notions and definitions used throughout this paper and we will introduce the new category **OVPT** of ordered pre-topological spaces of type V as objects and we will define its arrows. In Section 3, the construction of the T_0-reflection will be given. Finally, Section 4 will be devoted to the characterization of arrows in **OVPT** orthogonal to the full subcategory $\mathbf{OVPT}_0$ where objects are of type T_0.

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2. Preliminary

In this section, we will introduce definitions and notations which will be used throughout this paper. We start by the definition of a pre-topological space given using the Čech closure operator [7].

Definition 2.1. Let X be a nonempty set and a be a map from the power set of X, $\mathcal{P}(X)$ to itself. a is called a Čech closure operator on X if it satisfies:

(1) $a(\emptyset) = \emptyset$;
(2) $\forall A \in \mathcal{P}(X),\ A \subseteq a(A)$;
(3) $a(A \cup B) = a(A) \cup a(B)$ for all $A, B \in \mathcal{P}(X)$.

In this case, (X, a) is called a pre-topological space of type V or, for short, a pre-topological space.

When $\leq$ is a partial order on X, then the triplet $(X, a, \leq)$ is called an ordered pre-topological space.

In this paper, all pre-topological spaces are considered of type V.

Definition 2.2. Let (X, a) be an ordered pre-topological space and A a subset of X. Then,

(1) A is said to be a quasi-closed set (q-closed set for short) if there exists a subset B of X such that $A = a(B)$. The set of q-closed sets of X will be denoted by $QC(X)$.
(2) A is said to be a closed set if $a(A) = A$. The set of closed sets of X will be denoted by $C(X)$.
(3) A will be a quasi-open (q-open for short) set (resp. an open set) if $A^c \in QC(X)$ (resp. $A^c \in C(X)$). The set of q-open sets (resp. open sets) of X will be denoted by $QO(X)$ (resp. $O(X)$).

Remark 2.3. We have the following implication:

closed (resp. open) $\Longrightarrow$ q-closed (resp. q-open)

Example 2.4. Let $X = \{0, 1, 2\}$ and $a : \mathcal{P}(X) \to: \mathcal{P}(X)$ defined by $a(\emptyset) = \emptyset,\ a(\{0\}) = \{0\},\ a(\{1\}) = \{0, 1\},\ a(\{2\}) = \{1, 2\},\ a(\{0, 1\}) = \{0, 1\},\ a(\{0, 2\}) = a(\{1, 2\}) = a(X) = X$. (X, a) is an ordered pre-topological space and we have:

(1) $C(X) = \{\emptyset, \{0\}, \{0, 1\}, X\}$;
(2) $QC(X) = \{\emptyset, \{0\}, \{0, 1\}, \{1, 2\}, X\}$;
(3) $O(X) = \{\emptyset, \{2\}, \{1, 2\}, X\}$;
(4) $QO(X) = \{\emptyset, \{0\}, \{2\}, \{1, 2\}, X\}$.

Notation 2.5. Let $(X, a, \leq)$ be an ordered pre-topological space and A be a nonempty subset of X. The increasing (resp. decreasing) hull of A denoted by $i(A)$ (resp. $d(A)$) is defined by:

$$i(A) = \{y \in X : \exists x \in A \text{ with } x \leq y\}.$$

$$\left(resp.\ d(A) = \{y \in X : \exists x \in A \text{ with } y \leq x\}\right).$$

If $i(A) = A$ (resp. $d(A) = A$), then A is called an increasing (resp. decreasing) set (see [14]).

We also consider the following notations:

- We denote by $CI(X)$ (resp. $CD(X)$) the collection of all increasing (resp. decreasing) closed sets of X.
- We denote by $QCI(X)$ (resp. $QCD(X)$) the collection of all increasing (resp. decreasing) q-closed sets of X.
- We denote by $OI(X)$ (resp. $OD(X)$) the collection of all increasing (resp. decreasing) open sets of X.
- We denote by $QOI(X)$ (resp. $QOD(X)$) the collection of all increasing (resp. decreasing) q-open sets of X.
- We denote by $CM(X)$ the set $CI(X) \cup CD(X)$.
- $VI(A) = \bigcap\{F | F \in CI(X)\ \&\ A \subseteq F\}$.
- $VD(A) = \bigcap\{F | F \in CD(X)\ \&\ A \subseteq F\}$.
- We denote by $Co(A)$ the set $VI(A) \cap VD(A)$.

