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ON A SUBCLASS OF BI-CLOSE-TO-CONVEX FUNCTIONS BY MEANS OF THE GEGENBAUER POLYNOMIAL

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Abstract. In this paper, we express a new subcollection of bi-close-to-convex functions by means of Gegenbauer polynomials in the open unit disc $\mathbb{U}$. Further, several related outcomes such as coefficient bounds and Fekete–Szegő inequalities are obtained.

1. Introduction and preliminaries

The theory of orthogonal polynomials may be discussed in two major parts. These parts have many common features and the separation is quite unclear. It is more or less about algebra vs. analysis. These polynomials have a fundamental role in other branches of mathematics. For example, analytic functions (Bieberbach's conjecture), graph theory (matching numbers), operator theory (Jacobi operators), number theory (irrationality and transcendence), numerical analysis, approximation theory, special functions, combinatorics, random matrices, computer tomography (see [9, 10, 16]). Here, orthogonal polynomials are discussed by means of analytic functions (bi-univalent functions).

If $\{P_n(x)\}_{n=0}^{\infty}$ is a sequence of polynomials $P_n(x)$ with degree n such that

$$\int_a^b w(x)P_n(x)P_m(x)dx = 0 \qquad (n \neq m),$$

then $\{P_n(x)\}$ is named orthogonal with regard to the weight function w on the interval (a, b) $(a < b)$.

One of the distinctive cases of orthogonal polynomials is Gegenbauer polynomials. These polynomials are endowed with typically real functions $\mathcal{T}_R$ due to the relation $\mathcal{T}_R = \overline{co}\mathcal{S}_R$ [14]. This relation has an impressive role in the theory of geometric functions in estimating coefficient bounds. Here, $\mathcal{S}_R$ indicates the class of univalent functions with real coefficients and $\overline{co}\mathcal{S}_R$ indicates the closed convex hull of $\mathcal{S}_R$. Recently, many researchers have discussed these polynomials for a subfamily of bi-univalent functions (see [7, 8, 19]).

Gegenbauer polynomials $C_n^{\gamma}(x)$ are given as the coefficients of z^n for the generating function

$$H_{\gamma}(x, z) := \frac{1}{(1 - 2xz + z^2)^{\gamma}},$$

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where $x \in [-1,1]$, $\gamma \in (-1/2,\infty)\setminus\{0\}$ and , $z \in \mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For a fixed x, the function $H_\gamma(x,z)$ is analytic in $\mathbb{U}$ and has the following Taylor series expansion ([10, 17, 20, 21]):

$$\frac{1}{(1-2xz+z^2)^\gamma} = \sum_{n=0}^{\infty} C_n^\gamma(x) z^n. \tag{1.1}$$

For $\gamma = 0$, we arrive at

$$H_0(x,z) = 1 - \log\left(1-2xz+z^2\right) = \sum_{n=0}^{\infty} C_n^0(x) z^n.$$

Also, Gegenbauer polynomials can satisfy the recurrence relation

$$nC_n^\gamma(x) = 2x\,(n+\gamma-1)\,C_{n-1}^\gamma(x) - (n+2\gamma-2)\,C_{n-2}^\gamma(x),$$

with the initial values

$$C_0^\gamma(x) = 1, \quad C_1^\gamma(x) = 2\gamma x \quad \text{and} \quad C_2^\gamma(x) = 2\gamma\,(1+\gamma)\,x^2 - \gamma. \tag{1.2}$$

Remark 1.1. It is worthy to note that for $\gamma = 1$ and $\gamma = 1/2$, we get Chebyshev polynomials and Legendre polynomials, respectively.

In the survey, we study a new family of bi-close-to-convex functions by using Gegenbauer polynomials. Furthermore, some properties of the family such as the initial coefficients $|a_2|$ and $|a_3|$ for the functions belonging to the family and the celebrated Fekete–Szegő problem are considered. For this purpose, relevant definitions and some important concept details will be recalled.

