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THE CAUCHY COMPLETION OF A SYMMETRIC B-UNIFORM FILTER SPACE

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Abstract. b-uniform filter spaces are an appropriate tool for studying *convergence* from a higher point of view as demonstrated in two papers by the above mentioned authors. Fundamental properties of spaces in topology such as completeness, precompactness or compactness, respectively, can be newly defined and studied in the realm of these proper constructs. Moreover, we present a kind of completion, called Cauchy completion, which generalizes the corresponding concepts that arose in the past like the simple completion of semi-uniform convergence spaces in the sense of [10], the Wyler completion of separated uniform limit spaces [15], the Hausdorff completion of separated uniform spaces or the λ-completion of filter spaces in the sense of [2], and, last not least, the compactification of proximity spaces ([4], or [12], respectively).

1. Introduction

In this paper, we continue our studies of b-uniform filter spaces. As already shown in [8], these constructs represent a natural generalization of classical convergences like uniform convergences, point-convergences, filtermerotopies and Cauchy spaces and suited set-convergences as well. By the way, the category of b-uniform filter spaces and b-uniformly continuous maps forms a quasitopos in which quotient maps are closed under the formation of arbitrary products [9]. Its important full and isomorphism-closed subcategory of symmetric b-uniform filter spaces is bireflective and bicoreflective in this supercategory, and forms itself a quasitopos. Moreover, the fundamental properties of spaces in topology such as completeness, precompactness and compactness, respectively, can be newly defined and studied in the realm of these proper constructs. Now, our focus lies in the construction of a Cauchy completion for symmetric b-uniform filter spaces in such a manner that a lot of classical constructions turn out to be special cases. Moreover, we consider suitable separation axioms transformed by the mentioned Cauchy completion and, finally, we examine the compactification of a generalized proximity as a case of its Cauchy completion.

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In memoriam Professor Hans-Peter Künzi.

2. Basic notions

Firstly, let us repeat some basic notations.

Definition 2.1. For a set X, a pair $(\mathcal{B}^X, \mu)$ consisting of a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ and a non-empty set $\mu \subset FIL(X \times X)$ of uniform filters is called *b-uniform filter structure* on X, and the triple $(X, \mathcal{B}^X, \mu)$ a *b-uniform filter space*, provided that the following axioms are satisfied:

(buf$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;
(buf$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(buf$_3$) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \mu$;
(buf$_4$) $\mathcal{U} \in \mu$ and $\mathcal{U} \subset \mathcal{U}_1 \in FIL(X \times X)$ imply $\mathcal{U}_1 \in \mu$.

Here, $\underline{P}X$ denotes the power set of X, $FIL(X \times X)$ is the set of all so-called uniform filters (on X) and, for $B \in \mathcal{B}^X \backslash \{\emptyset\}$, $\overset{\bullet}{B} := \{A \subset X : A \supset B\}$.

Given a pair of b-uniform filter spaces $(X, \mathcal{B}^X, \mu_X)$, $(Y, \mathcal{B}^Y, \mu_Y)$, a map $f : X \longrightarrow Y$ is called *b-uniformly continuous*, shortly *buc*, if and only if f satisfies the following conditions:

(buc$_1$) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;
(buc$_2$) $\mathcal{U} \in \mu$ implies $(f \times f)(\mathcal{U}) \in \mu_Y$, where

$$(f \times f)(\mathcal{U}) := \{V \subset Y \times Y : \exists\, U \in \mathcal{U} \text{ s.t. } (f \times f)[U] \subset V\}$$

with

$$\begin{aligned}(f \times f)[U] &:= \{(f \times f)(x_1, x_2) : (x_1, x_2) \in U\} \\ &= \{(f(x_1), f(x_2)) : (x_1, x_2) \in U\}.\end{aligned}$$

By **b-UFIL** we are denoting the category of b-uniform filter spaces and b-uniformly continuous maps.

Remark 2.2. As already proved in [9], **b-UFIL** forms a quasitopos in which quotients are stable under products, that being a nice contribution to the study of strong topological universes in which convergence structures and uniform convergence structures are available [3].

Next, let us introduce some special classes of **b-UFIL** as follows:

Definition 2.3. A b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *symmetric* provided μ satisfies the following condition:

(s) $\mathcal{U} \in \mu$ implies $\mathcal{U}^{-1} \in \mu$, where

$$\mathcal{U}^{-1} := \{R^{-1} : R \in \mathcal{U}\}$$

with

$$R^{-1} := \{(z, x) \in X \times X : (x, z) \in R\}.$$

By **sb-UFIL**, we are denoting the full subcategory of **b-UFIL**, whose objects are the symmetric b-uniform filter spaces.

Remark 2.4. If $\mathcal{B}^X$ is discrete, meaning that $\mathcal{B}^X = \mathcal{D}^X := \{\emptyset\} \cup \{\{x\} : x \in X\}$, then **b-UFIL** and **PUCONV**, the category of preuniform convergence spaces and uniformly continuous maps in the sense of [10], are isomorphic. In addition,

sb-UFIL and **SUCONV**, the category of semi-uniform convergence spaces and uniformly continuous maps, are isomorphic too. Moreover, **sb-UFIL** also forms a strong topological universe in the sense of [10] (see [9]).

Definition 2.5. We call a b-uniform filter space $(X, \mathcal{B}^X, \mu)$ a *b-uniform limit space*, provided μ satisfies the following condition:

(bul) $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ implies $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mu$.

By **b-ULIM**, we are denoting the corresponding full subcategory of **b-UFIL** and by **sb-ULIM** its full subconstruct of symmetric b-uniform limit spaces.

Remark 2.6. If $\mathcal{B}^X$ is discrete, then **b-ULIM** and **PULIM** or **sb-ULIM** and **SULIM**, in the sense of [1], are isomorphic.

Definition 2.7. We call a b-uniform limit space $(X, \mathcal{B}^X, \mu)$ *b-uniform net space* provided μ satisfies the following condition:

(bun) $\mathcal{U}_1, \mathcal{U}_2 \in \mu$ implies $\mathcal{U}_1 \circ \mathcal{U}_2 \in \mu$, where $\mathcal{U}_1 \circ \mathcal{U}_2$ denotes the filter generated by the base

$$\{R_1 \circ R_2 : R_1 \in \mathcal{U}_1, R_2 \in \mathcal{U}_2\}$$

with

$$R_1 \circ R_2 := \{(x, z) \in X \times X : \exists y \in X \text{ s.t. } (x, y) \in R_2 \text{ and } (y, z) \in R_1\}.$$

By **b-UNET**, we are denoting the corresponding full subcategory of **b-ULIM** and by **sb-UNET** its full subconstruct of symmetric b-uniform net spaces.

Remark 2.8. If $\mathcal{B}^X$ is discrete, then **b-UNET** and **QULIM** or **sb-UNET** and **ULIM**, in the sense of Behling [1], are isomorphic.

Finally, in this context, let us mention the following important property:

Definition 2.9. A b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *generated* provided μ satisfies the following condition:

(g) $\exists\, \mathcal{U} \in \mu$ s.t. $\forall\, \mathcal{V} \in \mu,\ \mathcal{U} \subset \mathcal{V}$.

Then, $\mathcal{U}$ is called the *generator* of μ, because it is *uniquely* determined.

Remark 2.10. Firstly, let us note that condition (g) is equivalent to the following one:

$$\bigcap\{\mathcal{U} : \mathcal{U} \in \mu\} \in \mu.$$

If $\mathcal{B}^X$ is discrete, then the full subcategory **gb-UFIL** of **b-UFIL**, whose objects are the generated b-uniform filter spaces, is isomorphic to the full subcategory of **PUCONV**, whose objects are the principal preuniform convergence spaces. But the latter one is isomorphic to **PUNIF**, see [1].

Definition 2.11. By **gb-UNET**, we denote the (full and isomorphism closed) subcategory of **b-UNET**, whose objects are the generated b-uniform net spaces.

Remark 2.12. Next, we offer an interesting hierarchy of b-uniform filter spaces in the sense that each of the constructs in the following list

b-UFIL $\supset$ **b-ULIM** $\supset$ **b-UNET** $\supset$ **gb-UNET**

is a bireflective (full and isomorphism-closed) subconstruct of the preceding ones.

In fact, let $(X, \mathcal{B}^X, \mu)$ be a b-uniform filter space. Then, by setting $\mu^l := \{\mathcal{U} \in FIL(X \times X) : \exists\, \mathcal{U}_1, \ldots, \mathcal{U}_n \in \mu \text{ s.t. } \mathcal{U} \supset \cap\{\mathcal{U}_i : i \in \{1, \ldots, n\}\}$, we obtain a b-uniform limit space $(X, \mathcal{B}^X, \mu^l)$ such that $1_X : (X, \mathcal{B}^X, \mu) \longrightarrow (X, \mathcal{B}^X, \mu^l)$ is the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **b-ULIM**.

If $(X, \mathcal{B}^X, \mu)$ is a symmetric b-uniform filter space, then the b-uniform limit space $(X, \mathcal{B}^X, \mu^l)$ is also symmetric; thus, it offers a corresponding bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **sb-ULIM**.