Example 2.6. Let $X = \mathbb{R} \cup \{-\infty, +\infty\}$ with the natural order and a the closure operator defined, for all $A \subseteq X$, by:

- $a(A) = A \cup \{+\infty\}$ if $-\infty \in A$;
- $a(A) = A$ if $-\infty \notin A$ and A is finite;
- $a(A) = A \cup \{-\infty\}$ if $-\infty \notin A$ and A is infinite.

$(X, a, \leq)$ is an ordered pre-topological space. We have

$$a(\{+\infty\}) = \{+\infty\} \quad \text{and} \quad i(\{+\infty\}) = \{+\infty\}, \quad \text{so} \quad \{+\infty\} \in CI(X).$$

We have

$$d(\{-\infty\}) = \{-\infty\} \quad \text{but} \quad a(\{-\infty\}) = \{-\infty, +\infty\}, \quad \text{so} \quad \{+\infty\} \notin CD(X).$$

We have

$$a(\{0\}) = \{0\} \quad \text{but} \quad i(\{0\}) = \mathbb{R}_+ \cup \{+\infty\}, \quad \text{so} \quad \{0\} \notin CI(X).$$

Lemma 2.7. *Let $(X, a, \leq)$ be an ordered pre-topological space. Then, we have the following equivalence:*

$$Co(x) = Co(y) \Longleftrightarrow VI(x) = VI(y)\ \&\ VD(x) = VD(y).$$

Proof. Suppose that $Co(x) = Co(y)$. Without lost of generality it is sufficient to prove that $VI(x) \subseteq VI(y)$. Let $t \in VI(x)$ and F be an increasing closed set containing y. If $t \notin F$, then x must be outside F ($t \in VI(x)$), which contradicts the fact that $x \in Co(x) = Co(y) \subseteq F$. Then, we deduce that $I(x) \subseteq I(y)$.

The converse implication follows immediately. □

Definition 2.8. Let $(X, a, \leq)$ and $(Y, b, \subseteq)$ be two ordered pre-topological spaces. An increasing map f from X to Y is said to be V-continuous if the inverse image of a closed increasing (resp. decreasing) set of Y is a closed increasing (resp. decreasing) set of X.

Example 2.9. In $\mathbb{R}$, we define a and b, for all $F \subseteq \mathbb{R}$, by:

- If the infimum and the supremum of F exist: $a(F) =]-\alpha(F) - 1, \alpha(F) + 1[$ and $b(F) =]-\alpha(F), \alpha(F)[$ where $\alpha(F) = \max\{|\inf(F)|, |\sup(F)|\}$.
- else, $a(F) = b(F) = \mathbb{R}$.

Then, $(\mathbb{R}, a, \leq)$ and $(\mathbb{R}, b, \leq)$ are two ordered pre-topological spaces. Consider the map f defined from $(\mathbb{R}, a, \leq)$ to $(\mathbb{R}, b, \leq)$ by $f(x) = 2$ if $0 \leq x$ and $f(x) = 1$ if $0 \nleq x$.

It is clear that f is an increasing map. The unique monotone closed set of $(\mathbb{R}, b, \leq)$ is $\mathbb{R}$, so f is V-continuous.

We can also remark that $]-1.5, 1.5[$ is a closed set in $(\mathbb{R}, b)$ but $f^{-1}(]-1.5, 1.5[) =]-\infty, 0[$, which is not closed in $(\mathbb{R}, a)$. This example proves that the V-continuity does not imply the continuity between pre-topological spaces without regarding the order.

According to the definition of separation axioms for topological spaces, we introduce the following equivalent definition:

Definition 2.10. Let $(X, a, \leq)$ be an ordered pre-topological space. Then,

(1) $(X, a, \leq)$ is said to be T_0 (resp. QT_0) if, for all distinct two points of X, there exists a monotone open (resp. q-open) set containing one of the points which does not contain the other one.
(2) $(X, a, \leq)$ is said to be T_1 (resp. QT_1) if, for all $x \nleq y$, there exists an increasing open (resp. q-open) set containing x which does not contain y and there exists a decreasing open (resp. q-open) set containing y which does not contain x.
(3) $(X, a, \leq)$ is said to be T_2 (resp. QT_2) if, for all $x \nleq y$, there exists an increasing open (resp. q-open) set containing x disjoint from some decreasing open (resp. q-open) set containing y.

