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BIFURCATION OF POSITIVE PERIODIC SOLUTIONS TO NON-AUTONOMOUS UNDAMPED DUFFING EQUATIONS

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Abstract. We study a bifurcation of positive solutions to the parameter-dependent periodic problem

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega),$$

where $\lambda > 1$, $p, h, f \in L([0,\omega])$, and $\mu \in \mathbb{R}$ is a parameter. Both the coefficient p and the forcing term f may change their signs, $h \geq 0$ a. e. on $[0,\omega]$. We provide sharp conditions on the existence and multiplicity as well as non-existence of positive solutions to the given problem depending on the choice of the parameter μ.

1. Introduction

Consider the parameter-dependent problem

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega), \tag{1.1}$$

where $p, h, f \in L([0,\omega])$, $h \geq 0$ a. e. on $[0,\omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1.1), as usual, we understand a function $u\colon [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions.

We first note that the differential equation in (1.1) with $\lambda = 3$ is derived, for example, when approximating a non-linearity in the equation of motion of the oscillator illustrated in Fig. 1. Consider a forced undamped oscillator consisting of a mass body of weight m and a linear spring of characteristic k and non-deformed length ℓ. Assume that the mass body moves horizontally without any friction and the spring's base point B oscillates vertically, i.e., d is a positive ω-periodic function. This is a system with a single degree of freedom, described by the coordinate x, whose equation of motion is of the form

$$x'' = \frac{k}{m}\, x \left(\frac{\ell}{\sqrt{d^2(t) + x^2}} - 1 \right) + \frac{F(t)}{m}\,. \tag{1.2}$$

A classical approach to deriving Duffing equation is to approximate the non-linearity in (1.2) by a third-order Taylor polynomial centred at 0. We thus get the

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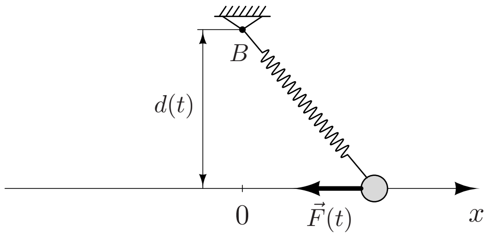

Figure 1. Forced undamped oscillator.

equation

$$x'' = \frac{k(\ell - d(t))}{md(t)}\, x - \frac{k\ell}{2md^3(t)}\, x^3 + \frac{F(t)}{m}\,, \tag{1.3}$$

which is a particular case of the differential equation in (1.1). It is worth mentioning that the results of the present paper can be applied, for instance, to the forcing terms

$$F(t) := -f_0, \qquad F(t) := A\left(\sin\frac{2\pi t}{\omega} - \frac{1}{2}\right), \tag{1.4}$$

where $f_0, A > 0$. Hence, Theorem 3.1 below provides information about the exact multiplicity of positive ω-periodic solutions to equation (1.3) depending on the value of f_0, resp. A (for discussion, see Section 6).

For the results covering the multiplicity and local/global bifurcations of periodic solutions to Duffing equations, we refer readers, for instance, to [2,4,5,8] (see also references therein). In [2,4,8], the authors study the parameter-dependent problems for second-order differential equations assuming a strong damped condition and a sign-constant forcing term. In the present paper, we consider an *undamped non-autonomous* Duffing equation with a linear part of the class $\mathcal{V}^-(\omega)$ (see Definition 2.1, Remark 3.2) and a forcing term f, which *may change its sign.* We use the results presented in [9] and show the existence and multiplicity as well as non-existence of *positive* solutions to problem (1.1) depending on the choice of the parameter μ.

Let us show, as a motivation, what happens in the autonomous case of (1.1). Hence, consider the equation

$$x'' = ax - b|x|^\lambda \operatorname{sgn} x - \mu, \tag{1.5}$$

where $a, b > 0$ and $\mu \in \mathbb{R}$. By direct calculation, the phase portraits of this equation can be elaborated depending on the choice of the parameter μ and, thus, one can prove the following proposition concerning the positive periodic solutions to equation (1.5).

Proposition 1.1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

(i) *If $\mu \le 0$, then equation* (1.5) *has a unique positive equilibrium (center) and non-constant positive periodic solutions with different periods.*

(ii) *If $0 < \mu < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (1.5) *possesses exactly two positive equilibria $x_2 > x_1$ (x_1 is a saddle and x_2 is a center) and non-constant* *positive periodic solutions with different periods. Moreover, all non-constant positive periodic solutions are greater than x_1 and oscillate around x_2.*

(iii) *If $\mu = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (1.5) *has a unique positive equilibrium (cusp) and no non-constant positive periodic solution occurs.*

(iv) *If $\mu > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (1.5) *has no positive periodic solution.*

Proposition 1.1 shows that, if we consider μ as a bifurcation parameter, then, crossing the value $\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, a bifurcation of positive periodic solutions to equation (1.5) occurs. In Section 3, we extend conclusions (ii)–(iv) of Proposition 1.1 for the non-autonomous problem (1.1) with the forcing term f satisfying $\int_0^\omega f(s)\mathrm{d}s < 0$.

2. Notation and definitons

The following notation is used throughout the paper:

- $\mathbb{R}$ is the set of real numbers. For $x \in \mathbb{R}$, we put $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.
- $C(I)$ denotes the set of continuous real functions defined on the interval $I \subseteq \mathbb{R}$. For $u \in C([a,b])$, we put $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.
- $AC^1([a,b])$ is the set of functions $u\colon [a,b] \to \mathbb{R}$ which are absolutely continuous together with their first derivatives.
- $AC_\ell([a,b])$ (resp. $AC_u([a,b])$) is the set of absolutely continuous functions $u\colon [a,b] \to \mathbb{R}$ such that u' admits the representation $u'(t) = \gamma(t) + \sigma(t)$ for a. e. $t \in [a,b]$, where $\gamma\colon [a,b] \to \mathbb{R}$ is absolutely continuous and $\sigma\colon [a,b] \to \mathbb{R}$ is a non-decreasing (resp. non-increasing) function whose derivative is equal to zero almost everywhere on $[a,b]$.
- $L([a,b])$ is the Banach space of Lebesgue integrable functions $p\colon [a,b] \to \mathbb{R}$ equipped with the norm $\|p\|_L = \int_a^b |p(s)|\mathrm{d}s$. The symbol $\operatorname{Int} A$ stands for the interior of the set $A \subset L([a,b])$.

