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AN INCREMENTAL METHOD FOR THE CONSTRUCTION OF THE BOX EXTENTS OF A CONTEXT

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Abstract. In this paper we are improving a method proposed in [2] for the construction of the box extents of a given formal context. We prove that the lattice of the box extents can be order-embedded in the lattice generated by the atomic extents of the given context.

1. Introduction

The notion of the box extents of a context and the box elements of a concept lattice are related to the application of Formal Concept Analysis [1] in data classification by defining partitions of a universe putting its elements with similar attributes into the same class, see [4, 6] and [2]. This task transparently appears in some clustering problems originating in database analysis, as well as in the engineering discipline Group Technology, see, e.g., [9] or [8]. This discipline exploits the similarities between technological objects and divides them into (relatively) homogeneous groups in order to optimize the manufacturing processes. In [2] it is proved that the box extents of a formal context form a complete atomistic lattice and two methods for the construction of this lattice are also presented. The second one has an incremental character, it is based on consecutive one-object extensions of a small context. Since, during a one-object extension, the number of box extents can even be doubled, we present a construction and an algorithm to avoid this problem. We also prove that the lattice of box extents can be order-embedded in the lattice generated by the atomic extents of a formal context. These results are presented in Section 3. Section 2 contains some preliminary notions and notations, i.e., the prerequisites of our investigations.

2. Preliminaries

For the lattice-theoretic terminology used we refer to [3]. A *formal context* is a triple $\mathcal{K} = (G, M, I)$, where G and M are sets and $I \subseteq G \times M$ is a binary relation. G is called the *object set* and M the *attribute set* of the context $\mathcal{K}$. The basic notions of Formal Concept Analysis can be found, e.g., in [1] or [10]. There

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are defined

$$A^I = \{m \in M \mid gIm \text{ for all } g \in A\},$$
$$B^I = \{g \in G \mid gIm \text{ for all } m \in B\}$$

for all subsets $A \subseteq G$ and $B \subseteq M$. We will use the notations $(A^I)^I = A^{II}$ and $(B^I)^I = B^{II}$. For any $g \in G$, we will write g^I and g^{II} instead of $\{g\}^I$ and $\{g\}^{II}$. The obtained maps $A \to A^I$, $A \subseteq G$ and $B \to B^I$, $B \subseteq M$ constitute a Galois connection between the power-set lattices $\wp(G)$ and $\wp(M)$, and the maps $A \to A^{II}$, $A \subseteq G$ and $B \to B^{II}$, $B \subseteq M$ are closure operators on $\wp(G)$ and $\wp(M)$, respectively. A *formal concept* of the context $\mathcal{K}$ is a pair $(A, B) \in \wp(G) \times \wp(M)$ with $A^I = B$ and $B^I = A$, where the set A is called the *extent* and B is called the *intent* of the concept (A, B). It is easy to check that a pair $(A, B) \in \wp(G) \times \wp(M)$ is a concept if and only if $(A, B) = (A^{II}, A^I) = (B^I, B^{II})$. The set of all concepts of the context (G, M, I) is denoted by $\mathcal{L}(G, M, I)$, and $\mathcal{E}(G, M, I)$ stands for the set of all concept extents of (G, M, I). Ordering $\mathcal{L}(G, M, I)$ as follows

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2,$$

we obtain a complete lattice, the *concept lattice of the context* $\mathcal{K} = (G, M, I)$. Similarly, ordering the extents of $\mathcal{K}$ by $\subseteq$, we get a lattice $(\mathcal{E}(G, M, I), \subseteq)$ isomorphic to the concept lattice $\mathcal{L}(G, M, I)$. Now, let us consider the concepts $\gamma(x) = (x^{II}, x^I)$ for any $x \in G$. It can be easily proved that for any concept $(A, B) \in \mathcal{L}(G, M, I)$, we have in $\mathcal{L}(G, M, I)$

$$(A, B) = \bigvee\{\gamma(x) \mid x \in A\}. \tag{2.1}$$

In view of (2.1), any atom in the lattice $\mathcal{L}(G, M, I)$ has the form (x^{II}, x^I), where $x \in G$. In this case, the extent x^{II} being also an atom in $\mathcal{E}(G, M, I)$ is called an *atomic extent*, and $x \in G$ is called an *atomic object*. If every nonzero element of a lattice L is a join of atoms of L, then L is called an *atomistic lattice*.