Now, let $(X, \mathcal{B}^X, \mu)$ be a b-uniform limit space, then $1_X : (X, \mathcal{B}^X, \mu) \longrightarrow (X, \mathcal{B}^X, (\mu^n)^l)$ is the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **b-UNET**, provided that

$$\mu^n := \{\mathcal{U} \in FIL(X \times X) : \exists\, \mathcal{U}_1, \ldots, \mathcal{U}_n \in \mu \text{ s.t. } \mathcal{U} \supset \mathcal{U}_1 \circ \cdots \circ \mathcal{U}_n\}.$$

If $(X, \mathcal{B}^X, \mu)$ is a symmetric b-uniform limit space, then the b-uniform net space $(X, \mathcal{B}^X, (\mu^n)^l)$ is also symmetric; thus, it offers a corresponding bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **sb-UNET**. In fact, let $\mathcal{U} \in (\mu^n)^l$; hence, we can find $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mu^n$ with $\mathcal{U} \supset \cap\{\mathcal{U}_i : i \in \{1, \ldots, n\}\}$. Thus, for each $i \in \{1, \ldots, n\}$, there exist $\mathcal{V}_i^1, \ldots, \mathcal{V}_i^m \in \mu$ with $\mathcal{U}_i \supset \mathcal{V}_i^1 \circ \cdots \circ \mathcal{V}_i^m$, and consequently $(\mathcal{V}_i^1)^{-1}, \ldots, (\mathcal{V}_i^m)^{-1} \in \mu$ follows. But

$$\mathcal{U}_i^{-1} \supset (\mathcal{V}_i^{-1} \circ \cdots \circ \mathcal{V}_i^m)^{-1} = (\mathcal{V}_i^m)^{-1} \circ \cdots \circ (\mathcal{V}_i^1)^{-1}$$

for each $i \in \{1, \ldots, n\}$ implying $\mathcal{U}_1^{-1}, \ldots, \mathcal{U}_n^{-1} \in \mu^n$, and $\mathcal{U}^{-1} \in (\mu^n)^l$ results by applying the hypothesis.

Now, finally, for a set X, a pair $(\mathcal{B}^X, \mathcal{U})$ consisting of a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ and an uniform filter $\mathcal{U} \in FIL(X \times X)$ is called *b-quasiuniformity* for X, and the triple $(X, \mathcal{B}^X, \mathcal{U})$ *b-quasiuniform space* provided that following axioms are satisfied:

(b-QU$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;
(b-QU$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(b-QU$_3$) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $B \times B \subset \cap\{R \subset X \times X : R \in \mathcal{U}\}$;
(b-QU$_4$) $R \in \mathcal{U}$ implies the existence of $U \in \mathcal{U}$ with $U \circ U \subset R$.

As easily seen, for a $\underline{B}$-set $\mathcal{B}^X$ and for each $B \in \mathcal{B}^X \backslash \{\emptyset\}$, the pair $(\mathcal{B}^X, \dot{B} \times \dot{B})$ defines a b-quasiuniformity (for X). Here, a $\underline{\mathrm{B}}$-set $\mathcal{B}^X$ is a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ satisfying the conditions (b-QU$_1$) and (b-QU$_2$).

Now, returning to our former list, let $(X, \mathcal{B}^X, \mu)$ be a b-uniform net space. By $(\mathcal{B}^X, Q^\mu)$, we are denoting the finest b-quasiuniformity for X which is coarser than each $(\mathcal{B}^X, \mathcal{U})$, $\mathcal{U} \in \mu$. Then, we put:

$$\mu^g := \{\mathcal{V} \in FIL(X \times X) : \mathcal{V} \supset Q^\mu\}.$$

Consequently, $1_X : (X, \mathcal{B}^X, \mu) \longrightarrow (X, \mathcal{B}^X, \mu^g)$ is the bireflection of $(X, \mathcal{B}^X, \mu)$ with respect to **gb-UNET**. Obviously, we can note that $(\mathcal{B}^X, Q^\mu)$ is the initial b-quasiuniformity with respect to $((X, 1_X^{\mathcal{V}}, (X, \mathcal{B}^X, \mathcal{V}))_{\mathcal{V} \in I}$, where $1_X^{\mathcal{V}} : X \longrightarrow (X, \mathcal{B}^X, \mathcal{V})$ is the identity map for each $\mathcal{V} \in I$ and $I := \{\mathcal{V} \subset \underline{P}(X \times X) : (\mathcal{B}^X, \mathcal{V})$ is b-quasiuniformity for X such that $\mathcal{V} \subset \mathcal{U}$ for each $\mathcal{U} \in \mu\}$.

We are closing our remark by mentioning the following important fact: Let us call a b-quasiuniformity $(\mathcal{B}^X, \mathcal{U})$ (for X) b-uniformity if and only if, in addition, $(\mathcal{B}^X, \mathcal{U})$ satisfies the following property:

(b-QU$_5$) $R \in \mathcal{U}$ implies $R^{-1} \in \mathcal{U}$.

Then, $(\mathcal{B}^X, \mathcal{U})$ is a b-uniformity for X if and only if $(\mathcal{B}^X, \{\mathcal{V} \in FIL(X \times X) : \mathcal{V} \supset \mathcal{U}\})$ is a symmetric b-uniform net structure (on X). If $\mathcal{B}^X$ is discrete, then b-uniformities and uniformities in the usual sense are essentially the same, and such a kind of space can also be described by its associated generated symmetric b-uniform net spaces.

3. B-filter spaces and set-convergence

Now, in approaching our goal of constructing a *Cauchy completion* for a symmetric b-uniform filter space with the announced applications, we need some additional concepts as follows:

Definition 3.1. For a set X, a pair $(\mathcal{B}^X, \tau)$ consisting of a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ and a non-empty subset $\tau \subset FIL(X)$ is called a *b-filter structure* on X, and the triple $(X, \mathcal{B}^X, \tau)$ a *b-filter space*, provided that the following axioms are satisfied:

(bf$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;
(bf$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(bf$_3$) $B \in \mathcal{B}^X \backslash \{\emptyset\}$ implies $\overset{\bullet}{B} \in \tau$;
(bf$_4$) $\mathcal{F} \in \tau$ and $\mathcal{F} \subset \mathcal{F}_1 \in FIL(X)$ imply $\mathcal{F}_1 \in \tau$.

Given a pair of b-filter spaces $(X, \mathcal{B}^X, \tau_X), (Y, \mathcal{B}^Y, \tau_Y)$, a map $f : X \longrightarrow Y$ is called *Cauchy continuous* if and only if f satisfies the following conditions:

(Chyc$_1$) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;
(Chyc$_2$) $f(\mathcal{F}) \in \tau_Y$ whenever $\mathcal{F} \in \tau_X$.

By **b-FIL**, we are denoting the category of b-filter spaces and Cauchy continuous maps.

As shown in our paper [8], **b-FIL** is isomorphic to the category **MERb-UFIL**, the full subcategory of **b-UFIL**, whose objects are the merotopic b-uniform filter spaces. Here, a b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *merotopic* if and only if μ satisfies the following condition:

(mer) $\mathcal{U} \in \mu$ implies the existence of $\mathcal{F} \in FIL(X)$ such that $\mathcal{F} \times \mathcal{F} \subset \mathcal{U}$ and $\mathcal{F} \times \mathcal{F} \in \mu$.

Moreover, we point out that merotopic b-uniform filter spaces are necessarily symmetric. Thus, **MERb-UFIL** can even be regarded as a full subcategory of **sb-UFIL**, and moreover it is bireflective as well as bicoreflective in **sb-UFIL**. Thus, **b-FIL** also forms a strong topological universe [10].

Finally, we point out that a merotopic b-uniform filter space $(X, \mathcal{B}^X, \mu)$ can be regarded as a space, where $(\mathcal{B}^X, \mu)$ is generated by all μ-Cauchy filters. Here, $\mathcal{F} \in FIL(X)$ is called *μ-Cauchy filter* if and only if $\mathcal{F} \times \mathcal{F} \in \mu$. The underlying b-filter space $(X, \mathcal{B}^X, \tau_\mu)$ of a b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is defined by setting:

$$\tau_\mu := \{\mathcal{F} \in FIL(X) : \mathcal{F} \times \mathcal{F} \in \mu\}.$$

On the other hand, every b-filter space $(X, \mathcal{B}^X, \tau)$ induces a set-convergence $(\mathcal{B}^X, q_\tau)$ and therefore a set-convergence space $(X, \mathcal{B}^X, q_\tau)$ by setting:

(1) $\mathcal{F}\ q_\tau\ \emptyset$ if and only if $\mathcal{F} = \underline{P}X$;

(2) $\mathcal{F}\ q_\tau\ B$ if and only if $\mathcal{F}\cap \overset{\bullet}{B}\in \tau$ for every $B \in \mathcal{B}^X\backslash\{\emptyset\}$.

Definition 3.2. Here, for a set X, a pair $(\mathcal{B}^X, q)$ consisting of a non-empty subset $\mathcal{B}^X \subset \underline{P}X$ and a relation $q \subset FIL(X) \times \mathcal{B}^X$ is called a *set-convergence* on X, and the triple $(X, \mathcal{B}^X, q)$ a *set -convergence space* provided that following axioms are satisfied:

(sc$_1$) $B_1 \subset B \in \mathcal{B}^X$ implies $B_1 \in \mathcal{B}^X$;
(sc$_2$) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;
(sc$_3$) $B \in \mathcal{B}^X$ implies $\overset{\bullet}{B}\ q\ B$;
(sc$_4$) $\mathcal{F}\ q\ \emptyset$, $\mathcal{F} \in FIL(X)$ implying $\mathcal{F} = \underline{P}X$;
(sc$_5$) $\mathcal{F}\ q\ B$, $B \in \mathcal{B}^X$ and $\mathcal{F} \subset \mathcal{F}_1 \in FIL(X)$ implying $\mathcal{F}_1\ q\ B$.

Here, the spelling $\mathcal{F}\ q\ B$ means that $(\mathcal{F}, B) \in q$ holds.

Given a pair of set-convergence spaces $(X, \mathcal{B}^X, q_X)$, $(Y, \mathcal{B}^Y, q_Y)$, a map $f : X \longrightarrow Y$ is called *b-continuous*, *bc* in short, if and only if f satisfies the following conditions:

(bc$_1$) $B \in \mathcal{B}^X$ implies $f[B] \in \mathcal{B}^Y$;
(bc$_2$) $B \in \mathcal{B}^X$ and $\mathcal{F}\ q\ B$ imply $f(\mathcal{F})\ q\ f[B]$.

By **SETCONV**, we are denoting the category of set-convergence spaces and b-continuous maps [14].