Indicate by $\mathcal{A}$ the family of functions with the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.3}$$

which are analytic in $\mathbb{U}$ and fulfill $f(0) = f'(0) - 1 = 0$. Indicate by $\mathcal{S}$ the family of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

If f_1 and f_2 are analytic in $\mathbb{U}$, then we say that f_1 is subordinate to f_2, showed by $f_1 \prec f_2$, for the Schwarz function

$$\varpi(z) = \sum_{n=1}^{\infty} \mathbf{c}_n z^n \qquad (\varpi(0) = 0, \quad |\varpi(z)| < 1),$$

analytic in $\mathbb{U}$ such that

$$f_1(z) = f_2(\varpi(z)) \qquad (z \in \mathbb{U}).$$

It is known that $|\mathbf{c}_n| \le 1$ for $\varpi(z)$ (see [11]).

Due to the Koebe One-Quarter Theorem [11], every function $f \in \mathcal{S}$ has an inverse map f^{-1} expressed by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f); \quad r_0(f) \ge \frac{1}{4}\right).$$

Thus, the inverse function is given by

$$F(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.4}$$

If f and f^{-1} are univalent in $\mathbb{U}$, then the function $f \in \mathcal{A}$ is named bi-univalent. We will denote the class which consists of functions f that are bi-univalent by Σ.

Gao and Zhou [12] established the subcollection $\mathcal{K}_s$ of close-to-convex functions as below.

Definition 1.2. [12] A function $f \in \mathcal{A}$ given by (1.3) is said to be in the class $\mathcal{K}_s$ if there exists a function $g \in \mathcal{S}^*(1/2)$ such that

$$\Re\left(\frac{z^2 f'(z)}{-g(z)g(-z)}\right) > 0 \qquad (z \in \mathbb{U}),$$

where $\mathcal{S}^*(1/2)$ indicates the class of starlike functions of order $1/2$.

Wang and Chen [23] extended that study by using the principle of subordination. After that, various subfamilies of close-to-convex functions have appeared [13,15,24,25]). Finally, Şeker and Sümer Eker [22] extended these definitions to the bi-univalent function class Σ. On the other hand, many researchers ([1–6, 18, 19]) recently investigated and introduced various subfamilies of bi-univalent functions.

Now, we define a new subfamily of bi-close-to-convex functions by using Gegenbauer polynomials as below.

Definition 1.3. Let $0 \leq \lambda \leq 1$, $x \in (1/2, 1)$ and $\gamma \in (-1/2, \infty) \setminus \{0\}$. A function $f \in \Sigma$ given by (1.3) is said to be in the class $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$ if there exist the functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} t_n w^n \in \mathcal{S}^*(1/2)$$

and the conditions are fulfilled

$$\frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} \prec H_\gamma(x, z) \qquad (z \in \mathbb{U}), \tag{1.5}$$

$$\frac{w^2 F'(w) + \lambda w^3 F''(w)}{-G(w)G(-w)} \prec H_\gamma(x, w) \qquad (w \in \mathbb{U}), \tag{1.6}$$

where the function $F = f^{-1}$ is defined by (1.4) and H_γ is the generating function of the Gegenbauer polynomial given by (1.1).

Remark 1.4. (i) If we set $\lambda = 0$, then the class $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$ reduces to the class $\mathcal{M}_{\Sigma_s}(H_\gamma)$, which consists of functions $f \in \Sigma$ satisfying

$$\frac{z^2 f'(z)}{-g(z)g(-z)} \prec H_\gamma(x, z) \qquad (z \in \mathbb{U}),$$

$$\frac{w^2 F'(w)}{-G(w)G(-w)} \prec H_\gamma(x, w) \qquad (w \in \mathbb{U}).$$

(ii) If we set $\lambda = 1$, then the class $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$ reduces to the class $\mathcal{N}_{\Sigma_s}(H_\gamma)$, which consists of functions $f \in \Sigma$ satisfying

$$\frac{z^2 f'(z) + z^3 f''(z)}{-g(z)g(-z)} \prec H_\gamma(x, z) \qquad (z \in \mathbb{U}),$$

$$\frac{w^2 F'(w) + w^3 F''(w)}{-G(w)G(-w)} \prec H_\gamma(x, w) \qquad (w \in \mathbb{U}).$$