An *extent partition* of a formal context (G, M, I) is a partition of the set G, all blocks of which are concept extents. Clearly, the *trivial partition* $\{G\}$ is always an extent partition. Note that, since the intersection of extents always yields an extent, the common refinements of extent partitions are still extent partitions. Therefore, the extent partitions of (G, M, I) form a complete $\cap$-subsemilattice of the partition lattice of G and, thus, a complete lattice which will be denoted by $\mathrm{Ext}(G, M, I)$. In particular, there is always a finest extent partition of the context denoted by $\pi_\square$. In [2], an algorithm for finding $\pi_\square$ was provided and it was pointed out that this can be done in $\mathbf{O}(\mid G \mid^2 \mid M \mid)$ steps. The next definition is also from [2]:

Definition 2.1. A set $E \subseteq G$ is called a *box extent* of be the context $\mathcal{K} = (G, M, I)$ if E is a class of some extent partition of $\mathcal{K}$ or $E = \varnothing$ if $\varnothing^{II} = \varnothing$.

(Note that if $\varnothing^{II} \neq \varnothing$, then $\{G\}$ is the only extent partition of the context (G, M, I)). The set of all box extents of $\mathcal{K}$ is denoted by $\mathcal{BE}(G, M, I)$. Observe that each object $g \in G$ is contained in a smallest box extent denoted by $g^{\square\square}$, which is the class of the finest extent partition $\pi_\square$ of the context (G, M, I) containing g.

Hence, according to [2],

$$E \in \mathcal{E}(G, M, I) \text{ is a box extent if and only if } g \in E \Longrightarrow g^{\Box\Box} \subseteq E.$$

As a consequence, we get that the intersection of box extents is also a box extent. As $g^{\Box\Box}$ is a box extent of (G, M, I), the inclusions $\{g\} \subseteq g^{II} \subseteq g^{\Box\Box}$ always hold. In [2], it was proved that, ordering the box extents by $\subseteq$, we obtain a complete atomistic lattice having as atoms the classes of the finest extent partition $\pi_\Box$. Moreover, this lattice $\mathcal{BE}(G, M, I)$ is a complete $\bigcap$-subsemilattice of $\mathcal{E}(G, M, I)$. If A is a box extent, then the concept (A, A^I) is called a *box element* of the lattice $\mathcal{L}(G, M, I)$, see [4]. The set of the box elements of (G, M, I) is denoted by $\mathcal{B}(G, M, I)$.

2.1. Box extents of a subcontext

Let $\mathcal{K} = (G, M, I)$ be a formal context and $H \subseteq G$. It is well-known that, for each extent E of (G, M, I), the restriction $E \cap H$ is an extent of the subcontext $(H, M, I \cap H \times M)$. Similarly, if E is a box extent of (G, M, I), then $E \cap H$ is a box extent of $(H, M, I \cap H \times M)$, according to [2]. In [2] it is also proved that the finite extent partition $\pi_\Box^H$ of $(H, M, I \cap H \times M)$ is a refinement of the restriction of $\pi_\Box$ to H. We say that the context (G, M, I) is a *one-object extension* of $(H, M, I \cap H \times M)$ if there exists an element $z \in G$ such that $H = G \setminus \{z\}$.

The following well-known lemma will be useful in our proofs.

Lemma 2.2. (i) *Any extent E of a finite formal context (G, M, I) contains an atomic object.*
(ii) *If $A \subseteq H \subseteq G$, then $A^{II} \cap H$ is an extent of the subcontext $(H, M, I \cap H \times M)$.*

Proof. (i) holds since any extent of a finite context contains an atomic extent. As A^{II} is an extent of (G, M, I), $A^{II} \cap H$ is an extent of $(H, M, I \cap H \times M)$, hence, (ii) holds. $\Box$

The next lemma follows from [2], however, to make our paper self-contained, we provide a direct proof of it.

Lemma 2.3. *Let $\mathcal{K} = (G, M, I)$ be a context with $\emptyset^{II} = \emptyset$, and $A \subseteq G$ the nonempty set of its atomic objects. Then, the box extents of the context $(A, M, I \cap A \times M)$ coincide with its extents and $\mathcal{E}(A, M, I \cap A \times M)$ is an atomistic lattice.*

Proof. First, observe that, for all $a \in A$, $a^{II} \subseteq A$. Indeed, let $a \in A$ and $g \in a^{II}$. Then, $g^{II} = a^{II}$ yields that g is also an atomic object and hence $g \in A$. As $\emptyset$ is an extent, the atomic extents a^{II} are mutually disjoint. Since $A = \bigcup\{a^{II} \mid a \in A\}$, we get that $\{a^{II} \mid a \in A\}$ is the least extent partition $\pi_\Box$ of $(A, M, I \cap A \times M)$, and for this context a^{II} coincides to $a^{\Box\Box}$, i.e., to that class of its $\pi_\Box$ which contains a. Now, let E be an arbitrary extent of the context $(A, M, I \cap A \times M)$. Then, $a \in E$ implies that $a^{\Box\Box} = a^{II} \subseteq E$ and this means that E is a box extent. The remaining part of the proof is an obvious consequence of the fact that $\mathcal{BE}(A, M, I \cap A \times M)$ is an atomistic lattice. $\Box$

3. One-object extensions of a context and the proposed incremental method

The following lemma shows that one-object extension can be used for an inductive construction of the box extents:

Lemma 3.1. *Let $\mathcal{K} = (G, M, I)$ be a finite formal context, $B \neq \emptyset$ a box extent of $\mathcal{K}$, $H \subseteq G$, and $E = B \cap H$. Then, E is a box extent of the subcontext*

$$(H, M, I \cap H \times M)$$

and it is extended by successive one-object extensions to B in such a way that, after each step, a box extent of the wider subcontext is obtained.