Definition 2.1. ([6, Definitions 0.1 and 15.1, Proposition 15.2]) We say that a function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^-(\omega)$ if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) \geq u'(\omega),$$

the inequality $u(t) \leq 0$ holds for $t \in [0,\omega]$.

Remark 2.2. Let $\omega > 0$. If $p(t) := p_0$ for $t \in [0,\omega]$, then one can show by direct calculation that $p \in \mathcal{V}^-(\omega)$ if and only if $p_0 > 0$. For non-constant functions $p \in L([0,\omega])$, efficient conditions guaranteeing the inclusion $p \in \mathcal{V}^-(\omega)$ are provided in [6] (see also [1,10]).

Remark 2.3. It is well known that, if the homogeneous problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.1}$$

has only the trivial solution, then, for any $f \in L([0,\omega])$, the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega) \tag{2.2}$$

possesses a unique solution u and this solution satisfies

$$|u(t)| \leq \Delta(p) \int_0^\omega |f(s)|\mathrm{d}s \quad \text{for } t \in [0,\omega],$$

where $\Delta(p)$, depending only on p, denotes a norm of the Green's operator of problem (2.1). Clearly, $\Delta(p) > 0$.

Remark 2.4. If $p \in \mathcal{V}^-(\omega)$, then problem (2.1) has only the trivial solution and the number $\Delta(p)$ defined in Remark 2.3 can be estimated, for example, by using a minimal value of the Green's function of problem (2.1) (see, e. g., [10]).

For instance, if $p(t) := p_0$ for $t \in [0,\omega]$ and $p_0 > 0$, then

$$\Delta(p) \leq \left(2\sqrt{p_0}\,\tanh\frac{\omega\sqrt{p_0}}{2}\right)^{-1} < \left(\frac{\omega p_0}{\cosh\frac{\omega\sqrt{p_0}}{2}}\right)^{-1}. \tag{2.3}$$

Definition 2.5 ([6, Definition 16.1])**.** Let $p, f \in L([0,\omega])$. We say that the pair (p,f) belongs to the set $\mathcal{U}(\omega)$ if problem (2.2) has a unique solution which is positive.

3. Main results

Theorem 3.1. *Let $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, and*

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad h(t) \not\equiv 0, \tag{3.1}$$

$$(p,f) \in \mathcal{U}(\omega), \qquad \int_0^\omega f(s)\mathrm{d}s < 0. \tag{3.2}$$

Then, there exists $\mu_0 \in\,]0,+\infty[$ such that the following conclusions hold:

(1) *If $\mu = 0$, then problem (1.1) has at least one positive solution and, for any couple of distinct positive solutions u_1, u_2 to (1.1), the conditions*

$$\begin{aligned} &\min\{u_1(t) - u_2(t) : t \in [0,\omega]\} < 0,\\ &\max\{u_1(t) - u_2(t) : t \in [0,\omega]\} > 0 \end{aligned} \tag{3.3}$$

hold. If, moreover,

$$\mathrm{e}^{-1+\sqrt{1+\omega\int_0^\omega p(s)\mathrm{d}s}}\left(-1+\sqrt{1+\omega\int_0^\omega p(s)\mathrm{d}s}\right) \leq 8\lambda^*,$$

where

$$\lambda^* := \begin{cases} \left\lfloor \frac{1}{\lambda-1} \right\rfloor & \text{for } \lambda \in\,]1,2], \\ \frac{1}{\lceil \lambda-1 \rceil} & \text{for } \lambda > 2, \end{cases}$$

in which $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor function and ceiling function, respectively, then problem (1.1) with $\mu = 0$ has a unique positive solution.

(2) *If $0 < \mu < \mu_0$, then problem (1.1) has solutions u_1, u_2 such that*

$$u_2(t) > u_1(t) > 0 \quad \text{for } t \in [0,\omega] \tag{3.4}$$

and, for any non-negative solution u to problem (1.1) satisfying

$$u(t) \not\equiv u_1(t), \qquad u(t) \not\equiv u_2(t), \tag{3.5}$$

the conditions

$$u(t) > u_1(t) \quad \textit{for } t \in [0, \omega] \tag{3.6}$$

and

$$\begin{aligned} &\min\{u(t) - u_2(t) : t \in [0, \omega]\} < 0, \\ &\max\{u(t) - u_2(t) : t \in [0, \omega]\} > 0 \end{aligned} \tag{3.7}$$

hold.

(3) *If* $\mu = \mu_0$, *then problem* (1.1) *has a unique positive solution.*

(4) *If* $\mu > \mu_0$, *then problem* (1.1) *has no positive solution.*

Open questions. The following two questions remain open in Theorem 3.1:

(1) Does there exist, for any $\omega > 0$, a positive solution u to problem (1.1) satisfying (3.5) in conclusion (2)?

(2) What happens in the case of $\mu < 0$?

Remark 3.2. By virtue of [6, Theorem 11.1], the hypothesis $p \in \mathcal{V}^-(\omega)$ of Theorem 3.1 is satisfied, for instance, if one of the following conditions hold:

(a)

$$p(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad p(t) \not\equiv 0,$$

(b)

$$0 < \int_0^\omega [p(s)]_- \mathrm{d}s < \frac{4}{\omega}, \qquad \int_0^\omega [p(s)]_+ \mathrm{d}s \geq \frac{\int_0^\omega [p(s)]_- \mathrm{d}s}{1 - \frac{\omega}{4} \int_0^\omega [p(s)]_- \mathrm{d}s}.$$

Other efficient conditions guaranteeing the inclusion $p \in \mathcal{V}^-(\omega)$ and their consequences for particular cases of the coefficient p are available in [6].

