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TOPOLOGICAL SOLUTIONS OF η-GENERALIZED VECTOR VARIATIONAL-LIKE INEQUALITY PROBLEMS

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Abstract. In this paper, we discuss several variants of the η-generalized vector variational-like inequality problem and provide existence theorems for their solutions via a topological approach. Several topological concepts like compactness, closedness, net theory and admissibility of function space topology are used for obtaining the main results. Finally, we give some topological properties of the solution set so obtained.

1. Introduction

In 1980, F. Gianessi [4] extended the concept of classical variational inequality (introduced by Stampacchia [23]) to vector variational inequality (VVI, in short) for vector valued functions in the setting of finite-dimensional Euclidean spaces. Further, VVI has been extended in various directions, in particular, the vector variational-like inequalities (VVLI, in short) [3, 8, 9, 18, 20]. VVI and their generalizations have been used extensively to solve vector optimization problems. Several researchers have established various relations between vector variational inequalities and vector optimization problems [12, 14, 24, 28].

In one direction, the concept of variational inequality was extended by Hanson [7] by introducing invex function (a generalization of convex function). Weir and Mond [25] and Noor [19] have studied some basic properties of preinvex and α-preinvex functions, respectively, along with their role in variational-like inequality problems and optimization problems. By assuming the condition of pseudo-invexity, Ruiz-Garzon *et al.* [20] have established some relations between vector variational-like inequality problems and optimization problems. In [8–10], Khan and others studied several variants of vector variational-like inequalities in the framework of Banach spaces. In 2017, Salahuddin [21] provided existence results for the solution of general set-valued vector variational inequalities. In the same year, Li and Yu [16] introduced a class of generalized invex functions, namely (α-ρ-η)-invex functions and provided the existence results for two types of vector variational-like inequalities. On the other hand, in 2018, Salahuddin [22] obtained the existence results for the solution of vector variational inequality problems by using sequentially continuous mapping. Recently, Gupta *et al.* [5] provided existence theorems for the solution of generalized non-linear vector variational-like

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inequality problems by using topological approach.

Variational-like inequalities have a wide range of applications, making it an interesting discipline for research. The flow equilibrium problem on a network using vector variational inequality has been discussed in [29]. Further, application of variational-like inequality in fuzzy setting for the optimization problem is discussed in [26]. Similar studies are available in the literature [17, 27, 30]. Motivated by these studies, here we study a generalized form of a vector variational-like inequality problem.

In [13], Lee *et al.* discussed the solvability of generalized weak vector variational-like inequalities (GWVVLI) in Banach spaces (reflexive Banach spaces) by using the Browder fixed point theorems and monotonicity of mappings and, in [15], Li *et al.* introduced a class of η-generalized vector variational-like inequalities (η-GVVLI) for Hausdorff topological vector spaces and gave two existence results for solution to the η-GVVLI problem under the assumption of η-hemicontinuity; in one result the compactness of K is considered, while in the other it is not so.

In the present paper, we consider a couple of η-GVVLI problems and prove the existence results for these problems in topological vector spaces using topological approach. We use the concept of closedness, compactness, and net theory along with the admissibility of a function space to obtain our results. The authors have found so far that the concept of admissibility of function spaces is not used extensively in the existing literature to obtain such results.

In the following, we will define two variants of the η-GVVLI problem:

Let X and Y be two topological vector spaces and $\mathcal{C}(X, Y)$ be the space of all continuous linear mappings from the space X to the space Y. Let $K \subseteq X$ be nonempty, closed and convex. Further, let $T : K \times K \times [0, 1] \to \mathcal{C}(X, Y)$ be a single-valued map defined by $T(x, z, \lambda) = T_{\lambda x+(1-\lambda)z}$ and $\eta : K \times K \to X$, $f : K \times K \to Y$ be two bifunctions.