Proof. Since G is finite, it has the form $G = H \cup \{z_1, ..., z_k\}$, $z_1, ..., z_k \notin H$. Hence, either $B = E$, or we have $B = E \cup \{z_1, ..., z_l\}$ with $1 \leq l \leq k$. Set $H_j := H \cup \{z_1, ..., z_j\}$, $1 \leq j \leq l$. Now, the proof is a trivial consequence of the fact that, for each $j \leq l$, $B \cap H_j = E \cup \{z_1, ..., z_j\}$ is a box extent of $(H_j, M, I \cap H_j \times M)$, and it is formed by a one-object extension with z_j from the box extent $E \cup \{z_1, ..., z_{j-1}\}$. □

In this section we use the results of the article [2], where the authors have shown how the box extents of a context are changed by a one-object extension.

Proposition 3.2. ([2, Prop. 4.5]) *If E is a box extent of (G, M, I) and $H = G \setminus \{z\}$ for some $z \in G$, then either*

(i) *E is a box extent of $(H, M, I \cap H \times M)$ with $E \cap z^{\square\square} = \emptyset$ or*
(ii) *$E \setminus \{z\}$ is a box extent of $(H, M, I \cap H \times M)$.*

In view of the above proposition, in case of a one-object extension, any box extent of the wider context (G, M, I) can be derived from a box extent of the subcontext $(H, M, I \cap H \times M)$. The next theorem shows exactly the conditions under which a box extent of the smaller context "gives life" to a box extent of (G, M, I).

Theorem 3.3. ([2, Thm 5.5]) *Let (G, M, I) be a context, E a box extent of the subcontext $(H, M, I \cap H \times M)$ with $H = G \setminus \{z\}$. Then*

(i) *E is a box extent of (G, M, I) if and only if $z^{\square\square} \cap E'' = \emptyset$;*
(ii) *$E^* = E \cup \{z\}$ is a box extent of (G, M, I) if and only if $z^{\square\square} \setminus \{z\} \subseteq E$ and $(E \cup \{z\})'' = E \cup \{z\}$.*

Observe that, in view of the above theorem, in case of a one-object extension, we have two possibilities:

1)E is also a box extent in the new context iff $z^{\square\square} \cap E'' = \emptyset$;

2) or $E \cup \{z\}$ is a box extent in the new context iff $z^{\square\square} \setminus \{z\} \subseteq E$ and $(E \cup \{z\})'' = E \cup \{z\}$.

By applying this theorem, in [2] an algorithm for determining the list of the box extents of (G, M, I) was constructed by using the list of the box extents of the subcontext $(H, M, I \cap H \times M)$. Unfortunately, the theorem shows that, during a one-object extension of the context $(H, M, I \cap H \times M)$, the box extents can be even doubled, for instance, in the case when $z^{\square\square} = \{z\}$. This means that the algorithm discussed in [2] can have an exponential time. In order to eliminate this problem, we first prove:

Proposition 3.4. *Let $\mathcal{K} = (G, M, I)$ be a finite formal context, $H \subseteq G$ a nonempty subset containing all the atomic objects of $\mathcal{K}$, E a box extent of the subcontext $(H, M, I \cap H \times M)$, and $z \in G \setminus H$. Set $H_* := H \cup \{z\}$ and $I_* = I \cap H_* \times M$. Then, the following cases exclude each other.*

1. *E is a box extent of (H_*, M, I_*);*
2. *$E \cup \{z\}$ is a box extent of (H_*, M, I_*).*

Proof. Assume that the cases (1) and (2) hold simultaneously and denote by $z^{\Box\Box_*}$ the class of the least extent partition of $\mathcal{K}_* := (H_*, M, I_*)$ that contains the element z. Then, in view of Theorem 3.3, both conditions $z^{\Box\Box_*} \cap E^{I_* I_*} = \emptyset$ and $z^{\Box\Box_*} \setminus \{z\} \subseteq E$ are satisfied. Hence, we get

$$z^{\Box\Box_*} \setminus \{z\} = (z^{\Box\Box_*} \setminus \{z\}) \cap E \subseteq z^{\Box\Box_*} \cap E^{I_* I_*} = \emptyset.$$