Remark 3.3. Let $p \in \mathcal{V}^-(\omega)$. It follows from [6, Theorem 16.2] that hypothesis (3.2) of Theorem 3.1 holds, provided that

$$\int_0^\omega [f(s)]_- \mathrm{d}s > \mathrm{e}^{\frac{\omega}{4} \int_0^\omega [p(s)]_+ \mathrm{d}s} \int_0^\omega [f(s)]_+ \mathrm{d}s. \tag{3.8}$$

In particular, if

$$f(t) \leq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad f(t) \not\equiv 0, \tag{3.9}$$

then (3.2) is satisfied.

We now provide lower and upper estimates of the number μ_0 appearing in the conclusion of Theorem 3.1.

Proposition 3.4. *Let* $\lambda > 1$, $p \in \mathcal{V}^-(\omega)$, h *satify* (3.1), *and* f *be such that* (3.8) *holds. Then, the number* μ_0 *appearing in the conclusion of Theorem 3.1 satisfies*

$$\mu_0 \geq \frac{(\lambda - 1)\, [\Delta(p)]^{-\frac{\lambda}{\lambda - 1}}}{\lambda \left[\lambda \int_0^\omega h(s) \mathrm{d}s\right]^{\frac{1}{\lambda - 1}} \int_0^\omega [f(s)]_- \mathrm{d}s}, \tag{3.10}$$

where Δ *is defined in Remark 2.3, and*

$$\mu_0 < \frac{(\lambda-1)\left[\mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+\mathrm{d}s}\int_0^\omega [p(s)]_+\mathrm{d}s - \int_0^\omega [p(s)]_-\mathrm{d}s\right]^{\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\left[\int_0^\omega [f(s)]_-\mathrm{d}s - \mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+\mathrm{d}s}\int_0^\omega [f(s)]_+\mathrm{d}s\right]}. \tag{3.11}$$

Remark 3.5. If $p \in \mathcal{V}^-(\omega)$, then [6, Proposition 10.8] yields $\int_0^\omega p(s)\mathrm{d}s > 0$. Therefore, inequality (3.11) in Proposition 3.4 is consistent.

Remark 3.6. Theorem 3.1 extends conclusions ((ii))–((iv)) of Proposition 1.1 for the non-autonomous Duffing equations with a sign-changing forcing term. Indeed, let $\omega > 0$ and

$$p(t) := a, \quad h(t) := b, \quad f(t) := -1 \quad \text{for } t \in [0,\omega],$$

where $a, b > 0$. Then, $p \in \mathcal{V}^-(\omega)$ (see Remark 2.2), h and f satisfy (3.1) and (3.9), respectively, and conclusions ((2))–((4)) of Theorem 3.1 coincide with those in Proposition 1.1. Moreover, the number $\Delta(p)$ satisfies (2.3) and, thus, the number μ_0 appearing in Proposition 3.4 satisfies

$$\left(\frac{1}{\cosh\frac{\omega\sqrt{a}}{2}}\right)^{\frac{\lambda}{\lambda-1}}\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}} < \mu_0 < \left(\mathrm{e}^{\frac{\omega^2 a}{4}}\right)^{\frac{\lambda}{\lambda-1}}\frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}};$$

compare it with the number appearing in Proposition 1.1.

4. Auxiliary statements

We first recall some results stated in [9].

Lemma 4.1 ([9, Theorem 3.6])**.** *Let* $\lambda > 1$*,* $\mu \in \mathbb{R}$*,* $p \in \mathcal{V}^-(\omega)$*,* $(p, \mu f) \in \mathcal{U}(\omega)$*, and* h *satisfy* (3.1)*. Let, moreover, there exist a positive function* $\beta \in AC_u([0,\omega])$ *such that*

$$\beta''(t) \le p(t)\beta(t) - h(t)\beta^\lambda(t) + \mu f(t) \quad \textit{for a. e. } t \in [0,\omega], \tag{4.1}$$

$$\beta(0) = \beta(\omega), \quad \beta'(0) = \beta'(\omega). \tag{4.2}$$

Then, problem (1.1) *has a positive solution* u_* *such that every non-negative solution* u *to problem* (1.1) *satisfies*

$$\textit{either} \quad u(t) > u_*(t) \quad \textit{for } t \in [0,\omega], \quad \textit{or} \quad u(t) \equiv u_*(t).$$

Moreover, for any couple of distinct positive solutions u_1*,* u_2 *to* (1.1) *satisfying*

$$u_1(t) \not\equiv u_*(t), \qquad u_2(t) \not\equiv u_*(t),$$

conditions (3.3) *hold.*

Lemma 4.2 ([9, Theorem 3.7])**.** *Let* $\lambda > 1$*,* $\mu \in \mathbb{R}$*,* $p \in \mathcal{V}^-(\omega)$*,* $(p, \mu f) \in \mathcal{U}(\omega)$*, and* h *satisfy* (3.1)*. Let, moreover, there exist functions* $\beta_1 \in AC^1([0,\omega])$ *and* $\beta_2 \in AC_u([0,\omega])$ *such that*

$$0 < \beta_1(t) < \beta_2(t) \quad \textit{for } t \in [0,\omega], \tag{4.3}$$

$$\beta_k''(t) \le p(t)\beta_k(t) - h(t)\beta_k^\lambda(t) + \mu f(t) \quad \textit{for a. e. } t \in [0,\omega], \quad k = 1, 2, \tag{4.4}$$

$$\beta_k(0) = \beta_k(\omega), \quad \beta_k'(0) = \beta_k'(\omega), \quad k = 1, 2. \tag{4.5}$$