Definition 3.3. Moreover, we call a set-convergence space $(X, \mathcal{B}^X, q)$ *set-limit space*, provided that the following condition is satisfied:

(lim) $\mathcal{F}_1\ q\ B$ and $\mathcal{F}_2\ q\ B$, $B \in \mathcal{B}^X$, $\mathcal{F}_1, \mathcal{F}_2 \in FIL(X)$ implying $\mathcal{F}_1 \cap \mathcal{F}_2\ q\ B$.

Now, the foregoing having been established, we are coming to the central definition of this paper:

Definition 3.4. A symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *Cauchy complete*, provided that each μ-Cauchy filter $\mathcal{F} \in FIL(X)$ converges in

$$(X, \mathcal{B}^X, q_{\tau_\mu}),$$

which means that $\mathcal{F}\cap \overset{\bullet}{B}\in \tau_\mu$ is valid (see also Definition 3.1).

Here, we note that the set-convergence $(\mathcal{B}^X, q_{\tau_\mu})$ possesses additional properties as follows:

Definition 3.5. A set-convergence $(\mathcal{B}^X, q)$ and the corresponding space

$$(X, \mathcal{B}^X, q)$$

is called

(i) *reordered* if and only if
  (ro) $\emptyset \neq B_1 \subset B \in \mathcal{B}^X$ and $\mathcal{F}\ q\ B$ imply $\mathcal{F}\ q\ B_1$;
(ii) *symmetric* if and only if
  (s) $B \in \mathcal{B}^X\backslash\{\emptyset\}$, $\mathcal{F}\ q\ B$ and $B_1 \subset \cap\mathcal{F}$, $B_1 \in \mathcal{B}^X\backslash\{\emptyset\}$ imply $\mathcal{F}\ q\ B_1$.

Then, a reordered set-convergence space $(X, \mathcal{B}^X, q)$ is called a *KENT set-convergence space* if and only if $(\mathcal{B}^X, q)$ satisfies the following condition:

(k) $B \in \mathcal{B}^X\backslash\{\emptyset\}$ and $\mathcal{F}\ q\ B$ imply $\mathcal{F}\cap \overset{\bullet}{B}\ q\ B$.

Now, in this context, we call a b-filter structure $(\mathcal{B}^X, \tau)$ and the corresponding space $(X, \mathcal{B}^X, \tau)$ *complete*, provided τ satisfies the following condition:

(cpl) $\mathcal{F} \in \tau$ implies the existence of $B \in \mathcal{B}^X \backslash \{\emptyset\}$ s.t. $\mathcal{F} \cap \overset{\bullet}{B} \in \tau$.

By taking the former definition into account, we can state that each Cauchy complete symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is complete with respect to τ_μ.

In continuing our previously created string, we call a symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ a *SYMCONV space*, provided μ satisfies the following condition:

(symc) $\mathcal{U} \in \mu$ implies the existence of $\mathcal{F} \in FIL(X)$ and $B \in \mathcal{B}^X \backslash \{\emptyset\}$ such that $\mathcal{U} \supset (\mathcal{F} \cap \overset{\bullet}{B}) \times (\mathcal{F} \cap \overset{\bullet}{B}) \in \mu$ (see [8]).

This definition, put into words, says that a SYMCONV space is a symmetric b-uniform filter space which is generated by its KENT set-convergent filters.

Remark 3.6. If $(X, \mathcal{B}^X, \mu)$ is a symmetric b-uniform net space, then the underlying symmetric KENT set-convergence relation q_{τ_μ} can also be described by $\mathcal{F}\ q_{\tau_\mu}\ B$ if and only if $\mathcal{F} \times \overset{\bullet}{B} \in \mu$ for each $B \in \mathcal{B}^X \backslash \{\emptyset\}$.

On the other hand, the full subcategory **RO-SETCONV** of **SETCONV**, whose objects are the reordered set-convergence spaces, is isomorphic to the full subcategory **SETb-UFIL** of **b-UFIL**, whose objects are the *setconvergent* b-uniform filter spaces.

Here, a b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *setconvergent*, provided $(\mathcal{B}^X, \mu)$ satisfies the following condition:

(sc) $\mathcal{U} \in \mu$ implies the existence of $B \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\mathcal{F} \in FIL(X)$ s.t. $\overset{\bullet}{B} \times \mathcal{F} \in \mu$ with $\overset{\bullet}{B} \times \mathcal{F} \subset \mathcal{U}$.

Then, for a symmetric b-uniform net space $(X, \mathcal{B}^X, \mu)$, we always get $q_{\tau_\mu} = q_\mu$, where

$$\mathcal{F}\ q_\mu\ \emptyset \quad \text{if and only if} \quad \mathcal{F} = \underline{P}X$$

and

$$\mathcal{F}\ q_\mu\ B \quad \text{if and only if} \quad \overset{\bullet}{B} \times \mathcal{F} \in \mu\ \forall B \in \mathcal{B}^X \backslash \{\emptyset\}.$$

In fact, $\mathcal{F}\ q_\mu\ B$ implies $\overset{\bullet}{B} \times \mathcal{F} \in \mu$; hence, $(\overset{\bullet}{B} \times \mathcal{F})^{-1} \in \mu$ is valid and $\mathcal{F} \times \overset{\bullet}{B} \in \mu$ follows, showing that $\mathcal{F}\ q_{\tau_\mu}\ B$ is true for each $B \in \mathcal{B}^X \backslash \{\emptyset\}$.

The converse can be deduced analogously. To this end, we note, for a set-convergence space $(X, \mathcal{B}^X, q)$ and for $A \in \underline{P}X$, that the *q-closure* of A is defined by setting:

$$cl_q(A) := \{x \in X : \exists\ \mathcal{F} \in FIL(X)\ \text{ s.t. } (\mathcal{F}\ q\ \{x\} \text{ and } A \in sec\mathcal{F})\}.$$

4. On certain separation axioms

A suitable convergence can be obtained by using specific separation properties. So, let us call a set-limit space $(X, \mathcal{B}^X, q)$ *pseudo-T_2* if and only if the existence of a filter $\mathcal{F} \in FIL(X)$ with $\mathcal{F}\ q\ B_1$ and $\mathcal{F}\ q\ B_2$, $B_1, B_2 \in \mathcal{B}^X$ implies $\{\mathcal{H} \in FIL(X) : \mathcal{H}\ q\ B_1\} = \{\mathcal{E} \in FIL(X) : \mathcal{E}\ q\ B_2\}$.

Lemma 4.1. *The underlying symmetric KENT set-convergence space*

$$(X, \mathcal{B}^X, q_{\tau_\mu})$$

of a symmetric b-uniform net space $(X, \mathcal{B}^X, \mu)$ is a pseudo-T_2 set-limit space.

Proof. Without restriction, let $\mathcal{F} \in FIL(X)$ with $\mathcal{F}\ q_{\tau_\mu}\ B_1$ and $\mathcal{F}\ q_{\tau_\mu}\ B_2$, $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$. If $\mathcal{H} \in FIL(X)$ with $\mathcal{H}\ q_{\tau_\mu}\ B_1$, then $\mathcal{H}\ q_{\tau_\mu}\ B_2$ because $\mathcal{H} \times \overset{\bullet}{B}_2 = (\mathcal{F} \times \overset{\bullet}{B}_2) \circ (\overset{\bullet}{B}_1 \times \mathcal{F}) \circ (\mathcal{H} \times \overset{\bullet}{B}_1) \in \mu$ (see Remark 3.5).

On the other hand, $(X, \mathcal{B}^X, q_{\tau_\mu})$ fulfills the property (lim) since

$$(\mathcal{F}_1 \cap \mathcal{F}_2) \times \overset{\bullet}{B} = (\mathcal{F}_1 \times \overset{\bullet}{B}) \cap (\mathcal{F}_2 \times \overset{\bullet}{B})$$

for $\mathcal{F}_1\ q_{\tau_\mu}\ B$, $\mathcal{F}_2\ q_{\tau_\mu}\ B$ and $B \in \mathcal{B}^X$. □

The next three definitions for set-convergences are natural generalizations of the corresponding concepts in general topology.

Definition 4.2. A set-convergence $(\mathcal{B}^X, q)$ is called

(i) T_0 *set-convergence* and the triple $(X, \mathcal{B}^X, q)$ a T_0 *set-convergence space* if and only if

(T$_0$) $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$, $\overset{\bullet}{B}_1\ q\ B_2$ and $\overset{\bullet}{B}_2\ q\ B_1$ implies $B_1 = B_2$;

(ii) T_1 *set-convergence* and the triple$(X, \mathcal{B}^X, q)$ a T_1 *set-convergence space* if and only if

(T$_1$) $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ and $\overset{\bullet}{B}_1\ q\ B_2$ implies $B_1 = B_2$;

(iii) T_2 *set-convergence* and the triple $(X, \mathcal{B}^X, q)$ a T_2 *set-convergence space* if and only if

(T$_2$) $\mathcal{F} \in FIL(X)$, $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\mathcal{F}\ q\ B_1$ and $\mathcal{F}\ q\ B_2$ implies $B_1 = B_2$.

Then, we call a symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ a T_0 *space* (T_1 *space*, T_2 *space*, respectively) if and only if $(X, \mathcal{B}^X, q_{\tau_\mu})$ is a T_0 set-convergence space (T_1 set-convergence space, T_2 set-convergence space, respectively).

Remark 4.3. A neighborhood space $(X, \mathcal{B}^X, \theta)$, especially supertopological space [13], is a T_0-neighborhood space (T_1-neighborhood space, T_2-neighborhood space, respectively) if and only if $(X, \mathcal{B}^X, q_\theta)$ is a T_0 set-convergence space (T_1 set-convergence space, T_2 set-convergence space, respectively), where $\mathcal{F}\ q_\theta\ B$ if and only if $\mathcal{F} \supset \theta(B)$. Here, $\theta(B)$ denotes the neighborhood system of B with respect to θ. If $\mathcal{B}^X$ is discrete, then the definitions of all the above mentioned separation axioms coincide with the usual ones in topology [7].