Remark 2.11. The following implications hold:

(1) $T_i \Rightarrow QT_i$ for all $i \in \{0, 1, 2\}$.
(2) $T_i \Rightarrow T_{i-1}$ for all $i \in \{1, 2\}$.
(3) $QT_i \Rightarrow QT_{i-1}$ for all $i \in \{1, 2\}$.

In ordered pre-topological spaces, the complement of an increasing (resp. decreasing) closed set is a decreasing (resp. increasing) open set. Then, replacing open by closed, increasing by decreasing and decreasing by increasing in Definition 2.10 results in the same definitions.

Proposition 2.12. *Let $(X, a, \leq)$ be an ordered pre-topological space. Then, the following statements are equivalent:*

(a) *$(X, a, \leq)$ is T_0;*
(b) *$VI(x) = VI(y)$ and $VD(x) = VD(y)$ implies $x = y$;*
(c) *$Co(x) = Co(y)$ implies $x = y$.*

Proof. (a) $\Rightarrow$ (b): Let $x \neq y \in X$. Then, there exists a monotone closed set F containing, for example, x which does not contain y. If F is increasing (resp. decreasing), then it contains $CI(x)$ (resp. $CD(x)$) and then $CI(x) \neq CI(y)$ (resp. $CD(x) \neq CD(y)$).

(b) $\Rightarrow$ (a): If $x \neq y \in X$, then, for example, $CI(x) \neq CI(y)$. So, $CI(x)$ (or $CI(y)$) is an increasing closed set containing one of the points and not containing the other one.

(b) $\Leftrightarrow$ (c): It is a direct consequence of Lemma 2.7. ☐

Proposition 2.13. *Let $(X, a, \leq)$ be an ordered pre-topological space. Then, the following statements are equivalent:*

(a) $(X, a, \leq)$ *is* T_1*;*
(b) $i(x)$ *and* $d(x)$ *are closed for all* $x \in X$*.*

Proof. (a) $\Rightarrow$ (b): Let $x \in X$. We have to prove that $a(i(x)) = i(x)$. Suppose that there exists $y \in X$ such that $y \in a(i(x)) \setminus i(x)$. Since $(X, a, \leq)$ is T_1, there exists an increasing closed set F containing x that does not contain y. $x \in F$ implies $i(x) \subseteq F$ and then $a(i(x)) \subseteq a(F) = F$ so that $y \in F$, which is a contradiction. We can deduce that $a(i(x)) = i(x)$.

(b) $\Rightarrow$ (a): Let $x \nleq y$. Then $d(y)^c$ is an increasing open set containing x that does not contain y and $i(x)^c$ is a decreasing open set containing y which does not contain x. □

Example 2.14. Let $X = \{0, 1\}$ and a be the closure operator defined on X by:

- $a(\emptyset) = \emptyset$;
- $a(\{0\}) = \{0\}$;
- $a(\{1\}) = X$;
- $a(X) = X$.

$(X, a, \leq)$ is an ordered pre-topological space where $\leq$ is the order induced from $\mathbb{N}$. So, we have:

(1) $CI(X) = \{X\}$;
(2) $CD(X) = \{\{0\}, X\}$.

Then, $(X, a, \leq)$ is T_0 but not T_1.

Ordered pre-topological spaces with increasing V-continuous functions form a category called **OVPT**. Its full subcategory of an ordered topological space of type T_0 will be denoted by $\mathbf{OVPT}_0$.

3. Reflection construction

The main goal of this section, and the paper, is to construct the T_0-reflection in **OVPT**.