Then, there exist solutions u_1, u_2 *to problem* (1.1) *such that* (3.4) *is fulfilled and, for any non-negative solution* u *to problem* (1.1) *satisfying* (3.5), *conditions* (3.6) *and* (3.7) *hold.*

Lemma 4.3 ([9, Corollary 3.9(ii)]). *Let* $\lambda > 1$, $\mu \in \mathbb{R}$, $p \in \mathcal{V}^-(\omega)$, $(p, \mu f) \in \mathcal{U}(\omega)$, *and* h *satisfy* (3.1). *If*

$$\int_0^\omega [\mu f(s)]_- \mathrm{d}s < \frac{\lambda - 1}{\lambda\,[\Delta(p)]^{\frac{\lambda}{\lambda-1}}\left[\lambda \int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}},$$

where Δ *is defined in Remark 2.3, then the conclusion of Lemma 4.2 holds.*

Lemma 4.4 ([9, Theorem 3.11]). *Let* $\lambda > 1$, $\mu \in \mathbb{R} \setminus \{0\}$, $p \in \mathcal{V}^-(\omega)$, h *satisfy* (3.1), *and*

$$\int_0^\omega [\mu f(s)]_- \mathrm{d}s - \mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+ \mathrm{d}s} \int_0^\omega [\mu f(s)]_+ \mathrm{d}s$$
$$\geq \frac{\lambda - 1}{\lambda} \, \frac{\left[\mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+ \mathrm{d}s} \int_0^\omega [p(s)]_+ \mathrm{d}s - \int_0^\omega [p(s)]_- \mathrm{d}s\right]^{\frac{\lambda}{\lambda-1}}}{\left[\lambda \int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}}.$$

Then, problem (1.1) *has no non-negative solution.*

Lemma 4.5 ([6, Theorem 16.2]). *Let* $p \in \mathcal{V}^-(\omega)$. *Then, there exists* $\nu > 0$ *such that, for any non-positive function* $q \in L([0, \omega])$, *the problem*

$$z'' = p(t)z + q(t); \quad z(0) = z(\omega),\ z'(0) = z'(\omega) \tag{4.6}$$

has a unique solution z *and this solution satisfies*

$$z(t) \geq \nu \int_0^\omega |q(s)|\mathrm{d}s \quad \text{for } t \in [0, \omega].$$

Lemma 4.6. *Let* $\lambda > 1$, *conditions* (3.1) *and* (3.2) *hold,* $\{\mu_n\}_{n=1}^\infty$ *be a sequence of positive numbers and let, for any* $n \in \mathbb{N}$, u_n *be a positive solution to problem* (1.1) *with* $\mu = \mu_n$. *Then, the sequences* $\{\|u_n\|_C\}_{n=1}^\infty$ *and* $\{\mu_n\}_{n=1}^\infty$ *are bounded.*

Proof. We first show that

$$\sup\left\{\|u_n\|_C : n \in \mathbb{N}\right\} < +\infty. \tag{4.7}$$

Suppose on the contrary that (4.7) does not hold. Then, we can assume without loss of generality that

$$\lim_{n \to +\infty} \|u_n\|_C = +\infty. \tag{4.8}$$

Put

$$v_n(t) := \frac{u_n(t)}{\|u_n\|_C} \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}.$$

Clearly,

$$\|v_n\|_C = 1, \quad v_n(t) > 0 \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}. \tag{4.9}$$

It follows from (1.1) with $\mu = \mu_n$ that, for any $n \in \mathbb{N}$,

$$v_n''(t) = p(t)v_n(t) - \|u_n\|_C^{\lambda-1} h(t) v_n^\lambda(t) + \frac{\mu_n}{\|u_n\|_C} f(t) \quad \text{for a.e. } t \in [0, \omega], \tag{4.10}$$

which yields

$$0 = \int_0^\omega p(s)v_n(s)\mathrm{d}s - \|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s + \frac{\mu_n}{\|u_n\|_C} \int_0^\omega f(s)\mathrm{d}s$$

for $n \in \mathbb{N}$. In view of (3.2) and (4.9), from the latter equality, we get

$$\|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s + \frac{\mu_n}{\|u_n\|_C} \left| \int_0^\omega f(s)\mathrm{d}s \right| \leq \int_0^\omega |p(s)|\mathrm{d}s \quad \text{for } n \in \mathbb{N}. \tag{4.11}$$

Put

$$A := \sup\left\{ \|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s : n \in \mathbb{N} \right\}, \quad B := \sup\left\{ \frac{\mu_n}{\|u_n\|_C} : n \in \mathbb{N} \right\}. \tag{4.12}$$

By virtue of (3.1), (3.2) and (4.9), it follows from (4.11) that $A \in]0, +\infty[$, $B \in]0, +\infty[$, and we can assume without loss of generality that

$$\lim_{n\to+\infty} \|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s = h_0, \qquad \lim_{n\to+\infty} \frac{\mu_n}{\|u_n\|_C} = \mu_0, \tag{4.13}$$

where

$$h_0 \geq 0, \qquad \mu_0 \geq 0.$$

For any $n \in \mathbb{N}$, we choose $t_n \in [0, \omega]$ such that $v_n'(t_n) = 0$. In view of (3.1), (4.9), and (4.12), integrating (4.10) from t_n to t, we get

$$\begin{aligned} |v_n'(t)| &= \left| \int_{t_n}^t \left[p(s)v_n(s) - \|u_n\|_C^{\lambda-1} h(s)v_n^\lambda(s) + \frac{\mu_n}{\|u_n\|_C} f(s) \right] \mathrm{d}s \right| \\ &\leq \int_0^\omega |p(s)|\mathrm{d}s + \|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s + \frac{\mu_n}{\|u_n\|_C} \int_0^\omega |f(s)|\mathrm{d}s \\ &\leq \int_0^\omega |p(s)|\mathrm{d}s + A + B \int_0^\omega |f(s)|\mathrm{d}s \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}. \end{aligned}$$