Here, in particular, we note that a discrete supertopological space $(X, \mathcal{B}^X, \theta)$ is essentially (up to isomorphism) a topological space generated by the well-known *surrounding axioms* of Hausdorff [5].

Lemma 4.4. *For a symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$, the following statements are equivalent:*

(i) $(X, \mathcal{B}^X, \mu)$ *is a T_0-space;*

(ii) $(X, \mathcal{B}^X, \mu)$ *is a T_1-space.*

If $(X, \mathcal{B}^X, \mu)$ is a symmetric b-uniform net space, then each of the above conditions is equivalent to

(iii) $(X, \mathcal{B}^X, \mu)$ *is a* T_2*-space.*

Proof. Evidently, (i) and (ii) are equivalent. Now, let $\mathcal{F} \in FIL(X)$ with $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ such that $\mathcal{F}\ q_{\tau_\mu}\ B_1$ and $\mathcal{F}\ q_{\tau_\mu}\ B_2$ are valid. Hence, $\mathcal{F} \cap \overset{\bullet}{B}\ q_{\tau_\mu}\ B_1$ and $\mathcal{F} \cap \overset{\bullet}{B}\ q_{\tau_\mu}\ B_2$. By applying 4.1., $\mathcal{F} \cap \overset{\bullet}{B_1}\ q_{\tau_\mu}\ B_2$ and $\mathcal{F} \cap \overset{\bullet}{B_2}\ q_{\tau_\mu}\ B_1$ follow. Consequently, $\overset{\bullet}{B_1}\ q_{\tau_\mu}\ B_2$ and $\overset{\bullet}{B_2}\ q_{\tau_\mu}\ B_1$ are valid and imply $B_1 = B_2$ by the hypothesis. □

Finally, we still consider another axiom of separation which is of interest in connection with reordered set-convergence spaces.

Definition 4.5. Let us call a set-convergence space $(X, \mathcal{B}^X, q)$ *point-* T_1 if and only if for each $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\overset{\bullet}{B_1}\ q\ B_2$, $B_1 \cap B_2 \neq \emptyset$.

Lemma 4.6. *For a reordered set-convergence space* $(X, \mathcal{B}^X, q)$*, the following statements are equivalent:*

(i) $(X, \mathcal{B}^X, q)$ *is point-*T_1*;*

(ii) $z \in X$ *and* $B \in \mathcal{B}^X \backslash \{\emptyset\}$ *with* $\overset{\bullet}{z}\ q\ B$ *imply* $\{z\} = B$*;*

(iii) $x, z \in X$ *with* $\overset{\bullet}{x}\ q\ \{z\}$ *implies* $x = z$*.*

Remark 4.7. Note that if $\mathcal{B}^X$ is discrete, then T_1 set-convergence spaces and point-T_1 set-convergence spaces are the same.

Proof of lemma 4.6. To (i) ⇒ (ii): For $z \in X$, $B \in \mathcal{B}^X \backslash \{\emptyset\}$, let $\overset{\bullet}{z}\ q\ B$; hence, $\{z\} \in \mathcal{B}^X \backslash \{\emptyset\}$. Choose $x \in B$, and consequently $\overset{\bullet}{z}\ q\ \{x\}$ is valid because $(\mathcal{B}^X, q)$ is reordered. But then the claim follows by applying the hypothesis.

To (ii) ⇒ (i): Let $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\overset{\bullet}{B_1}\ q\ B_2$. Choose $x_1 \in B_1$; hence, $\overset{\bullet}{x_1}\ q\ B_2$ implies $\{x_1\} = B_2$, and the claim follows.

To (i) ⇒ (iii): Evident.

To (iii) ⇒ (i): Let $B_1, B_2 \in \mathcal{B}^X \backslash \{\emptyset\}$ with $\overset{\bullet}{B_1}\ q\ B_2$. Choose $x_1 \in B_1, x_2 \in B_2$; hence, $\overset{\bullet}{x_1}\ q\ \{x_2\}$ follows, implying $x_1 = x_2$ by applying the hypothesis. □

5. The Cauchy completion

Now, in this section, the question is examined whether every symmetric b-uniform filter space can be *densely* embedded into a Cauchy complete symmetric b-uniform filter space, i.e., whether every symmetric b-uniform filter space has a Cauchy completion.

Definition 5.1. Let $(X, \mathcal{B}^X, \tau)$ be a b-filter space. Then, $\mathcal{F} \in \tau \backslash \{\underline{P}X\}$ is called τ*-independent* if and only if there is no filter $\mathcal{E} \in \tau \backslash \{\underline{P}X\}$ with $\cap \mathcal{E} \neq \emptyset$ and $\mathcal{E} \subset \mathcal{F}$.

Proposition 5.2. *Let* $(X, \mathcal{B}^X, \tau)$ *be a b-filter space and* $\mathcal{F} \in FIL(X)$*. Then, the following statements are equivalent:*

(i) $\mathcal{F}$ *is* τ*-independent;*

(ii) $\mathcal{F}$ *does not converge in the underlying KENT set-convergence space* $(X, \mathcal{B}^X, q_\tau)$ *(see also Definition 3.1).*

Proof. To (i) $\Rightarrow$ (ii): Let $\mathcal{F} \in \tau\backslash\{\underline{P}X\}$ with $\mathcal{F}\ q_\tau\ B$ for some $B \in \mathcal{B}^X\backslash\{\emptyset\}$; hence, $\mathcal{F}\cap \overset{\bullet}{B} \in \tau$ follows. But $\mathcal{F}\cap \overset{\bullet}{B} \subset \mathcal{F}\cap \overset{\bullet}{x}$ for some $x \in B$ implies $\mathcal{F}\cap \overset{\bullet}{x} \in \tau\backslash\{\underline{P}X\}$ with $\mathcal{F}\cap \overset{\bullet}{x} \subset \mathcal{F}$ and $\cap(\mathcal{F}\cap \overset{\bullet}{x}) \neq \emptyset$. Thus, $\mathcal{F}$ is not τ-independent.

To (ii) $\Rightarrow$ (i): If $\mathcal{F} \in \tau\backslash\{\underline{P}X\}$ is not τ-independent, then there exists $\mathcal{F}_0 \in \tau\backslash\{\underline{P}X\}$ with $\cap\mathcal{F}_0 \neq \emptyset$ and $\mathcal{F}_0 \subset \mathcal{F}$. Choose $x \in \cap\mathcal{F}_0$; hence, $\mathcal{F}_0\cap \overset{\bullet}{x} = \mathcal{F}_0 \subset \mathcal{F}\cap \overset{\bullet}{x}$ and $\mathcal{F}\cap \overset{\bullet}{x} \in \tau$ follows implying $\mathcal{F}\ q_\tau\ \{x\}$ with $\{x\} \in \mathcal{B}^X\backslash\{\emptyset\}$. □

Proposition 5.3. *If $(X, \mathcal{B}^X, \tau)$ is a b-filter space, then there is an equivalence relation $\approx$ defined by setting:*

$\mathcal{F} \approx \mathcal{E}$ if and only if there exists finitely many $\mathcal{F}_0, ..., \mathcal{F}_n \in \tau\backslash\{\underline{P}X\}$ with $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_n = \mathcal{E}$ such that $\mathcal{F}_{i-1} \subset sec\mathcal{F}_i$ for each $i \in \{1, ..., n\}$.

Proof. Evident (note that, for $\mathcal{F} \subset \underline{P}X$, $sec\mathcal{F} := \{A \subset X : A \cap F \neq \emptyset\ \forall\ F \in \mathcal{F}\}$). □

Theorem 5.4. *Let $(X, \mathcal{B}^X, \mu)$ be a symmetric b-uniform filter space. We put:*

$X_\mu := \{[\mathcal{F}] : \mathcal{F}$ is τ_μ- independent $\}$, where $[\mathcal{F}]$ denotes the equivalence class of $\mathcal{F}$ with respect to $\approx$, and $X^* := X \cup X_\mu$.

$\mathcal{B}_\mu := \{\{z\} : \exists\ \mathcal{F}\ \tau_\mu$-independent s.t. $z = [\mathcal{F}]\}$ and $\mathcal{B}^{X^*} := \mathcal{B}^X \cup \mathcal{B}_\mu$.

If $i : X \longrightarrow X^*$ denotes the inclusion map, then

$\mu_{X^*} := \{\mathcal{U} \in FIL(X^* \times X^*) : \exists\ \mathcal{V} \in b_{X^*}$ with $\mathcal{U} \supset \mathcal{V}\}$, where

$b_{X^*} := \{(i \times i)(\mathcal{H}) : \mathcal{H} \in \mu\} \bigcup \{(i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) : \mathcal{F}\ \tau_\mu$-independent$\}$.

$(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is a Cauchy complete symmetric b-uniform filter space containing $(X, \mathcal{B}^X, \mu)$ as a dense subspace (i.e., $cl_{q_{\tau_{\mu_{X^*}}}}(X) = X^*$), and the set of all μ_{X^*}-Cauchy filters is generated by $\{i(\mathcal{F}) : \mathcal{F}$ not $\tau_\mu -$ independent$\} \bigcup \{i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\} : \mathcal{F}\ \tau_\mu -$ independent$\} =: b_{\tau_\mu}$.

Proof. Evidently, $(\mathcal{B}^{X^*}, \mu_{X^*})$ satisfies (buf$_1$), (buf$_2$) and (buf$_4$), respectively.