- *η-generalized vector variational-like inequality problem I* (η-GVVLIP (I)): If C is a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, then η-GVVLIP (I) is to find $x_0 \in K$ such that there exist $z \in K$, $\lambda \in [0, 1]$ satisfying

$$T(x_0, z, \lambda)(\eta(y, x_0)) + f(y, x_0) \notin -\text{int}\, C \quad \forall y \in K.$$

- *η-generalized vector variational-like inequality problem II* (η-GVVLIP (II)): If $C : K \rightrightarrows Y$ is a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex pointed cone in Y with $\text{int}\, C(x) \neq \emptyset$, then η-GVVLIP (II) is to find $x_0 \in K$ such that there exist $z \in K$, $\lambda \in [0, 1]$ satisfying

$$T(x_0, z, \lambda)(\eta(y, x_0)) + f(y, x_0) \notin -\text{int}\, C(x_0) \quad \forall y \in K.$$

By considering a mapping $T : K \times K \times (0, 1] \to \mathcal{C}(X, Y)$ instead of $T : K \times K \times [0, 1] \to \mathcal{C}(X, Y)$ as in the above problems, we define two other variants of the η-GVVLI problem:

- *η-generalized vector variational-like inequality problem III* (η-GVVLIP (III)): If C is a closed convex pointed cone in Y with $\text{int}C \neq \emptyset$, then for some fixed $\lambda \in (0, 1]$, η-GVVLIP (III) is to find $x_0 \in K$ such that there exist $z \in K$ satisfying

$$T(x_0, z, \lambda)(\eta(y, x_0)) + f(y, x_0) \notin -\text{int}\, C \quad \forall y \in K.$$

- *η-generalized vector variational-like inequality problem IV* (η-GVVLIP (IV): If $C : K \rightrightarrows Y$ is a set-valued map such that for each $x \in K$, $C(x)$ is a closed convex pointed cone in Y with $\operatorname{int} C(x) \neq \emptyset$, then for some fixed $\lambda \in (0,1]$, η-GVVLIP (IV) is to find $x_0 \in K$ such that there exist $z \in K$ satisfying

$$T(x_0, z, \lambda)(\eta(y, x_0)) + f(y, x_0) \notin -\operatorname{int} C(x_0) \quad \forall y \in K.$$

The rest of the paper is organized in the following way: In Section 2, we recall some preliminaries required in the paper. In Section 3, we prove existence theorems for solutions to the η-GVVLIP (I) and η-GVVLIP (II). We then give an example to illustrate our results. Finally, we provide some properties of the solution sets so obtained.

2. Preliminaries

In this section, we recall some definitions and basic results which will be used later to obtain the main results.

Definition 2.1. Suppose $F : X \rightrightarrows Y$ is a set-valued map from X to Y. The *graph* of F, denoted by $\mathcal{G}(F)$, is

$$\mathcal{G}(F) = \{(x, y) \in X \times Y \mid x \in X, y \in F(x)\}.$$

Definition 2.2. ([2]) Let U be a nonempty subset of a topological vector space X. A set-valued map $F : U \rightrightarrows X$ is called a *KKM-mapping* if for every nonempty finite set $\{u_1, u_2, \ldots, u_n\}$ of U, we have

$$\operatorname{co}\{u_1, u_2, \ldots, u_n\} \subseteq \bigcup_{i=1}^{n} F(u_i),$$

where $\operatorname{co}\{u_1, u_2, \ldots, u_n\}$ denotes the convex hull of u_1, u_2, …, u_n.

The following result is taken from [2].

Lemma 2.3. (KKM-Theorem) *If U is a nonempty subset of a topological vector space X and $F : U \rightrightarrows X$ is a KKM-mapping such that for every $u \in U$, $F(u)$ is a closed subset of X and for at least one $u \in U$, $F(u)$ is compact, then $\bigcap_{u \in U} F(u) \neq \emptyset$.*

Definition 2.4. ([1,6]) Let(Y, μ_1) and (Z, μ_2) be two topological spaces. Let $\mathcal{C}(Y, Z)$ be the space of all continuous mappings from Y to Z. A topology τ on $\mathcal{C}(Y, Z)$ is called *admissible*, if the *evaluation map* $e : \mathcal{C}(Y, Z) \times Y \to Z$, defined by $e(f, y) = f(y)$, is continuous.