The main purpose of this paper is to obtain coefficient estimates for functions f belonging to the class $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$. Also, we find Fekete–Szegő type coefficient bounds for these functions.

Now, for proving the main outcomes, we shall look at the following lemma.

Lemma 1.5. ([12]) *If* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2)$, *then*

$$\psi(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1} \in \mathcal{S}^* \subset \mathcal{S},$$

where the coefficients of the odd-starlike function ψ *satisfy the condition*

$$|B_{2n-1}| = \left| 2b_{2n-1} - 2b_2 b_{2n-2} + \cdots + 2(-1)^n b_{n-1} b_{n+1} + (-1)^{n+1} b_n^2 \right| \leq 1$$

for $n \geq 2$.

2. Coefficient estimates

Throughout this paper, we assume that

$$x \in (1/2, 1), \quad 0 \leq \lambda \leq 1, \quad \gamma \in (-1/2, \infty) \setminus \{0\}$$

and

$$x^2 \neq \frac{(1+\lambda)^2}{2(1+\lambda)^2 - \gamma(1+2\lambda-2\lambda^2)}.$$

Theorem 2.1. *Let the function* f *given by* (1.3) *be in the class* $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$. *Then,*

$$|a_2| \leq \min\left\{ \frac{|\gamma| x}{1+\lambda}, \sqrt{\frac{|\gamma| x^2 (1+2|\gamma| x)}{|[\gamma(1+2\lambda-2\lambda^2) - 2(1+\lambda)^2] x^2 + (1+\lambda)^2|}} \right\}$$

and

$$|a_3| \leq \frac{\gamma^2 x^2}{(1+\lambda)^2} + \frac{1+2|\gamma| x}{3(1+2\lambda)}.$$

Proof. Let $f \in \mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$ be given by (1.3). Then, by Definition 1.3, there exist the functions

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^*(1/2), \qquad G(w) = w + \sum_{n=2}^{\infty} d_n w^n \in \mathcal{S}^*(1/2)$$

satisfying (1.5) and (1.6). Firstly, we will re-arrange the relations in (1.5) and (1.6) as follows:

$$\begin{aligned} \frac{z^2 f'(z) + \lambda z^3 f''(z)}{-g(z)g(-z)} &= \frac{z f'(z) + \lambda z^2 f''(z)}{\frac{-g(z)g(-z)}{z}} \\ &= \frac{z f'(z) + \lambda z^2 f''(z)}{\psi(z)} \\ &\preceq H_\gamma(x, z) \qquad (z \in \mathbb{U}) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\frac{w^2F'(w)+\lambda w^3F''(w)}{-G(w)G(-w)} &= \frac{wF'(w)+\lambda w^2F''(w)}{\frac{-G(w)G(-w)}{w}} \\ &= \frac{wF'(w)+\lambda w^2F''(w)}{\Omega(w)} \\ &\preceq H_\gamma(x,w) \qquad (w\in\mathbb{U}),\end{aligned} \tag{2.2}$$

respectively, where

$$\psi(z) := \frac{-g(z)g(-z)}{z} \qquad \text{and} \qquad \Omega(w) := \frac{-G(w)G(-w)}{w}.$$