In order to prove (buf$_3$): Let $B \in \mathcal{B}^{X^*}\backslash\{\emptyset\}$. If $B \in \mathcal{B}^X\backslash\{\emptyset\}$, then $\overset{\bullet}{B} \times \overset{\bullet}{B} \in \mu$ implies $(i \times i)(\overset{\bullet}{B} \times \overset{\bullet}{B}) \in b_{X^*}$, and the claim follows. If $B \in \mathcal{B}_\mu\backslash\{\emptyset\}$, $B = \{[\mathcal{F}]\}$ for some $\mathcal{F}\ \tau_\mu$-independent. We put

$$\mathcal{V} := (i(\mathcal{F}) \cap \{D \subset X^* : D \supset B\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : D \supset B\});$$

hence, $\mathcal{V} \in b_{X^*}$ follows. In showing $\overset{\bullet}{B} \times \overset{\bullet}{B} \supset \mathcal{V}$, it remains to verify $i(\mathcal{F}) \cap \{D \subset X^* : D \supset B\} \subset \overset{\bullet}{B}$. But $A \in i(\mathcal{F}) \cap \{D \subset X^* : D \supset B\}$ implies $A \in \overset{\bullet}{B}$, and the claim follows. $(X, \mathcal{B}^{X^*}, \mu_{X^*})$ is symmetric. So, let $\mathcal{U} \in \mu_{X^*}$; hence, we can find $\mathcal{V} \in b_{X^*}$ with $\mathcal{U} \supset \mathcal{V}$. In the first case, if $\mathcal{V} = (i \times i)(\mathcal{H})$ for some $\mathcal{H} \in \mu$, we get $\mathcal{H}^{-1} \in \mu$ with $\mathcal{U}^{-1} \supset \mathcal{V}^{-1} = ((i \times i)(\mathcal{H}))^{-1} \supset (i \times i)(\mathcal{H}^{-1}) \in b_{X^*}$; hence, $\mathcal{U}^{-1} \in \mu_{X^*}$ results. Secondly, if

$$\mathcal{V} = (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}),$$

$\mathcal{F}\ \tau_\mu$-independent, then $\mathcal{U}^{-1} \supset \mathcal{V}^{-1}$ is valid. But

$$((i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}))^{-1}$$
$$= (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\})$$

implies $\mathcal{U}^{-1} \in \mu_{X^*}$.

Evidently, $i : (X, \mathcal{B}^X, \mu) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is b-uniformly continuous. Furthermore, $(X, \mathcal{B}^X, \mu)$ is a b-uniform filter subspace of $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ because, firstly, $i : X \longrightarrow X^*$ implies $\{i[B] : B \in \mathcal{B}^X\} \subset \mathcal{B}^{X^*}$ and $\mathcal{B}_i^X := \{D \cap X : D \in \mathcal{B}^{X^*}\} \subset \mathcal{B}^X$ since $B \in \{D \cap X : D \in \mathcal{B}^{X^*}\}$ implies $B = D \cap X$ for some $D \in \mathcal{B}^{X^*}$. Then, if $D \in \mathcal{B}^X$, $B \in \mathcal{B}^X$ follows. If $D = \{[\mathcal{F}]\}$, $\mathcal{F}$ τ_μ-independent with $[\mathcal{F}] \in X$, $B \in \mathcal{B}^X$ follows, too. If $[\mathcal{F}] \notin X$, then $B = \{[\mathcal{F}]\} \cap X = \emptyset$, and $B \in \mathcal{B}^X$ is also true. Secondly, $\mathcal{H} \in \mu$ implies $(i \times i)(\mathcal{H}) \in \mu_{X^*}$ and

$$\mu_X^i := \{\mathcal{U} \in FIL(X \times X) : (i \times i)(\mathcal{U}) \in \mu_{X^*}\} \subset \mu$$

because $\mathcal{U} \in \mu_X^i$ implies $(i \times i)(\mathcal{U}) \in \mu_X^*$. Hence, we can find $\mathcal{V} \in b_{X^*}$ with $(i \times i)(\mathcal{U}) \supset \mathcal{V}$. If $\mathcal{V} = (i \times i)(\mathcal{H})$ for some $\mathcal{H} \in \mu$, then $(i \times i)(\mathcal{U}) \supset (i \times i)(\mathcal{H})$. But $\mathcal{U} = (i \times i)^{-1}((i \times i)(\mathcal{U})) \supset (i \times i)^{-1}((i \times i)(\mathcal{H})) = \mathcal{H}$, and $\mathcal{U} \in \mu$ follows. If

$$\mathcal{V} = (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}),$$

then $\mathcal{U} \supset (i \times i)^{-1}((i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\})) = i^{-1}(i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times i^{-1}(i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) = \mathcal{F} \times \mathcal{F}$. Since $\mathcal{F} \times \mathcal{F} \in \mu$, $\mathcal{U} \in \mu$ follows. But $(\mathcal{B}_i^X, \mu_X^i)$ is the *coarsest* b-uniform filter structure on X such that $i : (X, \mathcal{B}^X, \mu) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is b-uniformly continuous; hence, the claim follows.

Next, we will show that X is dense in X^*. Therefore, it suffices to verify that the inclusion $X^* \subset cl_{q_{\tau_{\mu^*}}}(X)$ holds. So, let $z \in X^*$. If $z \in X$, then

$$i(\dot{z}) = (i(\dot{z}))\ q_{\tau_{\mu_{X^*}}} \{i(z)\}$$

can be deduced with $X \in sec\ i(\dot{z})$, since $D \in i(\dot{z})$ implies $i(z) \in D \cap i[X] = X$. Consequently, $z \in cl_{q_{\tau_{\mu^*}}}(X)$ follows. If $z = [\mathcal{F}]$, $\mathcal{F}$ τ_μ-independent, then $i(\mathcal{F})\ q_{\tau_{\mu_{X^*}}}\ \{z\}$ because $i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\} \in \tau_{\mu_{X^*}}$. $X \in sec\ i(\mathcal{F})$ since $A \in i(\mathcal{F})$ implies $A \supset i[F]$ for some $F \in \mathcal{F}$. But $\emptyset \notin F$ implies $x \in F \cap X$ for some $x \in F$; hence, $A \cap X \neq \emptyset$ follows. Thus, $z \in cl_{q_{\tau_{\mu^*}}}(X)$.

Before showing that $(X, \mathcal{B}^{X^*}, \mu_{X^*})$ is Cauchy complete, we claim that the set of all proper μ_{X^*}-Cauchy filters is generated by the set $\{i(\mathcal{F}) : \mathcal{F}$ not $\tau_\mu -$ independent$\}$ $\bigcup\{i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\} : \mathcal{F}\ \tau_\mu -$ independent$\} =: b_{\tau_\mu}$, i.e., $\tau_{\mu_{X^*}} := \{\mathcal{E} \in FIL(X^*) \backslash \{\underline{P}X^*\} : \exists\ \mathcal{D} \in b_{\tau_\mu}\ \mathcal{E} \supset \mathcal{D}\} =: \tau^*$.

To " $\subset$ ": $\mathcal{E} \in \tau_{\mu_{X^*}}$ implies $\mathcal{E} \times \mathcal{E} \in \mu_{X^*}$. Firstly, let $\mathcal{E} \times \mathcal{E} \supset (i \times i)(\mathcal{H})$ for some $\mathcal{H} \in \mu$. Then, $X \in \mathcal{E}$ since $X \times X \in \mathcal{H} \subset (i \times i)(\mathcal{H}) \subset \mathcal{E} \times \mathcal{E}$ implying $X \times X \supset E \times E$ with $E \in \mathcal{E}$. Consequently, $X \supset E$; thus, $X \in \mathcal{E}$. Since $\mathcal{E}$ is a μ_{X^*}-Cauchy filter, $\mathcal{F} := i^{-1}(\mathcal{E})$ is a μ-Cauchy filter. Furthermore, $i(\mathcal{F}) = \mathcal{E}$ since $X \in \mathcal{E}$, and $\mathcal{E} \in \tau^*$ results.

Secondly, let

$$\mathcal{E} \times \mathcal{E} \supset (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}),$$

$\mathcal{F}$ τ_μ-independent. Thus, $\mathcal{E} \supset i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}$, i.e. $\mathcal{E} \in \tau^*$.

To " $\supset''$: If $\mathcal{E} \in \tau^*$ with $\mathcal{E} \supset i(\mathcal{F})$, $\mathcal{F}$ not τ_μ-independent, then $\mathcal{E} \times \mathcal{E} \supset i(\mathcal{F}) \times i(\mathcal{F}) = (i \times i)(\mathcal{F} \times \mathcal{F})$, i.e., $\mathcal{E} \times \mathcal{E} \in \mu_{X^*}$ since $\mathcal{F} \times \mathcal{F} \in \mu$. Thus, $\mathcal{E} \in \tau_{\mu_{X^*}}$. But the second case is evident.

In order to prove that $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is Cauchy complete, let $\mathcal{E} \in \tau_{\mu_{X^*}}$. Then, $\mathcal{E} = i(\mathcal{F})$ with $\mathcal{F}\ q_{\tau_\mu}(B)$ for some $B \in \mathcal{B}^X \backslash \{\emptyset\}$ or $\mathcal{E} = i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}$ with $\mathcal{F}$ τ_μ-independent. Since $(X, \mathcal{B}^X, q_{\tau_\mu})$ is a set-convergence subspace of $(X^*, \mathcal{B}^{X^*}, q_{\tau_{\mu^*}})$ and $i : (X, \mathcal{B}^X, q_{\tau_\mu}) \longrightarrow (X^*, \mathcal{B}^*, q_{\tau_{\mu_{X^*}}})$ is b-continuous, then

$$\mathcal{E} = i(\mathcal{F})\ q_{\tau_{\mu^*}}\ i(B) = B \in \mathcal{B}^{X^*} \backslash \{\emptyset\}$$

in the first case. In the second case, $\mathcal{E}\ q_{\tau_{\mu^*}}\ \{[\mathcal{F}]\}$ because

$$\mathcal{E} = i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}$$

for $\mathcal{F}$ τ_μ-independent; hence, $\mathcal{E} = \mathcal{E} \cap \{D \subset X^* : [\mathcal{F}] \in D\} \in \tau_{\mu_{X^*}}$. □

Definition 5.5. Let $(X, \mathcal{B}^X, \mu)$ be a symmetric b-uniform filter space and $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ the Cauchy complete, symmetric b-uniform filter space as constructed in 5.4. Then, the pair $(i, (X^*, \mathcal{B}^{X^*}, \mu_{X^*}))$ is called the *Cauchy completion* of $(X, \mathcal{B}^X, \mu)$ (sometimes only the space $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ will be called as above).