Lemma 2.5. ([6]) *A function space topology on $\mathcal{C}(X, Y)$, the collection of continuous mappings from the space X to the space Y, is admissible if and only if, for any net $\{f_n\}_{n \in D_1}$ in $\mathcal{C}(X, Y)$, the convergence of $\{f_n\}_{n \in D_1}$ to f implies the continuous convergence of $\{f_n\}_{n \in D_1}$ to f. That is, if $\{f_n\}_{n \in D_1}$ converges to f in $\mathcal{C}(X, Y)$ and $\{x_m\}_{m \in D_2}$ is any net in X converging to $x \in X$, then $\{f_n(x_m)\}_{(n,m) \in D_1 \times D_2}$ converges to $f(x)$ in Y.*

The above characterization of admissibility remains valid for the family of continuous linear mappings from X to Y, where X and Y are topological vector spaces.

Throughout the paper, 0_X and 0_Y denote the zero vectors in the space X and in the space Y, respectively.

3. Existence theorems for η-GVVLIP (I) and η-GVVLIP (II)

Theorem 3.1. *Let (X, τ_1) and (Y, τ_2) be any two topological vector spaces. Let $\mathcal{C}(X, Y)$ denote the space of all continuous linear mappings from X to Y, equipped with an admissible topology. Let $K \subseteq X$ be a nonempty closed convex compact subset of X. Let $C \subseteq Y$ be a closed convex pointed cone with $\operatorname{int} C \neq \emptyset$. Further, let $T : K \times K \times [0,1] \to \mathcal{C}(X, Y)$ be a single-valued continuous mapping. Suppose the maps $\eta : K \times K \to X$ and $f : K \times K \to Y$ are affine maps such that both are continuous in the second argument with $\eta(x, x) = 0_X$, $f(x, x) = 0_Y$ for all $x \in K$. Then, the η-GVVLIP (I) has a solution. That is, there exists $x_0 \in K$ such that, for some $z_0 \in K$ and for some $\lambda_0 \in [0,1]$, the following holds*

$$T(x_0, z_0, \lambda_0)(\eta(y, x_0) + f(y, x_0)) \notin -\operatorname{int} C \quad \forall y \in K.$$

Proof. Consider a set-valued map $F : K \rightrightarrows K$ defined as

$$F(y) = \{x \in K : \exists z \in K, \exists \lambda \in [0,1] \text{ s.t. } T(x, z, \lambda)(\eta(y, x)) + f(y, x) \notin -\operatorname{int} C\}.$$

Clearly, for each $y \in K$, $F(y)$ is nonempty as at least $y \in K$. For convenience, we divide the proof into two steps:

(i) *F is a KKM-map on K*: Let $U = \{u_1, u_2, \ldots, u_m\}$ be any finite subset of K. Let $v \in \operatorname{co}\{u_1, u_2, \ldots, u_m\}$ but $v \notin \bigcup_{i=1}^m F(u_i)$. Therefore, there exist $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\ldots$, $\lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ and $v = \sum_{i=1}^m \lambda_i u_i$. Since $v \notin F(u_i)$, for each $i = 1, 2, \ldots, m$, therefore $\forall z \in K$, $\forall \lambda \in [0,1]$, we have $T(v, z, \lambda)(\eta(u_i, v)) + f(u_i, v) \in -\operatorname{int} C$, for each $i = 1, 2, \ldots, m$. Since $-\operatorname{int} C$ is a convex set and $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$, therefore

$$\sum_{i=i}^{m} \lambda_i [T(v, z, \lambda)(\eta(u_i, v)) + f(u_i, v)] \in -\operatorname{int} C, \quad \forall z \in K,\ \forall \lambda \in [0,1].$$

Since η and f are affine, therefore

$$T(v, z, \lambda)(\eta(\sum_{i=1}^m \lambda_i u_i, v)) + f(\sum_{i=1}^m \lambda_i u_i, v) \in -\operatorname{int} C,\ \forall z \in K, \forall \lambda \in [0,1],$$

which implies

$$T(v, z, \lambda)(\eta(v, v)) + f(v, v) \in -\operatorname{int} C,\ \forall z \in K, \forall \lambda \in [0,1].$$

But we have $\eta(v, v) = 0_X$ and $f(v, v) = 0_Y$, therefore 0_Y belongs to $-\operatorname{int} C$, then 0_Y belongs to $\operatorname{int} C$, which is a contradiction as C is a pointed cone in Y.