Furthermore, by Lemma 1.5, we have the following equations:

$$\psi(z) = \frac{-g(z)g(-z)}{z} := z+\sum_{n=2}^{\infty} B_{2n-1}z^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |B_{2n-1}|\le 1, \tag{2.3}$$

$$\Omega(w) = \frac{-G(w)G(-w)}{w} := w+\sum_{n=2}^{\infty} D_{2n-1}w^{2n-1} \in \mathcal{S}^* \quad \text{and} \quad |D_{2n-1}|\le 1. \tag{2.4}$$

On the other hand, by means of the principle of subordination, we get from (2.1) and (2.2)

$$\frac{zf'(z)+\lambda z^2f''(z)}{\psi(z)} = H_\gamma(x,\varpi(z)) \tag{2.5}$$

and

$$\frac{wF'(w)+\lambda w^2F''(w)}{\Omega(w)} = H_\gamma(x,\varpi(w)), \tag{2.6}$$

respectively, where

$$\varpi(z) = \mathbf{c}_1 z+\mathbf{c}_2 z^2+\mathbf{c}_3 z^3+\cdots$$

and

$$\varpi(w) = \mathbf{d}_1 w+\mathbf{d}_2 w^2+\mathbf{d}_3 w^3+\cdots.$$

By virtue of the generating function of the Gegenbauer polynomial H_γ, the equations (2.5) and (2.6) can be expressed by

$$\frac{zf'(z)+\lambda z^2f''(z)}{\psi(z)} = 1+C_1^\gamma(x)\mathbf{c}_1 z+\left[C_1^\gamma(x)\mathbf{c}_2+C_2^\gamma(x)\mathbf{c}_1^2\right]z^2+\cdots$$

and

$$\frac{wF'(w)+\lambda w^2F''(w)}{\Omega(w)} = 1+C_1^\gamma(x)\mathbf{d}_1 w+\left[C_1^\gamma(x)\mathbf{d}_2+C_2^\gamma(x)\mathbf{d}_1^2\right])w^2+\cdots,$$

respectively. A direct calculation shows that

$$2(1+\lambda)a_2 = C_1^\gamma(x)\mathbf{c}_1, \tag{2.7}$$

$$3(1+2\lambda)a_3 - B_3 = C_1^\gamma(x)\mathbf{c}_2+C_2^\gamma(x)\mathbf{c}_1^2 \tag{2.8}$$

and

$$-2(1+\lambda)a_2 = C_1^\gamma(x)\mathbf{d}_1, \tag{2.9}$$

$$3(1+2\lambda)(2a_2^2-a_3) - D_3 = C_1^\gamma(x)\mathbf{d}_2+C_2^\gamma(x)\mathbf{d}_1^2. \tag{2.10}$$

From (2.7) and (2.9), we have

$$\mathbf{c}_1 = -\mathbf{d}_1 \tag{2.11}$$

and

$$8(1+\lambda)^2 a_2^2 = \left[C_1^\gamma(x)\right]^2 \left(\mathbf{c}_1^2 + \mathbf{d}_1^2\right). \tag{2.12}$$

Summing up (2.8) to (2.10), we get

$$6(1+2\lambda)a_2^2 - B_3 - D_3 = C_1^\gamma(x)\left(\mathbf{c}_2 + \mathbf{d}_2\right) + C_2^\gamma(x)\left(\mathbf{c}_1^2 + \mathbf{d}_1^2\right). \tag{2.13}$$

By using (2.12) in (2.13), we get

$$\left[6(1+2\lambda) - \frac{8(1+\lambda)^2 C_2^\gamma(x)}{[C_1^\gamma(x)]^2}\right] a_2^2 = B_3 + D_3 + C_1^\gamma(x)\left(\mathbf{c}_2 + \mathbf{d}_2\right). \tag{2.14}$$

By considering (1.2), (2.3) and (2.4), we get from (2.14) the desired inequality. Next, by subtracting (2.10) from (2.8) and using (2.11), we get

$$6(1+2\lambda)a_3 - 6(1+2\lambda)a_2^2 - B_3 + D_3 = C_1^\gamma(x)(\mathbf{c}_2 - \mathbf{d}_2). \tag{2.15}$$

By considering (2.12), (2.3) and (2.4), we get the desired inequality $|a_3|$. □

Letting $\lambda = 0$ in Theorem 2.1, we have the following consequence.