Therefore, the sequences $\{\|v_n\|_C\}_{n=1}^\infty$ and $\{\|v_n'\|_C\}_{n=1}^\infty$ are bounded and, thus, by the Arzelà–Ascoli theorem, we can assume without loss of generality that

$$\lim_{n\to+\infty} v_n(t) = v_0(t) \quad \text{uniformly on } [0, \omega], \tag{4.14}$$

where $v_0 \in C([0, \omega])$. From (4.9), we get

$$v_0(t) \geq 0 \quad \text{for } t \in [0, \omega], \qquad \|v_0\|_C = 1. \tag{4.15}$$

It follows from the hypothesis $(p, f) \in \mathcal{U}(\omega)$ that the problem

$$v'' = p(t)v + f(t); \quad v(0) = v(\omega),\ v'(0) = v'(\omega) \tag{4.16}$$

has a unique solution v which is positive. According to Lemma 4.5 (with $q(t) := -\|u_n\|_C^{\lambda-1} h(t)v_n^\lambda(t)$), there exists $\nu > 0$ such that, for any $n \in \mathbb{N}$, the problem

$$w'' = p(t)w - \|u_n\|_C^{\lambda-1} h(t)v_n^\lambda(t); \quad w(0) = w(\omega),\ w'(0) = w'(\omega)$$

has a unique solution w_n and

$$w_n(t) \geq \nu \|u_n\|_C^{\lambda-1} \int_0^\omega h(s)v_n^\lambda(s)\mathrm{d}s \quad \text{for } t \in [0, \omega],\ n \in \mathbb{N}. \tag{4.17}$$

It is clear that $v_n = w_n + \frac{\mu_n}{\|u_n\|_C} v$ for $n \in \mathbb{N}$. Therefore, (4.17) yields

$$v_n(t) \geq \nu \|u_n\|_C^{\lambda-1} \int_0^\omega h(s) v_n^\lambda(s) \mathrm{d}s + \frac{\mu_n}{\|u_n\|_C} v(t) \quad \text{for } t \in [0,\omega],\ n \in \mathbb{N},$$

and, thus, passing the limit for $n \to +\infty$ and taking into account (4.13) and (4.14), we get

$$v_0(t) \geq h_0 + \mu_0 v(t) \quad \text{for } t \in [0,\omega]. \tag{4.18}$$

Let us show that $h_0 + \mu_0 > 0$. Indeed, suppose on the contrary that $h_0 = 0$ and $\mu_0 = 0$. Then, by the hypothesis $p \in \mathcal{V}^-(\omega)$, it follows from (4.10) and Remarks 2.3 and 2.4 that

$$|v_n(t)| \leq \Delta(p) \left(\|u_n\|_C^{\lambda-1} \int_0^\omega h(s) v_n^\lambda(s) \mathrm{d}s + \frac{\mu_n}{\|u_n\|_C} \int_0^\omega |f(s)| \mathrm{d}s \right)$$

for $t \in [0,\omega]$, $n \in \mathbb{N}$, and, therefore, passing the limit for $n \to +\infty$ and taking into account (4.13) and (4.14), we obtain

$$|v_0(t)| \leq \Delta(p) \left(h_0 + \mu_0 \int_0^\omega |f(s)| \mathrm{d}s \right) = 0 \quad \text{for } t \in [0,\omega].$$

However, this contradicts (4.15). Hence, we have proved that $h_0 + \mu_0 > 0$, which, together with (4.18) and the positivity of v, leads to the condition

$$v_0(t) > 0 \quad \text{for } t \in [0,\omega]. \tag{4.19}$$

On the other hand, (4.11) yields

$$\int_0^\omega h(s) v_n^\lambda(s) \mathrm{d}s \leq \frac{1}{\|u_n\|_C^{\lambda-1}} \int_0^\omega |p(s)| \mathrm{d}s \quad \text{for } n \in \mathbb{N},$$

and, therefore, passing the limit for $n \to +\infty$ and taking into account (4.8) and (4.14), we get

$$\int_0^\omega h(s) v_0^\lambda(s) \mathrm{d}s \leq 0.$$

However, in view of (4.19), the latter inequality contradicts (3.1). The obtained contradiction proves that (4.7) holds.

Now we show that the sequence $\{\mu_n\}_{n=1}^\infty$ is bounded. Suppose on the contrary that $\sup\{\mu_n : n \in \mathbb{N}\} = +\infty$. Then, we can assume without loss of generality that

$$\lim_{n \to +\infty} \mu_n = +\infty. \tag{4.20}$$

Integrating the equation in (1.1) with $\mu = \mu_n$ over the interval $[0,\omega]$, we get

$$0 = \int_0^\omega p(s) u_n(s) \mathrm{d}s - \int_0^\omega h(s) u_n^\lambda(s) \mathrm{d}s + \mu_n \int_0^\omega f(s) \mathrm{d}s \quad \text{for } n \in \mathbb{N},$$

which, in view of (3.1) and the positivity of u_n and μ_n, yields

$$-\int_0^\omega f(s) \mathrm{d}s \leq \frac{\|u_n\|_C}{\mu_n} \int_0^\omega |p(s)| \mathrm{d}s \quad \text{for } n \in \mathbb{N}.$$

Taking into account (4.7), (4.20) and passing the limit for $n \to +\infty$, we obtain $-\int_0^\omega f(s) \mathrm{d}s \leq 0$, which contradicts the second condition in (3.2). ☐

Lemma 4.7. *Let $p \in \mathcal{V}^-(\omega)$ and $z \in AC^1([0,\omega])$ be such that*

$$z''(t) \le p(t)z(t) \quad \text{for a. e. } t \in [0,\omega], \quad z(0) = z(\omega), \quad z'(0) = z'(\omega), \tag{4.21}$$

$$\operatorname{meas}\big\{t \in [0,\omega] : z''(t) < p(t)z(t)\big\} > 0. \tag{4.22}$$

Then, $z(t) > 0$ for $t \in [0,\omega]$.