Theorem 5.6. *Let $(X, \mathcal{B}^X, \mu)$ be a symmetric b-uniform filter space. Then, the following statements are equivalent:*

(i) *$(X, \mathcal{B}^X, \mu)$ is merotopic, (see Section 3);*
(ii) *The Cauchy completion $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is merotopic;*
(iii) *$(X, \mathcal{B}^X, \mu)$ is a dense b-uniform filter subspace (with respect to **sb-UFIL**) of a SYMCONV space;*
(iv) *$(X, \mathcal{B}^X, \mu)$ is a b-uniform filter subspace (with respect to **sb-UFIL**) of a SYMCONV space.*

Proof. To (i) ⇒ (ii): Since $(X, \mathcal{B}^X, \mu)$ is merotopic $\mu = \{\mathcal{U} \in FIL(X \times X) : \exists\ \mathcal{F} \in \tau_\mu \text{ with } \mathcal{U} \supset \mathcal{F} \times \mathcal{F}\}$. We have to verify that $(\mathcal{B}^{X^*}, \mu_{X^*}) \leq (\mathcal{B}^{X^*}, \mu_{\tau_{\mu_{X^*}}})$ holds, which means

$$\mu_{X^*} \subset \mu_{\tau_{\mu_{X^*}}} = \{\mathcal{V} \in FIL(X^* \times X^*) : \exists\ \mathcal{E} \in \tau_{\mu_{X^*}} \text{ with } \mathcal{V} \supset \mathcal{E} \times \mathcal{E}\}.$$

Now, let $\mathcal{V} \in \mu_{X^*}$. Then, $\mathcal{V} \supset (i \times i)(\mathcal{H})$ with $\mathcal{H} \in \mu$ or

$$\mathcal{V} \supset (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}),$$

$\mathcal{F}$ τ_μ-independent. In the first case, we can find $\mathcal{F} \in \tau_\mu$ with $\mathcal{F} \times \mathcal{F} \subset \mathcal{H}$; thus, $(i \times i)(\mathcal{F} \times \mathcal{F}) = i(\mathcal{F}) \times i(\mathcal{F}) \subset (i \times i)(\mathcal{H}) \subset \mathcal{V}$. Since $i(\mathcal{F}) \in \tau_{\mu_{X^*}}$ (note that $i : (X, \mathcal{B}^X, \mu) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is Cauchy continuous) $\mathcal{V} \in \mu_{\tau_{\mu_{X^*}}}$ follows immediately. In the second case, $\mathcal{E} = i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}$, $\mathcal{F}$ τ_μ- independent belongs to $\tau_{\mu_{X^*}}$. But $\mathcal{E} \times \mathcal{E} \subset \mathcal{V}$ implies at once $\mathcal{V} \in \mu_{\tau_{X^*}}$.

To (ii) ⇒ (iii): By the hypothesis, $(X, \mathcal{B}^X, \mu)$ is a dense b-uniform filter subspace of the merotopic b-uniform filter space $(X^*, \mathcal{B}^{X^*}, \mu_{x^*})$. Moreover, since $(X, \mathcal{B}^{X^*}, \mu_{X^*})$ is also Cauchy complete, it is a SYMCONV space [8]. To (iii) ⇒ (iv): Evident.

To (iv) ⇒ (i): Every SYMCONV space is merotopic. Since MERb-UFIL is closed under the formation of subspaces, the claim follows immediately. □

Corollary 5.7. (i) *If $(X, \mathcal{B}^X, \mu)$ is a Cauchy complete, symmetric b-uniform filter space, then $(X, \mathcal{B}^X, \mu) = (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$;*

(ii) *If $(X, \mathcal{B}^X, \mu)$ is a SYMCONV space, then $(X, \mathcal{B}^X, \mu) = (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$.*

Proof. To (i): If $(X, \mathcal{B}^X, \mu)$ is Cauchy complete, then for each μ-Cauchy filter $\mathcal{F}\backslash\{\underline{P}X\}$ there is some $B \in \mathcal{B}^X\backslash\{\emptyset\}$ with $\mathcal{F}\ q_{\tau_\mu}\ B$. Consequently, $\mathcal{F}$ is not τ_μ-independent; hence, $(X, \mathcal{B}^X, \mu) = (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$.

To (ii): Every SYMCONV space is Cauchy complete. □

Remark 5.8. If $\mathcal{B}^X$ is discrete, then the Cauchy completion of a symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ and the simple completion of its underlying semi-uniform space essentially coincide (up to isomorphism) [10]. Additionally, by applying Theorem 5.6., we can also state that the Cauchy completion of a merotopic b-uniform filter space is Császár's λ-completion of a filter space. Also note that the categories **b-FIL** and **MERb-UFIL** are isomorphic, and therefore **FIL** and **b-FIL** can be identified.

Proposition 5.9. *A symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is a T_1-space if and only if its Cauchy completion $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is a T_1-space.*

Proof. Let $(X, \mathcal{B}^X, \mu)$ be a T_1-space and $z_1, z_2 \in X^*$ such that $\overset{\bullet}{z_1}\ \ q_{\tau_{\mu_{X^*}}}\ \{z_2\}$. This may only be possible if $z_1, z_2 \in X$ or $z_1 = z_2$. Since obviously b-uniform filter subspaces of T_1spaces are T_1-spaces, $(X, \mathcal{B}^X, \mu)$ is a T_1-space, provided that $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is a T_1-space. □

Remark 5.10. This result is of some importance since $\mathcal{E}$-connectedness, [11], is preserved by the Cauchy completion, provided that $\mathcal{E}$ is a class of T_1-spaces. But that may be examined in some future paper.

Lemma 5.11. *Let $(X, \mathcal{B}^X, \mu)$ be a symmetric b-uniform filter space,*

$$i : (X, \mathcal{B}^X, \mu) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu_{X^*})$$

its Cauchy completion, $(Y, \mathcal{B}^Y, \mu_Y)$ a Cauchy complete T_2-space and

$$f : (X, \mathcal{B}^X, \mu) \longrightarrow (Y, \mathcal{B}^Y, \mu_Y)$$

a b-uniformly continuous function. Then, there exists a unique b-uniformly continuous map $f^* : (X^*, \mathcal{B}^{X^*}, \mu_{X^*}) \longrightarrow (Y, \mathcal{B}^Y, \mu_Y)$ with $f^* \circ i = f$.

Proof. We define a map $f^* : X^* \longrightarrow Y$ by setting

$$f^*(z) := \begin{cases} f(z) & \text{if} \quad z \in X, \\ \ \ y & \text{if} \quad z \in X^*\backslash X, \end{cases}$$

where $f(\mathcal{F})\ q_{\tau_{\mu_Y}}\ \{y\}$ with $z = [\mathcal{F}]$, $\mathcal{F}$ τ_μ-independent.

Such a point $y \in Y$ exists because $(Y, \mathcal{B}^Y, \mu_Y)$ is Cauchy complete. Hence, $f(\mathcal{F})\ q_{\tau_{\mu_Y}}\ B$ for some $B \in \mathcal{B}^Y\backslash\{\emptyset\}$ implying that $f(\mathcal{F})\ q_{\tau_{\mu_Y}}\ \{y\}$ by choosing $y \in B$ and taking into account that $(\mathcal{B}^Y, q_{\tau_\mu})$ is reordered. The element y is uniquely determined since $(Y, \mathcal{B}^Y, \mu_Y)$ is a T_2-space and, furthermore, the above definition is independent of the choice of the representative $\mathcal{F}$.

Evidently, $f^* \circ i = f$. Moreover, without restriction, let $B \in \mathcal{B}^{X^*}\backslash\{\emptyset\}$. In the first case, $B \in \mathcal{B}^X\backslash\{\emptyset\}$ implies $f[B] \in \mathcal{B}^Y\backslash\{\emptyset\}$ with $f^*[B] = f[B]$ by the hypothesis. In the second case, $B = \{[\mathcal{F}]\}$, $\mathcal{F}$ τ_μ-independent implying $f^*[B] = \{f^*([\mathcal{F}])\} = \{y\}$ with $f(\mathcal{F})\ q_{\tau_{\mu_Y}}\ \{y\}$. But $\{y\} \in \mathcal{B}^Y\backslash\{\emptyset\}$ is valid, and the claim follows.

Next, let $\mathcal{U} \in \mu_{X^*}$; hence, there exists $\mathcal{V} \in b_{X^*}$ with $\mathcal{U} \supset \mathcal{V}$. Firstly, let $\mathcal{V} = (i \times i)(\mathcal{H})$ for some $\mathcal{H} \in \mu$; hence,

$$(f^* \times f^*)(\mathcal{U}) \supset (f^* \times f^*)((i \times i)(\mathcal{H})) = (f^* \circ i \times f^* \circ i)(\mathcal{H}) = (f \times f)(\mathcal{H}) \in \mu_Y.$$

Secondly, let

$$\mathcal{V} = (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}),$$

$\mathcal{F}$ τ_μ- independent. Consequently,

$$\begin{aligned}
&(f^* \times f^*)(\mathcal{U}) \\
&\supset (f^* \times f^*)((i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times (i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\})) \\
&= f^*(i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \times f^*(i(\mathcal{F}) \cap \{D \subset X^* : [\mathcal{F}] \in D\}) \\
&= (f(\mathcal{F}) \cap f^*(\{D \subset X^* : [\mathcal{F}] \in D\})) \times (f(\mathcal{F}) \cap f^*(\{D \subset X^* : [\mathcal{F}] \in D\})) \\
&= (f(\mathcal{F}) \cap \overset{\bullet}{y}) \times (f(\mathcal{F}) \cap \overset{\bullet}{y}) \in \mu_Y
\end{aligned}$$

since $f(\mathcal{F}) \cap \overset{\bullet}{y} \in \tau_{\mu_{Y'}}$, i.e., $f(\mathcal{F}) \; q_{\tau_{\mu_Y}} \; \{y\}$. Thus, $(f^* \times f^*)(\mathcal{U}) \in \mu_Y$ follows. Since now $f^* : X^* \longrightarrow Y$ is b-uniformly continuous, it is also b-continuous with respect to the underlying KENT set-convergence space. But f^* is uniquely determined by $f^* \circ i = f$ because X is dense in X^* and $(Y, \mathcal{B}^Y, \mu_Y)$ is a T_2-space. □

Definition 5.12. Let $(X, \mathcal{B}^X, \mu)$ be a T_2-symmetric b-uniform net space and $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ its Cauchy completion. Then, the pair $(i, (X^*, \mathcal{B}^{X^*}, \mu^l_{X^*}))$ is called the *supercompletion* of $(X, \mathcal{B}^X, \mu)$, where

$$1_{X^*} : (X^*, \mathcal{B}^{X^*}, \mu_{X^*}) \longrightarrow (X^*, \mathcal{B}^{X^*}, \mu^l_{X^*})$$

denotes the bireflection of $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ with respect to **b-ULIM** (see also Remark 2.10).