Corollary 2.2. *Let the function f given by* (1.3) *be in the class $\mathcal{M}_{\Sigma_s}(H_\gamma)$. Then,*

$$|a_2| \le \min\left\{|\gamma|\, x,\ \sqrt{\frac{|\gamma|\, x^2\,(1+2\,|\gamma|\, x)}{|(\gamma-2)\, x^2 + 1|}}\right\}$$

and

$$|a_3| \le \gamma^2 x^2 + \frac{1+2|\gamma| x}{3}.$$

Letting $\lambda = 1$ in Theorem 2.1, we have the following consequence.

Corollary 2.3. *Let the function f given by* (1.3) *be in the class $\mathcal{N}_{\Sigma_s}(H_\gamma)$. Then,*

$$|a_2| \le \min\left\{\frac{|\gamma|\, x}{2},\ \sqrt{\frac{|\gamma|\, x^2\,(1+2\,|\gamma|\, x)}{|(\gamma-8)\, x^2 + 4|}}\right\}$$

and

$$|a_3| \le \frac{\gamma^2 x^2}{4} + \frac{1+2\,|\gamma|\, x}{9}.$$

3. Fekete–Szegő problem

Theorem 3.1. *Let the function f given by* (1.3) *be in the class $\mathcal{K}_{\Sigma_s}(\lambda, H_\gamma)$. Then, for some $\eta \in \mathbb{R}$,*

$$\left|a_3 - \eta a_2^2\right| \le \begin{cases} \frac{|\gamma| x^2 |1-\eta|}{|[\gamma(1+2\lambda-2\lambda^2)-2(1+\lambda)^2]x^2+(1+\lambda)^2|} + \frac{1+2|\gamma|x}{3(1+2\lambda)} \\ \quad \text{if} \quad \left|I_\gamma^\lambda(x,\eta)\right| \le \frac{1}{3(1+2\lambda)} \\ \frac{|\gamma| x^2 (1+2|\gamma|x)|1-\eta|}{|[\gamma(1+2\lambda-2\lambda^2)-2(1+\lambda)^2]x^2+(1+\lambda)^2|} + \frac{1}{3(1+2\lambda)} \\ \quad \text{if} \quad \left|I_\gamma^\lambda(x,\eta)\right| \ge \frac{1}{3(1+2\lambda)} \end{cases},$$

where

$$I_{\gamma}^{\lambda}(x,\eta)=\frac{\gamma x^{2}}{\left[\gamma(1+2\lambda-2\lambda^{2})-2(1+\lambda)^{2}\right]x^{2}+(1+\lambda)^{2}}(1-\eta).$$

Proof. Let $f\in\mathcal{K}_{\Sigma_s}(\lambda,H_{\gamma})$ be given by (1.3). In the light of the proof of Theorem 2.1, by using (2.14), (2.15) and (1.2) for some $\eta\in\mathbb{R}$, we get

$$a_3-\eta a_2^2=\frac{\gamma x^{2}\left(B_3+D_3\right)(1-\eta)}{2\left[\gamma(1+2\lambda-2\lambda^{2})-2(1+\lambda)^{2}\right]x^{2}+2(1+\lambda)^{2}}+\frac{B_3-D_3}{6(1+2\lambda)}$$

$$+\gamma x\left[\left(I_{\gamma}^{\lambda}(x,\eta)+\frac{1}{3(1+2\lambda)}\right)\mathbf{c}_2+\left(I_{\gamma}^{\lambda}(x,\eta)-\frac{1}{3(1+2\lambda)}\right)\mathbf{d}_2\right],$$

where

$$I_{\gamma}^{\lambda}(x,\eta)=\frac{\gamma x^{2}}{\left[\gamma(1+2\lambda-2\lambda^{2})-2(1+\lambda)^{2}\right]x^{2}+(1+\lambda)^{2}}(1-\eta).$$