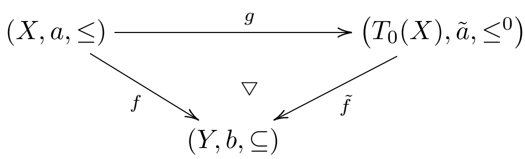
By Maclane [16], to show that the full subcategory $\mathbf{OVPT}_0$ is reflective in **OVPT**, it will be sufficient to prove that, for each $(X, a, \leq) \in \mathbf{OVPT}$, there exists an object $(T_0(X), \tilde{a}, \leq^0) \in \mathbf{OVPT}_0$ and an arrow

$$g : (X, a, \leq) \to (T_0(X), \tilde{a}, \leq^0)$$

such that, for every $(Y, b, \subseteq) \in \mathbf{OVPT}_0$ and each increasing V-continuous map $f : (X, a, \leq) \to (Y, b, \subseteq)$, there exists an increasing V-continuous map

$$\tilde{f} : (T_0(X), \tilde{a}, \leq^0) \to (Y, b, \subseteq)$$

rendering commutative the following diagram:



Let $(X, a, \leq)$ be an ordered pre-topological space. The relation defined on X by $x \sim y$ if and only if $VI(x) = VI(y)$ and $VD(x) = VD(y)$ is an equivalence relation. The quotient set of this equivalence relation is denoted by $X/\sim$ and μ_X denotes the canonical surjection from X onto $X/\sim$.

Let $\tilde{a}$ be the map defined on $\mathcal{P}(X/\sim)$ to itself by

$$\tilde{a}(A) = \mu_X(a(\mu_X^{-1}(A))).$$

Proposition 3.1. *$(X/\sim, \tilde{a})$ is a pre-topological space of type V.*

Proof. One can easily see that $\tilde{a}$ satisfies the conditions in Definition 2.1. □

In the ordered pre-topological space $(X/\sim, \tilde{a})$, we will take the finite step order $\leq^0$, which is defined by: $\overline{z_0} \leq^0 \overline{z_n}$ if and only if:

$$\exists\ \overline{z_1}, \dots, \overline{z_{n-1}} \quad \text{and} \quad \exists\ z_i^{'}, z_i^{*} \in \overline{z_i} \quad (i = 0, 1, \dots, n)$$
$$\text{with} \quad z_i^{'} \leq z_{i+1}^{*} \quad \forall i = 0, 1, \dots, n-1. \tag{3.1}$$

Proposition 3.2. (1) *If $A \in CI(X)$, then $\mu_X(A) \in CI(X/\sim)$.*
(2) *If $B \in CI(X/\sim)$, then $\mu_X^{-1}(B) \in CI(X)$.*

Proof. We start by proving that $A \in CI(X)$ if and only if $\mu_X^{-1}(\mu_X(A)) = A$. The inclusion $A \subseteq \mu_X^{-1}(\mu_X(A))$ is straightforward. Conversely, let $y \in \mu_X^{-1}(\mu_X(A))$ and $x \in A$ such that $\mu_X(x) = \mu_X(y)$. Since A is increasing closed, $CI(x) \subseteq A$ and $y \in CI(y) = CI(x) \subseteq A$. This fact proves that $A \in CI(X)$ implies $\mu_X^{-1}(\mu_X(A)) = A$. The converse implication is easy to prove.

(1) Suppose that $A \in CI(X)$. The equality $\mu_X^{-1}(\mu_X(A)) = A$ implies

$$\tilde{a}(\mu_X(A)) = \mu_X(a(\mu_X^{-1}(\mu_X(A)))) = \mu_X(a(A)) = \mu_X(A).$$

Then, $\mu_X(A)$ is V-closed. The definition of the finite step order implies the increase of $\mu_X(A)$. Thus, $\mu_X(A) \in CI(X/\sim)$.

(2) Let $B \in CI(X/\sim)$. We have $\mu_X^{-1}(B) \subseteq a(\mu_X^{-1}(B))$. On the other hand,

$$a(\mu_X^{-1}(B)) \subseteq \mu_X^{-1}(\mu_X(a(\mu_X^{-1}(B)))) = \mu_X^{-1}(\tilde{a}(B)) = \mu_X^{-1}(B).$$

Thus, $a(\mu_X^{-1}(B)) = \mu_X^{-1}(B)$ and then $\mu_X^{-1}(B) \in C(X)$. Let $x \in \mu_X^{-1}(B)$ and $y \in X$ such that $x \leq y$. So, $\bar{x} \in B$ and $\bar{x} \leq^0 \bar{y}$, which proves that $\bar{y} \in B$ because B is increasing, $y \in \mu_X^{-1}(B)$ and, finally, $\mu_X^{-1}(B) \in CI(X)$.