(ii) *$F(y)$ is closed for each $y \in K$*: Let $\{x_\alpha\}_{\alpha \in D}$ be a net in $F(y)$ with $\{x_\alpha\}$ converging to $\bar{x}$ in X. As K is closed, $\bar{x} \in K$. We have to show that $\bar{x} \in F(y)$, that is, there exist $\bar{z} \in K$, $\bar{\lambda} \in [0,1]$ such that $T(\bar{x}, \bar{z}, \bar{\lambda})(\eta(y, \bar{x}) + f(y, \bar{x})) \notin -\operatorname{int} C$. Since $x_\alpha \in F(y)$, therefore, there exist nets $\{z_\alpha\}_{\alpha \in D}$, $\{\lambda_\alpha\}_{\alpha \in D}$ such that $T(x_\alpha, z_\alpha, \lambda_\alpha)(\eta(y, x_\alpha)) + f(y, x_\alpha) \notin -\operatorname{int} C$. Since $z_\alpha \in K$ and K is compact, therefore there exists a subnet $\{z_{\alpha_l}\}_{\alpha_l \in D_1}$ of $\{z_\alpha\}$ such that $\{z_{\alpha_l}\}$ converges to some z in K. As $\lambda_\alpha \in [0,1]$, by Bolzano–Weierstrass Theorem, there exists a subnet $\{\lambda_{\alpha_m}\}_{\alpha_m \in D_2}$ such that $\{\lambda_{\alpha_m}\}$ converges to some $\lambda \in [0,1]$. Without loss of generality, suppose $z = \bar{z}$ and $\lambda = \bar{\lambda}$.

Now, we form a directed set $D_3 \subseteq D$ in the following manner: By the virtue of the order property of the directed set D, for each pair $\alpha_l, \alpha_m \in D$, there exists some $\alpha_\delta \in D$ such that $\alpha_\delta \geq \alpha_l$, $\alpha_\delta \geq \alpha_m$. Let D_3 be the compilation of such α_δ's. It is easy to verify that D_3 is a directed set under the induced ordering of D.

Thus, we have subnets $\{x_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$, $\{z_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$ and $\{\lambda_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$ of $\{x_\alpha\}$, $\{z_\alpha\}$ and $\{\lambda_\alpha\}$, respectively, such that $\{x_{\alpha_\delta}\}$, $\{z_{\alpha_\delta}\}$, $\{\lambda_{\alpha_\delta}\}$ converge to $\bar{x} \in K$, $\bar{z} \in K$ and $\bar{\lambda} \in [0,1]$, respectively. As T is continuous, $T(x_{\alpha_\delta}, z_{\alpha_\delta}, \lambda_{\alpha_\delta}) \to T(\bar{x}, \bar{z}, \bar{\lambda})$. Also, as $\eta(y, \cdot)$ and $f(y, \cdot)$ are continuous in second argument, $\eta(y, x_{\alpha_\delta}) \to \eta(y, \bar{x})$ and $f(y, x_{\alpha_\delta}) \to f(y, \bar{x})$. Since the function space has admissible topology, therefore

$$T(x_{\alpha_\delta}, z_{\alpha_\delta}, \lambda_{\alpha_\delta})(\eta(y, x_{\alpha_\delta})) \to T(\bar{x}, \bar{z}, \bar{\lambda})(\eta(y, \bar{x})).$$

Thus,

$$T(x_{\alpha_\delta}, z_{\alpha_\delta}, \lambda_{\alpha_\delta})(\eta(y, x_{\alpha_\delta})) + f(y, x_{\alpha_\delta}) \to T(\bar{x}, \bar{z}, \bar{\lambda})(\eta(y, \bar{x})) + f(y, \bar{x}).$$

Now, if

$$T(\bar{x}, \bar{z}, \bar{\lambda})(\eta(y, \bar{x})) + f(y, \bar{x}) \in -\operatorname{int} C,$$

then

$$T(x_{\alpha_\delta}, z_{\alpha_\delta}, \lambda_{\alpha_\delta})(\eta(y, x_{\alpha_\delta})) + f(y, x_{\alpha_\delta}) \in -\operatorname{int} C$$

eventually, which leads to a contradiction. Hence,

$$T(\bar{x}, \bar{z}, \bar{\lambda})(\eta(y, \bar{x})) + f(y, \bar{x}) \notin -\operatorname{int} C,$$

that is, $\bar{x} \in F(y)$.