So, we conclude that

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{|\gamma|x^{2}|1-\eta|}{\left|\left[\gamma(1+2\lambda-2\lambda^{2})-2(1+\lambda)^{2}\right]x^{2}+(1+\lambda)^{2}\right|}+\frac{1+2|\gamma|x}{3(1+2\lambda)}\\ \quad\text{if}\quad\left|I_{\gamma}^{\lambda}(x,\eta)\right|\leq\frac{1}{3(1+2\lambda)}\\ \frac{|\gamma|x^{2}(1+2|\gamma|x)|1-\eta|}{\left|\left[\gamma(1+2\lambda-2\lambda^{2})-2(1+\lambda)^{2}\right]x^{2}+(1+\lambda)^{2}\right|}+\frac{1}{3(1+2\lambda)}\\ \quad\text{if}\quad\left|I_{\gamma}^{\lambda}(x,\eta)\right|\geq\frac{1}{3(1+2\lambda)}\end{cases}.$$

□

Letting $\lambda=0$ in Theorem 3.1, we have the following consequence.

Corollary 3.2. *Let the function f given by* (1.3) *be in the class $\mathcal{M}_{\Sigma_s}(H_{\gamma})$. Then, for some $\eta\in\mathbb{R}$,*

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{|\gamma|x^{2}|1-\eta|}{|(\gamma-2)x^{2}+1|}+\frac{1+2|\gamma|x}{3}, & \left|I_{\gamma}(x,\eta)\right|\leq\frac{1}{3}\\ \frac{|\gamma|x^{2}(1+2|\gamma|x)|1-\eta|}{|(\gamma-2)x^{2}+1|}+\frac{1}{3}, & \left|I_{\gamma}(x,\eta)\right|\geq\frac{1}{3}\end{cases},$$

where

$$I_{\gamma}(x,\eta)=\frac{\gamma x^{2}(1-\eta)}{(\gamma-2)x^{2}+1}.$$

Letting $\lambda=1$ in Theorem 3.1, we have the following consequence.

Corollary 3.3. *Let the function f given by* (1.3) *be in the class $\mathcal{N}_{\Sigma_s}(H_{\gamma})$. Then, for some $\eta\in\mathbb{R}$,*

$$\left|a_3-\eta a_2^2\right|\leq\begin{cases}\frac{|\gamma|x^{2}|1-\eta|}{|(\gamma-8)x^{2}+4|}+\frac{1+2|\gamma|x}{9}, & \left|I_{\gamma}(x,\eta)\right|\leq\frac{1}{9}\\ \frac{|\gamma|x^{2}(1+2|\gamma|x)|1-\eta|}{|(\gamma-8)x^{2}+4|}+\frac{1}{9}, & \left|I_{\gamma}(x,\eta)\right|\geq\frac{1}{9}\end{cases},$$

where

$$I_{\gamma}(x,\eta)=\frac{\gamma x^{2}}{(\gamma-8)\,x^{2}+4}(1-\eta).$$

4. Conclusion

The survey introduces a new subcollection of bi-close-to-convex functions by means of Gegenbauer polynomials. Also, the approach presented here has been extended to establish new subfamilies of bi-univalent functions with the other special functions. The related outcomes may be left to the the researchers for practice.