Proof. It follows from the hypotheses of the lemma that z is a solution to problem (4.6), where $q(t) \le 0$ for a. e. $t \in [0,\omega]$ and $q(t) \not\equiv 0$. Therefore, Lemma 4.5 yields $z(t) > 0$ for $t \in [0,\omega]$. □

Lemma 4.8. *Let $\lambda > 1$, $\mu_0 > 0$, $p \in \mathcal{V}^-(\omega)$, $(p,f) \in \mathcal{U}(\omega)$, h satisfy (3.1), and there exist a positive function $\beta \in AC^1([0,\omega])$ such that (4.1) with $\mu = \mu_0$ and (4.2) hold. Then, for any $\mu \in]0,\mu_0[$, there exist functions $\beta_1, \beta_2 \in AC^1([0,\omega])$ satisfying conditions (4.3), (4.4), and (4.5).*

Proof. Let $\mu \in]0,\mu_0[$ be arbitrary and put $\beta_2(t) := \frac{\mu}{\mu_0}\beta(t)$ for $t \in [0,\omega]$. It follows from (4.1) with $\mu = \mu_0$ and (4.2) that $\beta_2 \in AC^1([0,\omega])$ and

$$\beta_2(t) > 0 \quad \text{for } t \in [0,\omega], \tag{4.23}$$

$$\beta_2(0) = \beta_2(\omega), \quad \beta_2'(0) = \beta_2'(\omega), \tag{4.24}$$

$$\begin{aligned}\beta_2''(t) &\le p(t)\beta_2(t) - \left(\frac{\mu_0}{\mu}\right)^{\lambda-1} h(t)\beta_2^\lambda(t) + \mu f(t) \\ &\le p(t)\beta_2(t) - h(t)\beta_2^\lambda(t) + \mu f(t) \quad \text{for a. e. } t \in [0,\omega],\end{aligned} \tag{4.25}$$

and

$$\operatorname{meas}\big\{t \in [0,\omega] : \beta_2''(t) < p(t)\beta_2(t) - h(t)\beta_2^\lambda(t) + \mu f(t)\big\} > 0, \tag{4.26}$$

because $0 < \mu < \mu_0$ and h satisfies (3.1). By the hypothesis $(p,f) \in \mathcal{U}(\omega)$ and the condition $\mu > 0$, the problem

$$v'' = p(t)v + \mu f(t); \quad v(0) = v(\omega), \ v'(0) = v'(\omega) \tag{4.27}$$

has a unique solution v which is positive. In view of (3.1) and (4.23), conditions (4.25) and (4.27) yield

$$v''(t) \ge p(t)v(t) - h(t)v^\lambda(t) + \mu f(t) \quad \text{for a. e. } t \in [0,\omega] \tag{4.28}$$

and

$$\big(\beta_2(t) - v(t)\big)'' \le p(t)\big(\beta_2(t) - v(t)\big) \quad \text{for a. e. } t \in [0,\omega].$$

Therefore, by (4.24), (4.27), and the hypothesis $p \in \mathcal{V}^-(\omega)$, we get

$$v(t) \le \beta_2(t) \quad \text{for } t \in [0,\omega]. \tag{4.29}$$

Now, by virtue of (4.24), (4.25), (4.27), (4.28), and (4.29), we conclude that the functions v and β form a well-ordered pair of lower and upper functions of (1.1) and, thus, problem (1.1) has a solution β_1 such that

$$v(t) \le \beta_1(t) \le \beta_2(t) \quad \text{for } t \in [0,\omega] \tag{4.30}$$

(see, e. g., [3, Chapter I]). Consequently, the functions β_1, β_2 satisfy conditions (4.4) and (4.5). We finally show that (4.3) holds as well. Indeed, let $z(t) :=$

$\beta_2(t) - \beta_1(t)$ for $t \in [0, \omega]$. Since β_1 is a solution to problem (1.1) and β_2 satisfies (4.24), (4.25), and (4.26), we get

$$z(0) = z(\omega), \quad z'(0) = z'(\omega),$$
$$z''(t) = p(t)z(t) - h(t)\Big(\beta_2^\lambda(t) - \beta_1^\lambda(t)\Big) - \ell(t) \quad \text{for a. e. } t \in [0, \omega],$$

where $\ell \in L([0, \omega])$ is such that

$$\ell(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \quad \ell(t) \not\equiv 0.$$

Therefore, in view of (3.1) and (4.30), the function z satisfies (4.21) and (4.22). Consequently, Lemma 4.7 implies that $\beta_2(t) > \beta_1(t)$ for $t \in [0, \omega]$, which, together with (4.30) and the positivity of v, results in (4.3). □

Lemma 4.9. *Let $\lambda > 1$, $\mu_0 > 0$, $p, h, f \in L([0, \omega])$, h satisfy* (3.1), *and there exist functions $\beta_1, \beta_2 \in AC^1([0, \omega])$ such that* (4.3), (4.4) *with $\mu = \mu_0$, and* (4.5) *hold. Then, there exist $\mu > \mu_0$ and a positive function $\beta \in AC^1([0, \omega])$ satisfying* (4.1) *and* (4.2).