□

Corollary 3.3. *$\mu_X : (X, a) \to (X/\sim, \tilde{a})$ is an increasing V-continuous map.*

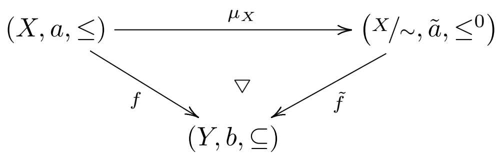
Theorem 3.4. *$(X/\sim, \tilde{a}, \leq^0)$ is T_0.*

Proof. Let $\mu_X(x) \neq \mu_X(y) \in X/\sim$. There exists an increasing V-closed set A containing, for example, x which does not contain y. Thus, $\mu_X(A)$ is an increasing closed set of $X/\sim$ containing $\mu_X(x)$ which does not contain $\mu_X(y)$. This proves that $(X/\sim, \tilde{a}, \leq^0)$ is T_0. □

Theorem 3.5. $\mathbf{OVPT}_0$ *is reflective in* **OVPT**.

Proof. It suffices to prove that, for any ordered pre-topological space $(X, a, \leq)$, $(X/{\sim}, \tilde{a}, \leq^0)$ is the T_0-reflection of $(X, a, \leq)$.

For this, using the characterization given by MacLane, we must prove that, for every $(Y, b, \subseteq)$ which is T_0 and every increasing V-continuous map from $(X, a, \leq)$ to $(Y, b, \subseteq)$, there exists a unique increasing V-continuous map $\tilde{f}$ rendering the following diagram commutative:



Uniqueness: Clearly, if $\tilde{f}$ exists, then it is unique and naturally defined by $\tilde{f}\,(\mu_X(x)) = f\,(x)$.

$\tilde{f}$ *is well defined:* Suppose $x, y \in X$ and $\mu_X(x) = \mu_X(y)$. If $f\,(x) \neq f\,(y)$ and since $(Y, b, \subseteq)$ is T_0, then there exists a monotone closed set A which contains for example $f(x)$ which does not contain $f(y)$. So, $f^{-1}(A)$ is a monotone closed set which contains x and contains y. This is a contradiction.

$\tilde{f}$ *is V-continuous:* Let F be an increasing closed set of $(Y, b, \subseteq)$. We have $\mu_X^{-1}\left(\tilde{f}^{-1}\,(F)\right) = f^{-1}\,(F) \in CI\,(X)$, then, by Proposition 3.2, we have

$$\mu_X\left(\mu_X^{-1}(\tilde{f}^{-1}\,(F))\right) = \tilde{f}^{-1}\,(F) \in CI\,({}^{X}\!/\!{\sim}).$$

Then, $\tilde{f}$ is V-continuous.

$\tilde{f}$ *is increasing:* Let $\bar{x} \leq^0 \bar{y} \in {}^{X}\!/\!{\sim}$. Using the fact that $\overline{z_i'} = \overline{z_i^*}$ implies $f\,(z_i') = f\,(z_i^*)$, we can see that $f\,(x) \subseteq f\,(z_1) \subseteq \cdots \subseteq f\,(z_{n-1}) \subseteq f\,(y)$. So, $\tilde{f}\,(\bar{x}) \subseteq \tilde{f}\,(\bar{y})$ and $\tilde{f}$ is increasing. □

In the category **Top**, an object is said to be $T_{(i,j)}$ if its T_i-reflection is a T_j-space ($i \in \{0, 1, 2\}$). This definition can be found in [3]. Similarly, in our category, **OVPT** is said to be an ordered $T_{(0,1)}$ if ${}^{X}\!/\!{\sim}$ is T_1. Our goal now is to characterize those objects. First, let us introduce the following definition.

Definition 3.6. Let $(X, a, \leq)$ be an ordered pre-topological space. The relative order of $(X, a, \leq)$ is denoted by $\preceq_{(X,a,\leq)}$ ($\preceq$ for short) and defined on X by:

$x \preceq y$ if and only if there exist $k_i, k_i', k_i^* \in X$ $(0 \leq i \leq n)$ satisfying

- $Co\,(k_0) = Co\,(x)$;
- $Co\,(k_n) = Co\,(y)$;
- $Co\,(k_i) = Co\,(k_i') = Co\,(k_i^*)\ \forall\, 0 \leq i \leq n$;
- $k_i' \leq k_{i+1}^*\ \forall\, 0 \leq i < n$.