Now, $F(y)$, being a closed subset of a compact set K, is compact. Therefore, by KKM-Theorem, $\bigcap_{y \in K} F(y) \neq \emptyset$. Hence, there exists $x_0 \in F(y)$ for all $y \in K$, that is, there exist $z_0 \in K$, $\lambda_0 \in [0,1]$, such that $T(x_0, z_0, \lambda_0)(\eta(y, x_0)) + f(y, x_0) \notin -\operatorname{int} C$, for every $y \in K$.

□

In the next theorem, we provide the existence condition for the solution of η-GVVLIP (II).

Theorem 3.2. *Let (X, τ_1) and (Y, τ_2) be two topological vector spaces and $\mathcal{C}(X, Y)$ be the space of all continuous linear mappings from X to Y, equipped with an admissible topology. Let K be a nonempty closed convex compact subset of X. Let $C : K \rightrightarrows Y$ be a set-valued map such that, for every $x \in K$, $C(x)$ is a closed convex pointed cone with $\operatorname{int} C(x) \neq \emptyset$. Suppose the set-valued map $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus (-\operatorname{int} C(x))$ has a closed graph $\mathcal{G}(W)$ in $X \times Y$. Let $T : K \times K \times [0,1] \to \mathcal{C}(X, Y)$ be a single-valued continuous mapping. Suppose the maps $\eta : K \times K \to X$ and $f : K \times K \to Y$ are affine mappings such that both are continuous in the second argument with $\eta(x, x) = 0_X$, $f(x, x) = 0_Y$, for all $x \in K$. Then, the η-GVVLIP (II) has a solution. That is, there exists $x_0 \in K$ such that, for some $z_0 \in K$ and for some $\lambda_0 \in [0,1]$, the following holds*

$$T(x_0, z_0, \lambda_0)(\eta(y, x_0)) + f(y, x_0) \notin -\operatorname{int} C(x_0) \quad \forall y \in K.$$

Proof. For each $y \in K$, we define a set-valued map $F : K \rightrightarrows K$ as

$$F(y) = \{x \in K : \exists z \in K, \exists \lambda \in [0,1] \text{ s.t. } T(x,z,\lambda)(\eta(y,x)) + f(y,x) \notin -\operatorname{int} C(x)\}.$$

Clearly, for each $y \in K$, $F(y)$ is nonempty as at least $y \in K$. The proof of the theorem is divided into two steps:

(i) F is a KKM-map on K;

(ii) $F(y)$ is closed for each $y \in K$.

We are avoiding the proof of step (i) as it is similar to that of Theorem 3.1.

Let $\{x_\alpha\}_{\alpha \in D}$ be a net in $F(y)$ converging to some $\bar{x} \in X$. As $K \subseteq X$ is closed, $\bar{x} \in K$. We have to show $\bar{x} \in F(y)$, that is, there exist some $\bar{z} \in K$, $\bar{\lambda} \in [0,1]$ such that

$$T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})) + f(y,\bar{x}) \notin -\operatorname{int} C(\bar{x}).$$

Since $x_\alpha \in F(y)$, therefore there exist nets $\{z_\alpha\}_{\alpha \in D}$ in K and $\{\lambda_\alpha\}_{\alpha \in D}$ in $[0,1]$ such that $T(x_\alpha,z_\alpha,\lambda_\alpha)(\eta(y,x_\alpha)) + f(y,x_\alpha) \notin -\operatorname{int} C(x_\alpha)$, which implies

$$T(x_\alpha,z_\alpha,\lambda_\alpha)(\eta(y,x_\alpha)) + f(y,x_\alpha) \in W(x_\alpha),$$

which gives

$$\{(x_\alpha, T(x_\alpha,z_\alpha,\lambda_\alpha)(\eta(y,x_\alpha)) + f(y,x_\alpha))\} \in \mathcal{G}(W).$$