Proof. It is clear that there exist the numbers $d_1 > c_1 > 0$ and $d_2 > c_2 > 0$ such that

$$c_1 \leq \beta_1(t) \leq d_1, \quad c_2 \leq \beta_2(t) - \beta_1(t) \leq d_2 \quad \text{for } t \in [0, \omega]. \tag{4.31}$$

Let $\vartheta \in]0, 1[$ be arbitrary. Put

$$M := \big\{(x_1, x_2) \in \mathbb{R}^2 : c_1 \leq x_1 \leq d_1,\ c_2 \leq x_2 - x_1 \leq d_2\big\}$$

and

$$\ell(x_1, x_2) := \frac{\vartheta x_1^\lambda + (1-\vartheta)x_2^\lambda}{\big[\vartheta x_1 + (1-\vartheta)x_2\big]^\lambda} \quad \text{for } (x_1, x_2) \in M.$$

Since the function $x \mapsto x^\lambda$ is strictly convex on $]0, +\infty[$, we have

$$\big[\vartheta x_1 + (1-\vartheta)x_2\big]^\lambda < \vartheta x_1^\lambda + (1-\vartheta)x_2^\lambda \quad \text{for } 0 < x_1 < x_2,$$

which implies that $\ell(x_1, x_2) > 1$ for $(x_1, x_2) \in M$. The function ℓ is continuous on the compact set M and, thus, there exists $\varepsilon > 1$ such that

$$\varepsilon^{\lambda-1}\big[\vartheta x_1 + (1-\vartheta)x_2\big]^\lambda \leq \vartheta x_1^\lambda + (1-\vartheta)x_2^\lambda \quad \text{for } (x_1, x_2) \in M. \tag{4.32}$$

We now put

$$\beta(t) := \varepsilon\vartheta\beta_1(t) + \varepsilon(1-\vartheta)\beta_2(t) \quad \text{for } t \in [0, \omega].$$

In view of (4.3) and the conditions $\vartheta \in]0, 1[$ and $\varepsilon > 1$, the function β is positive and satisfies (4.2). Moreover, from (3.1), (4.4) with $\mu = \mu_0$, (4.31), and (4.32), we get

$$\begin{aligned}
\beta''(t) &\leq p(t)\beta(t) - h(t)\varepsilon\Big[\vartheta\beta_1^\lambda(t) + (1-\vartheta)\beta_2^\lambda(t)\Big] + \varepsilon\mu_0 f(t) \\
&\leq p(t)\beta(t) - h(t)\varepsilon^\lambda\big[\vartheta\beta_1(t) + (1-\vartheta)\beta_2(t)\big]^\lambda + \varepsilon\mu_0 f(t) \\
&= p(t)\beta(t) - h(t)\beta^\lambda(t) + \varepsilon\mu_0 f(t) \quad \text{for a. e. } t \in [0, \omega],
\end{aligned}$$

i. e., β satisfies also (4.1) with $\mu = \varepsilon\mu_0 > \mu_0$. □

5. Proofs of main results

Proof of Theorem 3.1. Conclusion (1) of the theorem follows immediately from [7, Corollary 2.11].

Put

$$\mathcal{A} := \big\{\mu > 0 : \text{problem (1.1) has a positive solution}\big\}.$$

In view of Lemma 4.3, it is clear that $\mathcal{A} \neq \emptyset$. Let

$$\mu_0 := \sup \mathcal{A}. \tag{5.1}$$

Then, $\mu_0 > 0$ and Lemma 4.6 implies that $\mu_0 < +\infty$. Therefore, conclusion (4) of the theorem holds.

We now show that

$$\mu_0 \in \mathcal{A}. \tag{5.2}$$

Indeed, let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\mu_n \in \mathcal{A} \quad \text{for } n \in \mathbb{N}, \qquad \lim_{n\to+\infty} \mu_n = \mu_0.$$

Moreover, for any $n \in \mathbb{N}$, let u_n be a positive solution to problem (1.1) with $\mu = \mu_n$. Then, Lemma 4.6 yields (4.7). By the standard arguments used in the proof of a well-posedness of the periodic problem for second-order ODEs, one can show that there exists a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to+\infty} u_{n_k}^{(i)}(t) = u_0^{(i)}(t) \quad \text{ uniformly on } [0,\omega],\ i = 0,1,$$

where $u_0 \in AC^1([0,\omega])$ is a solution to problem (1.1) with $\mu = \mu_0$. All the functions u_{n_k} are positive and, thus, it is clear that

$$u_0(t) \geq 0 \quad \text{for } t \in [0,\omega]. \tag{5.3}$$

By virtue of the hypothesis $(p,f) \in \mathcal{U}(\omega)$, problem (4.16) has a unique solution v which is positive. Since u_0 is a solution to problem (1.1) with $\mu = \mu_0$, by (3.1), (5.3), and (4.16), we get (4.21), where $z(t) := u_0(t) - \mu_0 v(t)$ for $t \in [0,\omega]$. Therefore, the hypothesis $p \in \mathcal{V}^-(\omega)$ yields $z(t) \geq 0$ for $t \in [0,\omega]$. Hence, we have

$$u_0(t) \geq \mu_0 v(t) > 0 \quad \text{for } t \in [0,\omega]$$

and, thus, condition (5.2) holds.

Having a positive solution u_0 to problem (1.1) with $\mu = \mu_0$, it is clear that all the hypotheses of Lemma 4.8 (with $\beta(t) := u_0(t)$) are fulfilled. Consequently, for any $\mu \in]0,\mu_0[$, $(p,\mu f) \in \mathcal{U}(\omega)$ and there exist functions $\beta_1, \beta_2 \in AC^1([0,\omega])$ satisfying (4.3), (4.4), and (4.5). Therefore, conclusion (2) of the theorem follows from Lemma 4.2.