Theorem 3.7. *Let $(X, a, \leq)$ be an ordered pre-topological space. Then, the following statements are equivalent:*

(a) $\left({}^{X}\!/\!{\sim}, \tilde{a}, \leq^0\right)$ *is* T_1*;*

(b) $x \npreceq y$ *implies that there exist* $O_1 \in OI\,(X)$ *and* $O_2 \in OD\,(X)$ *such that* $x \in O_1 \setminus O_2$ *and* $y \in O_2 \setminus O_1$*;*

(c)

$$i(T_x) = \bigcap\{O \mid O \in OI(X) \;\&\; x \in O\}$$

and

$$d(T_x) = \bigcap\{O \mid O \in OD(X) \;\&\; x \in O\},$$

where $T_x = \{y \in X \mid Co(y) = Co(x)\}$.

Proof. (a) ⇔ (b): It is sufficient to see that $x \preceq y$ if and only if $\bar{x} \leq^0 \bar{y}$ and $O \in OI(X)$ (resp. $O \in OD(X)$) if and only if $\mu_X(O) \in OI(X/\sim)$ (resp. $\mu_X(O) \in OI(X/\sim)$).

(a) ⇒ (c): If $z \in i(T_x)$, then there exists $y \in T_x$ such that $y \leq z$. Now, let $O \in OI(X)$ such that $x \in O$. Since $y \in T_x$, $y \in O$ and $z \in O$, which proves that $i(T_x) \subseteq \bigcap\{O \mid O \in OI(X) \;\&\; x \in O\}$.

Conversely, let $y \in \bigcap\{O \mid O \in OI(X) \;\&\; x \in O\}$. Suppose that $y \notin i(T_x)$. By definition of $\leq^0$, we could see that $\bar{x} \not\leq^0 \bar{y}$ in $X/\sim$. Since $(X/\sim, \tilde{a}, \leq^0)$ is T_1, there exists $\widetilde{O} \in OI(X/\sim)$ such that $\bar{x} \in \widetilde{O}$ and $\bar{y} \notin \widetilde{O}$. So, $\mu_X^{-1}(\widetilde{O}) \in OI(X)$, $x \in \mu_X^{-1}(\widetilde{O})$ and $y \notin \mu_X^{-1}(\widetilde{O})$, which is a contradiction. We can deduce that $y \in i(T_x)$ and

$$\bigcap\{O \mid O \in OI(X) \;\&\; x \in O\} \subseteq i(T_x).$$

(c) ⇒ (a): Let $\bar{x}, \bar{y} \in X/\sim$ such that $\bar{x} \not\leq^0 \bar{y}$. Then, $y \notin i(T_x)$ and $x \notin d(T_y)$. So, there exists $O_1 \in OI(X)$ and $O_2 \in OD(X)$ such that $x \in O_1 \setminus O_2$ and $y \in O_2 \setminus O_1$. This implies $\mu_X(O_1) \in OI(X/\sim)$, $\mu_X(O_2) \in OD(X/\sim)$, $\bar{x} \in \mu_X(O_1) \setminus \mu_X(O_2)$ and $\bar{y} \in \mu_X(O_2) \setminus \mu_X(O_1)$. We conclude that $(X/\sim, \tilde{a}, \leq^0)$ is T_1. □

4. Orthogonality

A morphism $f : A \longrightarrow B$ and an object X in a category C are called orthogonal [11], if the mapping $hom_C(f; X) : hom_C(B; X) \longrightarrow hom_C(A; X)$ that takes g to gf is bijective. For a class of morphisms Σ (resp., a class of objects D), we denote by $\Sigma^\perp$ the class of objects orthogonal to every f in Σ (resp., by $D^\perp$ the class of morphisms orthogonal to all X in D) [11].

The orthogonality class of morphisms $D^\perp$ associated with a reflective subcategory D of a category C satisfies the following identity $D^{\perp\perp} = D$ [2, Proposition 2.6]. Thus, it is of interest to give explicitly the class $D^\perp$. Note also that, if $I : D \longrightarrow C$ is the inclusion functor and $F : C \longrightarrow D$ is a left adjoint functor of I, then the class $D^\perp$ is the collection of all morphisms of C rendered invertible by the functor F (i.e. $D^\perp = \{f \in hom_C : F(f)$ is an isomorphism of $D\}$) [2, Proposition 2.3].