Now, following the lines of the proof of Theorem 3.1, we get subnets $\{x_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$, $\{z_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$ and $\{\lambda_{\alpha_\delta}\}_{\alpha_\delta \in D_3}$ of $\{x_\alpha\}$, $\{z_\alpha\}$ and $\{\lambda_\alpha\}$, respectively, such that $\{x_{\alpha_\delta}\}$, $\{z_{\alpha_\delta}\}$, $\{\lambda_{\alpha_\delta}\}$ converge to $\bar{x} \in K$, $\bar{z} \in K$ and $\bar{\lambda} \in [0,1]$, respectively.

As T is continuous, $T(x_{\alpha_\delta},z_{\alpha_\delta},\lambda_{\alpha_\delta}) \to T(\bar{x},\bar{z},\bar{\lambda})$. Since the functions η and f are continuous in the second argument, therefore

$$\eta(y,x_{\alpha_\delta}) \to \eta(y,\bar{x}), \quad f(y,x_{\alpha_\delta}) \to f(y,\bar{x}).$$

As the function space $C(X,Y)$ is admissible,

$$T(x_{\alpha_\delta},z_{\alpha_\delta},\lambda_{\alpha_\delta})(\eta(y,x_{\alpha_\delta})) \to T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})).$$

Thus,

$$T(x_{\alpha_\delta},z_{\alpha_\delta},\lambda_{\alpha_\delta})(\eta(y,x_{\alpha_\delta})) + f(y,x_{\alpha_\delta}) \to T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})) + f(y,\bar{x}).$$

The graph $\mathcal{G}(W)$ of W is closed, therefore

$$(\bar{x}, T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})) + f(y,\bar{x})) \in \mathcal{G}(W),$$

which implies

$$T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})) + f(y,\bar{x}) \in W(\bar{x}),$$

which leads to

$$T(\bar{x},\bar{z},\bar{\lambda})(\eta(y,\bar{x})) + f(y,\bar{x}) \notin -\operatorname{int} C(\bar{x}).$$

Hence, $\bar{x} \in F(y)$.

Thus, for each $y \in K$, $F(y)$ is a closed subset of a compact set K, so $F(y)$ is compact. Now, by KKM-Theorem, we have $\bigcap_{y \in K} F(y) \neq \emptyset$. Hence, there exists $x_0 \in K$ such that $x_0 \in F(y)$, for all $y \in K$, that is, there exist $z_0 \in K$, $\lambda_0 \in [0,1]$ such that $T(x_0,z_0,\lambda_0)(\eta(y,x_0)) + f(y,x_0) \notin -\operatorname{int} C(x_0)$, for every $y \in K$. □

Remark 3.3. The existence results for the solution of problems η-GVVLIP (III) and η-GVVLIP (IV) can be proved along similar lines to those of Theorem 3.1 and those of Theorem 3.2, respectively.

Remark 3.4. If we take $\eta(x,y) = x-y$, $f(x,y) = 0_Y$ for all $(x,y) \in K\times K$, $z = 0_X$ and $\lambda = 0$, then the η-generalized vector variational-like inequality problems reduce to the vector variational inequality problems discussed in [11]. Also, we have

(i) Theorem 3.1 reduces to Theorem 3.1 of [11];
(ii) Theorem 3.2 reduces to Theorem 3.2 of [11].

Here, we provide an example to illustrate our results as well as to show that our results are independent of the result obtained by Li *et al.* [15].