Remark 5.13. Here, we point out that in the discrete case (i.e., $\mathcal{B}^X = \mathcal{D}^X$) the supercompletion essentially coincides with the Wyler completion of the separated uniform limit space (X, μ) (see Remark 2.8., [10] and [15], respectively).

Moreover, if $(X, \mathcal{B}^X, \mu)$ is a generated T_2-symmetric b-uniform net space with generator $\mathcal{U}^\mu$ such that $\mathcal{B}^X = \mathcal{D}^X$, then the underlying b-uniformity $(\mathcal{B}^{X^*}, \mathcal{U}^{\mu^*})$ of its supercompletion is (up to isomorphism) the *Hausdorff completion* of $(X, \mathcal{U}^\mu)$, (see Definition 2.7 and 2.9, Remark 2.8 and 2.10, and [10], respectively).

6. The compactification of b-proximity spaces

In this section, the question is examined whether every b-proximity space can be densely embedded into a Cauchy complete b-proximity space, i.e., whether every b-proximity space has a compactification.

Definition 6.1. A symmetric b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *precompact* if and only if every ultrafilter $\mathcal{F} \in FIL(X)$ is a μ-Cauchy filter [8].

Remark 6.2. In our former paper [8], we have shown that a symmetric b-uniform net space is compact if and only if it is Cauchy complete and precompact. Consequently, we are coming to the next natural definition.

Definition 6.3. Objects of **b-UNIF** are the b-uniform spaces and its morphisms are defined as bounded maps which additionally preserve the associated entourages. Then, we denote by **b-PROX** the full subcategory of **b-UNIF** whose objects are the b-proximity spaces, regarded as precompact b-uniform spaces.

Remark 6.4. Further, in the discrete case, b-proximity spaces and EF-proximity spaces (Efremovič proximity spaces), [4], are essentially (up to isomorphism) the same. Further, we can state that **b-PROX** is bireflective in **b-UNIF** since **b-UNIF** is isomorphic to **gsb-UNET**, the full subcategory of **sb-UNET** whose objects are generated. **PCsb-UFIL**, the full subcategory of **sb-UNET** whose objects are precompact, and **b-UNIF** are both bireflective in **sb-UFIL**. **b-PROX** is closed under the formation of initial b-uniform filter structures in **b-UNIF**, i.e., **b-PROX** is bireflective in **b-UNIF**. In obtaining the *Smirnov-compactification* [12], of a separated EF-proximity space as a special case of our general concept, we introduce the term separated as follows:

Definition 6.5. A b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *separated*, provided that μ satisfies the following condition:

(sep) $\bigcap\{R \subset X \times X : R \in \cap\mu\} \subset \Delta$, where $\bigcap \mu := \bigcap\{\mathcal{U} \in FIL(X \times X) : \mathcal{U} \in \mu\}$ and $\Delta := \{(x, x) : x \in X\}$.

Remark 6.6. Here, we note that in the discrete case the above defined property coincides with the usual one for preuniform spaces (see Remark 2.10).

Lemma 6.7. *For a generated symmetric b-uniform net space, the following statements are equivalent:*

(i) $(X, \mathcal{B}^X, \mu)$ *is a* T_2 *space;*
(ii) $(X, \mathcal{B}^X, \mu)$ *is separated.*

Proof. To (i) $\Rightarrow$ (ii): Let us suppose for $x, z \in X$, $x \neq z$; hence, $(\dot{z}, \{x\}) \notin q_{\tau_\mu}$ by the hypothesis and applying Lemma 4.4 and 4.6, respectively. Consequently, $\dot{z} \times \dot{x} \notin \mu$ by Definition 3.5. Hence, $\cap\mu$ is not a subset of $\dot{z} \times \dot{x}$. Thus, we can find $R' \in \cap\mu$ with $R' \notin \dot{z} \times \dot{x}$ implying $(z, x) \notin R'$, and $(z, x) \notin \bigcap\{R \subset X \times X : R \in \cap\mu\}$ results.

To (ii) $\Rightarrow$ (i): Conversely, let $\dot{z}\ q_{\tau_\mu}\ \{x\}$ and suppose $z \neq x$. By the hypothesis, we have $(z, x) \notin \bigcap\{R \subset X \times X : R \in \cap\mu\}$, and $\dot{z} \times \dot{x} \in \mu$. Then, we can find $R' \in \cap\mu$ with $(z, x) \notin R'$. On the other hand, $R' \in \dot{z} \times \dot{x}$ follows, which contradicts, and therefore $x = z$ results. □

Next, we will show that the supercompletion preserves precompactness as follows:

Lemma 6.8. *The supercompletion* $(X^*, \mathcal{B}^{X^*}, \mu^l_{X^*})$ *of a precompact* T_2*-symmetric b-uniform net space* $(X, \mathcal{B}^X, \mu)$ *is a compact* T_2*-space (see Remark 6.2).*

Proof. First, we should note that the supercompletion $(X^*, \mathcal{B}^*, \mu^l_{X^*})$ of a given T_2-symmetric b-uniform net space is a Cauchy complete, T_2-symmetric b-uniform net space and is characterized by a universal property (compare with Lemma 5.11).

Now, let $\mathcal{F}^* \in FIL(X^*)$ be an ultrafilter, then we have to verify that $\mathcal{F}^* \times \mathcal{F}^* \in (\mu_{X^*})^l$ is valid. We put $\mathcal{F}_X := \{F^* \cap X : F^* \in \mathcal{F}^*\}$; hence, $\mathcal{F}_X \in FIL(X)$ follows.

Consequently, there exists an ultrafilter $\mathcal{F} \in FIL(X)\backslash\{\underline{P}X\}$ with $\mathcal{F}_X \subset \mathcal{F}$. But $\mathcal{F}$ is a μ-Cauchy filter by the hypothesis, so that $\mathcal{F} \times \mathcal{F} \in \mu$ is true. Thus, $i(\mathcal{F}) \times i(\mathcal{F}) = (i \times i)(\mathcal{F} \times \mathcal{F}) \in \mu_{X^*}$ results.

Now, our goal is to verify that the inclusion $i(\mathcal{F}) \subset sec\mathcal{F}^*$ holds. $A \in i(\mathcal{F})$ and $F^* \in \mathcal{F}^*$ imply $A \supset i[F]$ for some $F \in \mathcal{F}$. Hence, $F^* \cap X \in \mathcal{F}_X$ implies $\emptyset \notin F^* \cap F = (F^* \cap X) \cap F \in \mathcal{F}$. Consequently, $A \in sec\mathcal{F}^*$ follows. Since $\mathcal{F}^*$ is an ultrafilter, we have $sec\mathcal{F}^* \subset \mathcal{F}^*$ and $i(\mathcal{F}) \subset \mathcal{F}^*$ results. But then, $\mathcal{F}^* \times \mathcal{F}^* \in \mu_{X^*} \subset \mu^l_{X^*}$ are valid, showing that $\mathcal{F}^*$ is a $\mu^l_{X^*}$-Cauchy filter. Thus, by applying 6.2. the claim follows. □

Theorem 6.9. *For a separated b-proximity space $(X, \mathcal{B}^X, \mu)$, the space*

$$(X^*, \mathcal{B}^{X^*}, (\mu^l_{X^*})^g)$$

is a separated compact b-proximity space.

Proof. By taking the previous results into account, we will only show that $(X^*, \mathcal{B}^*, (\mu^l_{X^*})^g)$ is separated and compact. So, let $z_1, z_2 \in X^*$ such that

$$\dot{z}_1 \ q_{\tau_{(\mu^l_{X^*})^g}} \ \{z_2\};$$

hence, $\dot{z}_1 \times \dot{z}_2 \in (\mu^l_{X^*})^g$. Consequently, $\dot{z}_1 \times \dot{z}_2 \supset \mathcal{U}^{\mu^l_{X^*}}$.

By choosing $\mathcal{V} \in \mu_{X^*}$, we get $\mathcal{V} \in \mu^l_{X^*}$; hence, $\mathcal{V} \subset \dot{z}_1 \times \dot{z}_2$ by the definition of $\mathcal{U}^{\mu^l_{X^*}}$. Consequently, $\dot{z}_1 \times \dot{z}_2 \in \mu_{X^*}$ is valid, and $\dot{z}_1 \ q_{\mu_{X^*}} \ \{z_2\}$ follows. But then, $z_1 = z_2$ or $z_1, z_2 \in X$ are valid, but $(X^*, \mathcal{B}^{X^*}, \mu_{X^*})$ is a T_1-space by the hypothesis, and the claim is true.