Since u_0 is a positive solution to problem (1.1) with $\mu = \mu_0$, to prove conclusion ((3)) of the theorem, it is sufficient to show that problem (1.1) with $\mu = \mu_0$ has at most one positive solution. Suppose on the contrary that there exists a positive solution to problem (1.1) with $\mu = \mu_0$ different from u_0. Since $\mu_0 > 0$, (3.2) yields $(p,\mu_0 f) \in \mathcal{U}(\omega)$ and, thus, it follows from Lemma 4.1 (with $\beta(t) := u_0(t)$ and $\mu := \mu_0$) that problem (1.1) with $\mu = \mu_0$ possesses solutions u_*, u^* such that

$$u^*(t) > u_*(t) > 0 \quad \text{for } t \in [0,\omega].$$

Therefore, Lemma 4.9 (with $\beta_1(t) := u_*(t)$ and $\beta_2(t) := u^*(t)$) guarantees that there exist $\tilde{\mu} > \mu_0$ and a positive function $\beta \in AC^1([0,\omega])$ satisfying (4.1) with $\mu = \tilde{\mu}$ and (4.2). Consequently, in view of the hypothesis $(p,f) \in \mathcal{U}(\omega)$ and the positivity of $\tilde{\mu}$, it follows from Lemma 4.1 (with $\mu := \tilde{\mu}$) that problem (1.1) with $\mu = \tilde{\mu}$ has at least one positive solution. However, this implies $\tilde{\mu} \in \mathcal{A}$, which contradicts (5.1). ☐

Proof of Proposition 3.4. By Remark 3.3, it follows from (3.8) that condition (3.2) holds. Let μ_0 be the number appearing in the conclusion of Theorem 3.1.

We first show that μ_0 satisfies (3.10). Suppose on the contrary that (3.10) does not hold, i. e.,

$$\mu_0 < \frac{(\lambda-1)\,[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\int_0^\omega [f(s)]_-\mathrm{d}s}\,.$$

Then, it follows from Lemmas 4.3 and 4.2 that problem (1.1) with $\mu = \mu_0$ has at least two positive solutions, which contradicts conclusion (3) of Theorem 3.1.

Now we show that μ_0 satisfies (3.11). Suppose on the contrary that (3.11) does not hold, i. e.,

$$\mu_0 \geq \frac{(\lambda-1)\left[\mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+\mathrm{d}s}\int_0^\omega [p(s)]_+\mathrm{d}s - \int_0^\omega [p(s)]_-\mathrm{d}s\right]^{\frac{\lambda}{\lambda-1}}}{\lambda\left[\lambda\int_0^\omega h(s)\mathrm{d}s\right]^{\frac{1}{\lambda-1}}\left[\int_0^\omega [f(s)]_-\mathrm{d}s - \mathrm{e}^{\frac{\omega}{4}\int_0^\omega [p(s)]_+\mathrm{d}s}\int_0^\omega [f(s)]_+\mathrm{d}s\right]}\,.$$

Then, it follows from Lemma 4.4 that problem (1.1) with $\mu = \mu_0$ has no positive solution, which contradicts conclusion (3) of Theorem 3.1. ☐

6. Model examples

In this section, we consider the model equation (1.3) with F given by the relations in (1.4) in order to demonstrate a possible use of Theorem 3.1.

Let us choose $\omega > 0$ and consider the equation

$$x'' = \frac{k(\ell - d(t))}{md(t)}\,x - \frac{k\ell}{2md^3(t)}\,x^3 - \frac{f_0}{m}\,, \tag{6.1}$$

where $m, k, \ell, f_0 > 0$ and $d\colon \mathbb{R} \to]0,+\infty[$ is a positive ω-periodic function such that $d(t) \not\equiv \ell$ and

$$\int_0^\omega \left[\frac{\ell - d(s)}{d(s)}\right]_- \mathrm{d}s < \frac{4m}{\omega k}\,, \qquad \int_0^\omega \left[\frac{\ell - d(s)}{d(s)}\right]_+ \mathrm{d}s < \frac{\int_0^\omega \left[\frac{\ell-d(s)}{d(s)}\right]_-\mathrm{d}s}{1 - \frac{\omega k}{4m}\int_0^\omega \left[\frac{\ell-d(s)}{d(s)}\right]_-\mathrm{d}s}\,.$$

Observe that equation (6.1) is a Duffing equation with non-constant coefficients and a constant forcing term. By Remarks 3.3 and 3.5, we get

$$\frac{k(\ell - d(\cdot))}{md(\cdot)} \in \mathcal{V}^-(\omega), \qquad \left(\frac{k(\ell - d(\cdot))}{md(\cdot)}, -\frac{1}{m}\right) \in \mathcal{U}(\omega).$$

Therefore, assuming that f_0 is a bifurcation parameter, it follows from Theorem 3.1 that there exists a critical value $f_0^* > 0$ of f_0 such that, crossing the value f_0^*, a bifurcation of positive ω-periodic solutions to (6.1) occurs.

As a second example, we consider the equation

$$x'' = \frac{k(\ell - d_0)}{md_0}\, x - \frac{k\ell}{2md_0^3}\, x^3 + \frac{A}{m}\left(\sin\frac{2\pi t}{\omega} - \frac{1}{2}\right), \tag{6.2}$$

where $A, \omega > 0$ and $m, k, \ell, d_0 > 0$ such that $d_0 < \ell$ and

$$\int_0^\omega \left[\sin\frac{2\pi s}{\omega} - \frac{1}{2}\right]_- \mathrm{d}s > \mathrm{e}^{\frac{\omega^2 k(\ell - d_0)}{4md_0}} \int_0^\omega \left[\sin\frac{2\pi s}{\omega} - \frac{1}{2}\right]_+ \mathrm{d}s.$$

Unlike the first example, equation (6.2) is a Duffing equation with constant coefficients and a sign-changing forcing term. By Remarks 2.2 and 3.5, we get

$$\frac{k(\ell - d_0)}{md_0} \in \mathcal{V}^-(\omega), \qquad \left(\frac{k(\ell - d_0)}{md_0}, \frac{g(\cdot)}{m}\right) \in \mathcal{U}(\omega),$$

where $g(t) := \sin\frac{2\pi t}{\omega} - \frac{1}{2}$. Therefore, if we consider A as a bifurcation parameter, it follows from Theorem 3.1 that there exists a critical value $A^* > 0$ of A such that, crossing the value A^*, a bifurcation of positive ω-periodic solutions to (6.2) occurs.

We finally mention that Proposition 3.4 provides lower and upper estimates of the critical values f_0^* and A^* of bifurcation parameters f_0 and A.

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