Example 3.5. Consider $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $K = [0,1]\times[0,1]$. Clearly, K is closed convex and compact. Let $C : K \rightrightarrows Y$ be defined by $C(x) = \mathbb{R}^+ \cup \{0\}$, for every $x \in K$. Then, $C(x)$ is a closed convex pointed cone with $\text{int}C(x) \neq \emptyset$, and $-\text{int}\, C(x) = (-\infty, 0)$, for each $x \in K$. Let $\eta : K \times K \to X$ and $f : K\times K \to Y$ be defined by $\eta(y,x) = y - x$ and $f(y,x) = 3(\|y\| - \|x\|)$, respectively. Further, let $T : K \times K \times [0,1] \to \mathcal{C}(X,Y)$ be defined by $T_x(u) = -\langle x, u\rangle$, where $x = (x_1, x_2)$ and $u = (u_1, u_2)$ are in K. That the induced topology of $\mathcal{C}(X,Y)$ is admissible can be verified by the fact that if $\{x_n\}$ converges to x in X and $\{h_n\}$ converges to h in $\mathcal{C}(X,Y)$, then we have

$$\begin{aligned}\|h_n(x_n) - h(x)\| &= \|h_n(x_n) - h_n(x) + h_n(x) - h(x)\|\\ &\le \|h_n(x_n) - h_n(x)\| + \|h_n(x) - h(x)\|\\ &\le \|h_n\|\|x_n - x\| + \|h_n(x) - h(x)\|.\end{aligned}$$

Hence, $h_n(x_n) \to h(x)$.

We take $x_0 = (0,0)$. Then, for any $y = (y_1, y_2)$, $z = (z_1, z_2)$ in K, and $\lambda \in [0,1]$, we have

$$\begin{aligned}T_{\lambda x_0+(1-\lambda)z}(\eta(y,x_0)) + f(y,x_0) &= -\langle \lambda x_0 + (1-\lambda)z, \eta(y,x_0)\rangle + f(y,x_0)\\ &= (\lambda-1)(z_1y_1 + z_2y_2) + 3\|y\| \ge 0,\end{aligned}$$

for $z = (\frac{1}{2}, \frac{1}{2})$, $\lambda = \frac{1}{2}$ and for all $y \in K$. Therefore,

$$T_{\lambda x_0+(1-\lambda)z}(\eta(y,x_0)) + f(y,x_0) \notin -\text{int}\, C(x_0).$$

Hence, x_0 is a solution for the η-generalized vector variational-like inequality problem.

T is not η-monotone in C: Let $T : K \to \mathcal{C}(X,Y)$ and $\eta : K\times K \to K$ be two mappings and suppose $C = \bigcap_{x\in K} C(x) \neq \emptyset$. T is called *η-monotone in C* ([15]) if and only if, for every pair $x, y \in K$, we have $\langle T(x) - T(y), \eta(x,y)\rangle \in C$.

Now,

$$\langle T(x) - T(y), \eta(x,y)\rangle = T_{x-y}(\eta(x,y)) = -\langle x-y, x-y\rangle = -\|x-y\|^2 < 0.$$

Thus, $T_{x-y}(\eta(x,y)) \notin C$. Hence, T is not η-monotone in C.

In the following result, we discuss some topological properties of the solution sets obtained above.

Theorem 3.6. *The solution set for the η-GVVLIP (I) (or η-GVVLIP (II)) obtained in Theorem 3.1 (or Theorem 3.2) is closed as well as compact.*

Proof. Let $F : K \rightrightarrows K$ be the set-valued map defined by

$$F(u) = \{x \in K : \exists z \in K, \exists \lambda \in [0,1] \text{ s.t. } T(x,z,\lambda)(\eta(u,x)) + f(u,x) \notin -\operatorname{int} C\}.$$

Then, by Theorem 3.1, the solution set $\mathcal{S}$ of the η-GVVLIP (I) is given by $\mathcal{S} = \bigcap_{u\in K} F(u)$. As shown in Theorem 3.1 (or Theorem 3.2), $F(u)$ is closed for every $u \in K$. Therefore, $\bigcap_{u\in K} F(u)$ is closed, that is, $\mathcal{S}$ is closed. Also, $\mathcal{S}$, being a closed subset of a compact set K, is compact. $\square$

Conclusion

In this study, we have provided existence theorems for the solution of two variants of the η-generalized vector variational-like inequality problem by adopting a topological approach, a significantly different one from those in the existing literature so far. The admissibility of function space topology and net theory are the major tools of achieving the main results. These tools have not been extensively used earlier in literature to obtain such results. It would be interesting to see whether this approach may be used for other variants of variational inequality problems.

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