References

[1] A. G. AlAmoush, *Coefficient estimates for a new subclasses of lambda-pseudo bi-univalent functions with respect to symmetrical points associated with the Horadam Polynomials*, Turk. J. Math. **3** (2019), 2865–2875.
[2] A. G. AlAmoush, *Coefficient estimates for certain subclass of bi-Bazilevic functions associated with Chebyshev polynomials*, Acta Univ. Apulensis **60** (2019), 53–59.
[3] A. G. AlAmoush, *On subclass of analytic bi-close-to-convex functions*, Int. J. Open Problems Complex Analysis **13** (2021), 10–18.
[4] A. G. AlAmoush, *On a subclass of bi-univalent functions associated to Horadam polynomials*, Int. J. Open Problems Complex Analysis **12** (2020), 58–65.
[5] A. G. AlAmoush and M. Darus, *On coefficient estimates for new generalized subclasses of bi-univalent functions*, AIP Conf. Proc. **1614** (2014), 844–850.
[6] Ş. Altınkaya, *Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers*, Turk. J. Math. **44** (2020), 553–560.
[7] A. Amourah, A. G. AlAmoush and M. Al-Kaseasbeh, *Gegenbauer polynomials and bi-univalent functions*, Palest. J. Math. **10** (2021), 625–632.
[8] A. Amourah, B. A. Frasin and A. Thabet, *Fekete–Szegő inequality for analytic and bi-univalent functions subordinate to Gegenbauer polynomials*, J. Funct. Spaces **2021** (2021), Article ID 5574673, 7 pp.
[9] H. Bateman, *Higher Transcendental Functions*, McGraw-Hill, 1953.
[10] B. Doman, *The Classical Orthogonal Polynomials*, World Scientific, 2015.
[11] P. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer-Verlag, New York, 1983.
[12] C. Y. Gao and S. Q. Zhou, *On a class of analytic functions related to the starlike functions*, Kyungpook Math. J. **45** (2005), 123–130.
[13] P. Goswami, S. Bulut and T. Bulboaca, *Certain properties of a new subclass of close-to-convex functions*, Arab. J. Math. **1** (2012), 309–317.
[14] K. Kiepiela, I. Naraniecka and J. Szynal, *The Gegenbauer polynomials and typically real functions*, J. Comput. Appl. Math. **153** (2003), 273–282.
[15] J. Kowalczyk and E. Les-Bomba, *On a subclass of close-to-convex functions*, Appl. Math. Lett. **23** (2010), 1147–1151.
[16] A. M. Legendre, *Recherches sur quelques objets d'analyse indéterminée et particuliérement sur le théorème de Fermat*, Mém. Acad. Roy. Sci. Institut France **6** (1823), 1–60.
[17] W. Liu and L. Wang, *Asymptotics of the generalized Gegenbauer functions of fractional degree*, J. Approx. Theory **253** (2020), Article ID 105378, 25 pp.
[18] N. Magesh, C. Abirami and Ş. Altınkaya, *Initial bounds for certain classes of bi-univalent functions defined by the (p,q)-Lucas polynomials*, TWMS J. Appl. Eng. Math. **11** (2021), 282–288.
[19] G. Murugusundaramoorthy, V. Kaliappan and H. Ö. Güney, *Coefficient bound for certain subclasses of bi-starlike functions associated with the Gegenbauer polynomials*, Proc. Jangjeon Math. Soc., in press.

[20] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, 2010.

[21] M. Reimer, *Multivariate Polynomial Approximation*, ISNM International Series of Numerical Mathematics **144**, Birkhäuser, 2003.

[22] B. Şeker and S. Sümer Eker, *On subclasses of bi-close-to-convex functions related to the odd-starlike functions*, Palest. J. Math. **6** (2017), 215–221.

[23] Z.-G. Wang, C.-Y. Gao and S.-M. Yuan, *On certain subclasses of close-to-convex and quasi-convex functions with respectto k-symmetric points*, J. Math. Anal. Appl. **322** (2006), 97–106.

[24] Z.-G. Wang and D. Z. Chen, *On a subclass of close-to-convex functions*, Hacet. J. Math. Stat. **38** (2009), 95–101.

[25] Z.-G. Wang, C. Y. Gao and S.-M. Yuan, *On certain subclass of close-to-convex functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) **22** (2006), 171–177.

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