This section is devoted to the study of the orthogonal class $\mathbf{OVPT_0^\perp}$; hence, we will give a characterization of morphisms rendered invertible by the functor T_0 in the category **OVPT**.

Firstly, in an ordered pre-topological space $(X, a, \leq)$, a subset A of X is said to be saturated if we have the following implication:

$$x \in A \Rightarrow \{y \in X \mid Co(x) = Co(y)\} \subseteq A.$$

We denote by $CS(X)$ the set of all saturated subsets of X.

The next definition presents an equivalent definition of quasi-homeomorphism given by Grothendiek in [12].

Definition 4.1. Let $f : (X, a, \leq) \longrightarrow (Y, b, \subseteq)$ be an increasing V-continuous map. f is said to be a quasi-isomorphism in **OVPT** if the map

$$\begin{array}{rccc} \varphi_f : & CS(Y) & \longrightarrow & CS(X) \\ & A & \longmapsto & f^{-1}(A) \end{array}$$

is bijective.

Example 4.2. (1) $\mu_X : (X, a, \leq) \longrightarrow ({}^{X}\!/\!\sim, \tilde{a}, \leq^0)$ is a quasi-isomorphism in **OVPT**.
(2) Every isomorphism in **OVPT** is a quasi-isomorphism.

The following result is immediate.

Proposition 4.3. *Let* $f : (X, a, \leq) \longrightarrow (Y, b, \subseteq)$, $g : (Y, b, \subseteq) \longrightarrow (Z, c, \preceq)$ *be two increasing V-continuous maps and* $h = g \circ f$. *Then, the set* $\{f, g, h\}$ *cannot contain exactly two quasi-isomorphisms in* **OVPT**.

Proposition 4.4. *Let* $f : (X, a, \leq) \longrightarrow (Y, b, \subseteq)$ *be a quasi-isomorphism in* **OVPT**. *If* $(X, a, \leq)$ *is* T_0, *then* f *is one to one.*

Proof. Let $x, y \in X$ such that $f(x) = f(y)$. Suppose that $x \neq y$. Since $(X, a, \leq)$ is T_0, there exists $A \in CM(X)$ containing for example x and not y. Since f is a quasi-isomorphism, there is $T \in CS(Y)$ such that $f^{-1}(T) = A$ and $f(x) \in T$ and $f(y) \notin T$, which is a contradiction. □

Definition 4.5. Let $f : (X, a, \leq) \longrightarrow (Y, b, \subseteq)$ be an increasing V-continuous map.
(1) f is said to be V-one-to-one if, for all $x, y \in X$, we have $Co(f(x)) = Co(f(y))$ implies $Co(x) = Co(y)$.
(2) f is said to be V-onto if, for every $y \in Y$, there exists $x \in X$ such that $Co(f(x)) = Co(y)$.
(3) f is called V-bijective if it is both V-one-to-one and V-onto.

Remarks 4.6. (1) If f is one-to-one (resp. onto), then f is V-one-to-one (resp. V-onto).
(2) If $(X, a, \leq)$ is not T_0, then $\mu_X : (X, a, \leq) \longrightarrow ({}^{X}\!/\!\sim, \tilde{a}, \leq^0)$ is V-one-to-one but not one-to-one.

Let $f : (X, a, \leq) \longrightarrow (Y, b, \subseteq)$ be an increasing V-continuous map. We denote by $T_0(f)$ the map defined on $({}^{X}\!/\!\sim, \tilde{a}, \leq^0)$ to $({}^{Y}\!/\!\sim, \tilde{b}, \subseteq^0)$ rendering commutative the following diagram:

$$\begin{array}{ccc} (X, a, \leq) & \xrightarrow{\ f\ } & (Y, b, \subseteq) \\ {\scriptstyle \mu_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \mu_Y} \\ ({}^{X}\!/\!\sim, \tilde{a}, \leq^0) & \xrightarrow[T_0(f)]{} & ({}^{Y}\!/\!\sim, \tilde{b}, \subseteq^0) \end{array}$$

Lemma 4.7. *We have the following equivalences:*

(1) *f is V-one-to-one if and only if $T_0(f)$ is one-to-one;*
(2) *f is V-onto if and only if $T_0(f)$ is onto;*
(3) *f is V-bijective if and only if $T_0(f)$ is bijective.*

Definition 4.8. Let $f:(X,a,\leq)\longrightarrow(Y,b,\subseteq)$ be an increasing V-continuous map. f is said to be V_0-increasing if $y\subseteq z\in Y$ implies that there exist $a,b\in X$ such that $a\preceq b$, $Co(f(a))=Co(y)$ and $Co(f(b))=Co(z)$.