Next, let $\mathcal{F}^* \in FIL(X^*)$ be an ultrafilter. Our goal is to verify the existence of some $B \in \mathcal{B}^{X^*}\backslash\{\emptyset\}$ such that $\mathcal{F}^* \ q_{(\mu^l_{X^*})^g} \ B$ is true, which is equivalent to $\dot{B} \times \mathcal{F}^* \in (\mu^l_{X^*})^g$. If setting $\mathcal{F}_X := \{F^* \cap X : F^* \in \mathcal{F}^*\}$, then we can find an ultrafilter $\mathcal{F} \in FIL(X)\backslash\{\underline{P}X\}$, where $\mathcal{F}_X \subset \mathcal{F}$. But then, $\mathcal{F}^* \times \mathcal{F}^* \in \mu^l_{X^*}$-Cauchy filter (see the previous proving). Hence, there exists $B \in \mathcal{B}^{X^*}\backslash\{\emptyset\}$ such that $\mathcal{F}^* \ q_{\mu^l_{X^*}} \ B$ with respect to Lemma 6.8. But this means $\dot{B} \times \mathcal{F} \in \mu^l_{X^*} \subset (\mu^l_{X^*})^g$, and the claim follows. □

Remark 6.10. In the discrete case, we call the pair $(i, X^*, \mathcal{B}^{X^*}, (\mu^l_{X^*})^g)$ the T_2-compactification of the separated b-proximity space $(X, \mathcal{B}^X, \mu)$ and it represents (up to isomorphisms) the *Smirnov compactification* of a separated EF-proximity space.

7. Some open questions

Now, it seems to be of interest to consider the case of *completion* for a given not necessarily symmetric b-uniform filter space. Thus, we have to alter the definition of a μ-Cauchy filter in such a way that it represents a *generalization* of the given one and, moreover, if considering the symmetrical case, the definitions should coincide. As a first step in this direction, let us consider the following notions:

Definition 7.1. Let $(X, \mathcal{B}^X, \mu)$ be a b-uniform filter space. $\mathcal{F} \in FIL(X)$ is called a *pre-Cauchy filter* (in $(\mathcal{B}^X, \mu)$), provided $\mathcal{F}$ satisfies the following condition:

(pChy) $\exists\, \mathcal{U} \in \mu$ s.t. $\forall\, R \in \mathcal{U}, \exists\, B \in \mathcal{B}^X \backslash \{\emptyset\}$ s.t. $R(B) \in \mathcal{F}$.

Here, $R(B) := \{z \in X : \exists\, x \in B \text{ s.t. } (x,z) \in R\}$.

Remark 7.2. As we already know, a filter $\mathcal{F} \in FIL(X)$ is μ-convergent to $B \in \mathcal{B}^X \backslash \{\emptyset\}$ if and only if $\overset{\bullet}{B} \times \mathcal{F} \in \mu$ is valid. It is easy to verify that every μ-convergent filter $\mathcal{F} \in FIL(X)$ is a pre-Cauchy filter in $(\mathcal{B}^X, \mu)$.

Then, a more restricted definition leads to so-called *semi-Cauchy filters* as follows:

Definition 7.3. Let $(X, \mathcal{B}^X, \mu)$ be a b-uniform filter space. $\mathcal{F} \in FIL(X)$ is called a *semi-Cauchy filter* (in μ), provided $\mathcal{F}$ satisfies the following condition:

(SChy) $\exists\, \mathcal{U} \in \mu, \forall\, R \in \mathcal{U}, \exists x \in X$ s.t. $R(\{x\}) \in \mathcal{F}$.

Remark 7.4. As easily seen, every semi-Cauchy filter is a pre-Cauchy filter in $(\mathcal{B}^X, \mu)$. In addition, we note that every μ-Cauchy filter $\mathcal{F} \in FIL(X) \backslash \{\underline{P}X\}$ is a semi-Cauchy filter in μ. Next, we will see under which condition these two definitions coincide.

Lemma 7.5. *For a symmetric b-uniform net space $(X, \mathcal{B}^X, \mu)$ and a filter $\mathcal{F} \in FIL(X) \backslash \{\underline{P}X\}$, the following statements are equivalent:*

(i) *$\mathcal{F}$ is a semi-Cauchy filter;*
(ii) *$\mathcal{F}$ is a μ-Cauchy filter.*

Proof. It remains to verify the case (i) $\Rightarrow$ (ii): Choose $\mathcal{U} \in \mu$; hence, $\mathcal{U} \circ \mathcal{U}^{-1} \in \mu$ can be deduced. Our goal is to verify that $\mathcal{U} \circ \mathcal{U}^{-1} \subset \mathcal{F} \times \mathcal{F}$ is valid. $R \in \mathcal{U} \circ \mathcal{U}^{-1}$ implies $R \supset U \circ U^{-1}$ for some $U \in \mathcal{U}$. By the hypothesis, we can find an element $x \in X$ such that $U(\{x\}) \in \mathcal{F}$ is true. But this implies $U(\{x\}) \times U(\{x\}) \in \mathcal{F} \times \mathcal{F}$. So, it suffices to show $U(\{x\}) \times U(\{x\}) \subset U \circ U^{-1}$. $(y,z) \in U(\{x\}) \times U(\{x\})$ implies $y, z \in U(\{x\})$; hence, $(x,z) \in U$ and $(y,x) \in U^{-1}$ are valid, showing that $(y,z) \in U \circ U^{-1}$ is true, and the claim follows. □

Remark 7.6. If returning to the *discrete* case, then, for a symmetric b-uniform net space and for a filter $\mathcal{F} \in FIL(X) \backslash \{\underline{P}X\}$, we get $\mathcal{F}$ is a pre-Cauchy filter if and only if $\mathcal{F}$ is a semi-Cauchy filter if and only if $\mathcal{F}$ is a μ-Cauchy filter.

So, it seems to be *quite natural* to introduce a more general concept of completeness being characterized by its corresponding Cauchy filters.

Further, it seems to be of some relevance to examine the generated case, here especially that of b-quasiuniformities.

Here, we still only mention the concept of completeness, which seems to be appropriate for further studies as follows:

Definition 7.7. A b-uniform filter space $(X, \mathcal{B}^X, \mu)$ is called *ultracomplete*, provided it satisfies the following condition:

(ucpl) Every pre-Cauchy filter $\mathcal{F}$ in $(\mathcal{B}^X, \mu)$ is μ-convergent.

Lemma 7.8. *Every non-empty finite setconvergent b-uniform filter space is ultracomplete.*

Proof. So, let $\mathcal{F} \in FIL(X)$ be a pre-Cauchy filter. Choose $\mathcal{U} \in \mu$; hence, we can find a bounded set $B_1 \in \mathcal{B}^X \setminus \{\emptyset\}$ and a filter $\mathcal{F}_1 \in FIL(X)$ such that $\mathcal{U} \supset \overset{\bullet}{B_1} \times \mathcal{F}_1 \in \mu$ are valid. It remains to show that the inclusion $\mathcal{F}_1 \subset \mathcal{F}$ is valid. $F_1 \in \mathcal{F}_1$ implies $B_1 \times F_1 \in \overset{\bullet}{B_1} \times \mathcal{F}_1$; hence, there exists $B \in \mathcal{B}^X \setminus \{\emptyset\}$ with $(B_1 \times F_1)(B) \in \mathcal{F}$. Since X is finite, $\cap \mathcal{F} \in \mathcal{F}$ can be deduced. But $\cap \mathcal{F} \subset (B_1 \times F_1)(B) \subset F_1$ hold. Thus, $\mathcal{F}$ is μ-convergent to B_1, and so $(X, \mathcal{B}^X, \mu)$ is ultracomplete. □

Remark 7.9. By transforming this result to the set-convergence spaces, we can state that, roughly speaking, finite reordered set-convergence spaces are ultracomplete. Finally, we note that in the discrete case, for symmetric b-uniform net spaces, the concepts of Cauchy completeness and ultracompleteness are the same.

Now, we will pause because the further studies announced are not the aim of this paper (see also [6]).

References

[1] A. Behling, *Einbettung uniformer Räume in topologische Universen*, Ph.D. thesis, FU Berlin, FB Math., 1992.

[2] Á. Császár, *λ-complete filter spaces*, Acta Math. Hungar. **70** (1996), 75–87.

[3] S. Dolecki and F. Mynárd, *Convergence Foundations of Topology*, World Scientific, 2016.

[4] V. A. Efremovič, *The geometry of proximity*, Mat. Sb. **32** (1952), 189–200.

[5] F. Hausdorff, *Grundzüge der Mengenlehre*, Veit & Comp., Leipzig, 1914.

[6] D. Leseberg, *On topologically induced b-convergences*, Topology Proc. **37** (2011), 293–313.

[7] D. Leseberg, *Supertopologies as starting points for generalized continuity structures*, in: W. Gähler *et al.* (eds.), Categorical structures and their applications, Proceedings of the North-West European category seminar, Berlin, Germany, March 28–29, 2003, World Scientific (2004), 189–206.

[8] D. Leseberg and Z. Vaziry, *Categories of several convergences*, `https://www.researchgate.net/publication/347424178`, 2020.

[9] D. Leseberg and Z. Vaziry, *The quasitopos of b-uniform filter spaces*, Math. Appl. **7** (2018), 155–171.

[10] G. Preuss, *Foundation of Topology, An Approach to Convenient Topology*, Dordrecht, Kluwer Acad. Publ., 2002.

[11] G. Preuss, *E-Zusammenhängende Räume*, Manuscr. Math. **3** (1970), 331–342.

[12] J. M. Smirnov, *On proximity spaces in the sense of V. A. Efremovič*, Dokl. Akad. Nauk. SSSR **84**, 895–898.

[13] A. Tozzi and O. Wyler, *On categories of supertopological spaces*, Acta Univ. Carol., Math. Phys. **28** (1987), 137–149.

[14] O. Wyler, *On convergence of filters and ultrafilters to subsets*, Categorical Methods in Computer Science With Aspects from Topology, Lecture Notes in Computer Science **393**, 1989, pp. 340–350.

[15] O. Wyler, *Ein Komplettierungsfunktor für uniforme Limesräume*, Math. Nachr. **46** (1970), 1–12.

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