Theorem 4.9. *Let $f:(X,a,\leq)\longrightarrow(Y,b,\subseteq)$ be an increasing V-continuous map. Then, the following statements are equivalent:*

(a) *f is a V-onto V_0-increasing quasi-isomorphism in* **OVPT**.
(b) *$T_0(f)$ is an isomorphism in* **OVPT**.

Proof. (b) $\Rightarrow$ (a): μ_X,μ_Y and $T_0(f)$ are quasi-isomorphisms and we have $T_0(f)\circ\mu_X=\mu_Y\circ f$. Then, by Proposition 4.3, f is a quasi-isomorphism.

Since $T_0(f)$ is onto, f is V-onto by Lemma 4.7.

Let $y_1\subseteq y_2\in Y$. Since f is V-onto, there exists $x_1,x_2\in X$ such that $\overline{f(x_1)}=\overline{y_1}$ and $\overline{f(x_2)}=\overline{y_2}$. Since μ_Y is increasing, we have $\overline{f(x_1)}\subseteq^0\overline{f(x_2)}$, equivalently, $T_0(f)(\overline{x_1})\subseteq^0 T_0(f)(\overline{x_2})$ so that $\overline{x_1}\leq^0\overline{x_2}$. By the definition of $\leq^0$, there exist $k_i,k_i',k_i^*\in X$ $(0\leq i\leq n)$ satisfying

$$\begin{aligned}&Co(k_0)=Co(x_1)\\&Co(k_n)=Co(x_2)\\&Co(k_i)=Co(k_i')=Co(k_i^*)\ \forall\, 0\leq i\leq n\\&k_i'\leq k_i^*\ \forall\, 0\leq i<n.\end{aligned}$$

Finally, $t_0=x_1,t_n=x_n$, $t_i=k_i$ for $0<i<n$ and $t_i'=k_i'$, $t_i^*=k_i^*$ for $0\leq i\leq n$ satisfy the conditions in Definition 4.8.

We deduce that f is V_0-increasing.

$(a)\Rightarrow(b)$: $T_0(f)$ is V-continuous because μ_X,μ_Y and $T_0(f)$ are V-continuous and $T_0(f)\circ\mu_X=\mu_Y\circ f$.

Let $\bar{x}\leq^0\bar{y}$ and z_i',z_i^* satisfy the condition in (3.1).

We can see that $f(z_i'),f(z_i^*)$ satisfy $\overline{f(z_i')}=\overline{f(z_i^*)}$ and $f(z_i')\subseteq f\left(z_{i+1}^*\right)$, which is sufficient to prove that $\overline{f(x)}\subseteq^0\overline{f(y)}$, or equivalently $T_0(f)(\bar{x})\subseteq^0 T_0(f)(\bar{y})$. We can deduce that $T_0(f)$ is increasing.

We know that $T_0(f)$ is a quasi-isomorphism because μ_X,μ_Y and f are quasi-isomorphisms and $T_0(f)\circ\mu_X=\mu_Y\circ f$. Then, by Proposition 4.4, $T_0(f)$ is one-to-one.

Since f is V-onto, $T_0(f)$ is onto by Lemma 4.7.

$T_0(f)^{-1}$ is V-continuous. Let $U\in CI\left({}^{X}/_{\sim}\right)$, then $\mu_X^{-1}(U)\in CS(X)$. Since f is a quasi-isomorphism, there exists $T\in CS(Y)$ such that $f^{-1}(T)=\mu_X^{-1}(U)$. Finally, $\mu_Y(T)\in CI\left({}^{Y}/_{\sim}\right)$ and

$$\begin{aligned}\mu_Y(T)&=\mu_Y\left(f\left(f^{-1}(T)\right)\right)\\&=T_0(f)\left(\mu_X\left(\mu_X^{-1}(U)\right)\right)\\&=T_0(f)(U),\end{aligned}$$

which proves that $T_0(f)^{-1}$ is V-continuous.

$T_0(f)^{-1}$ is increasing because f is V_0-increasing. □

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