Since $\{z\} \subseteq z^{I_* I_*} \subseteq z^{\Box\Box_*}$, we obtain $z^{\Box\Box_*} = z^{I_* I_*} = \{z\}$. On the other hand, because $\mathcal{K}$ is a finite context, there exists at least one atomic object $a \in G$ with $a^{II} \subseteq z^{II}$. Now, $a \in H$, according to our assumption. Thus, we obtain:
$a \in a^{II} \cap H \subseteq z^{II} \cap H_* = z^{I_* I_*} = \{z\}$, i.e., $z = a$,
which is a contradiction because $z \notin H$ by our construction. □

Another problem, which arises when the box extents of a formal context $\mathcal{K} = (G, M, I)$ are constructed by successive one-element extensions from the box extents of a subcontext $(H, M, I \cap H \times M)$, is the requirement to construct in each step the class $z^{\Box\Box_*}$, i.e., the class of the least extent partition of the extended context (H_*, M, I_*) containing the new element z. The next proposition shows that $z^{\Box\Box_*}$ can be always replaced by the class $z^{\Box\Box}$ of the extent partition $\pi_\Box$ of the whole $\mathcal{K}$.

Proposition 3.5. *Let B be a box extent of the context $\mathcal{K} = (G, M, I)$ $H \subseteq G$ a nonempty subset, B a box extent of $\mathcal{K}$, and $z \in G \setminus H$ arbitrary. Set $E := B \cap H$, and $H_* := H \cup \{z\}$, $I_* := I \cap H_* \times M$. If $z^{\Box\Box} \cap E^{I_* I_*} \neq \emptyset$, then $E \cup \{z\}$ is a box extent of the subcontext (H_*, M, I_*) and the conditions $\big(z^{\Box\Box} \cap H_*\big) \setminus \{z\} \subseteq E$, $(E \cup \{z\})^{I_* I_*} = E \cup \{z\}$ hold.*

Proof. Clearly, E is a box extent of the subcontext $(H, M, I \cap H \times M)$. Since $E^{I_* I_*} \subseteq E^{II} \subseteq B^{II} = B$, by our assumption, we have $z^{\Box\Box} \cap B \neq \emptyset$. Since B is a box extent of $\mathcal{K}$, it is a union of some classes of the finest extent partition $\pi_\Box$ of $\mathcal{K}$ and, hence, $z^{\Box\Box} \cap B \neq \emptyset$ yields $z \in z^{\Box\Box} \subseteq B$. Thus, we obtain:

$$B \cap H_* = B \cap (H \cup \{z\}) = (B \cap H) \cup \{z\} = E \cup \{z\},$$

and hence $E \cup \{z\}$ is a box extent of (H_*, M, I_*). Then $(E \cup \{z\})^{I_* I_*} = E \cup \{z\}$ also holds. Now, suppose by contradiction that $\big(z^{\Box\Box} \cap H_*\big) \setminus \{z\} \subsetneq E$. Then, there is a $g \in z^{\Box\Box} \cap H_*$, $g \neq z$ with $g \notin E$. However $g \in H_* \setminus \{z\} = H$ yields $g \in z^{\Box\Box} \cap H \subseteq B \cap H = E$, which is a contradiction. Thus, $\big(z^{\Box\Box} \cap H_*\big) \setminus \{z\} \subseteq E$. □

Remark 3.6. Assume now that conditions of Proposition 3.5 are satisfied, let $\pi^*_\Box$ stand for the finest extent partition of the context $\mathcal{K}(H_*) = (H_*, M, I_*)$ and denote the class of the element z in $\pi^*_\Box$ by $z^{\Box\Box_*}$ and its class in $\pi_\Box$ by $z^{\Box\Box}$. We show that, during the construction of the box extents of $\mathcal{K} = (G, M, I)$ by successive one-object extensions, $z^{\Box\Box_*}$ can be replaced by $z^{\Box\Box}$ in checking the condition $z^{\Box\Box} \cap E^{I_* I_*} = \emptyset$, $\left(z^{\Box\Box} \cap H_*\right) \setminus \{z\} \subseteq E$, $\left(E \cup \{z\}\right)^{I_* I_*} = E \cup \{z\}$ (where $E \subseteq H_*$), respectively, and this will not alter the list of the box extents obtained.

Indeed, $\pi^*_\Box$ is a refinement of the restriction of $\pi_\Box$ to H_*, hence $z^{\Box\Box_*} \subseteq z^{\Box\Box}$, and $z^{\Box\Box_*} \setminus \{z\} \subseteq \left(z^{\Box\Box} \cap H_*\right) \setminus \{z\}$. Then, $z^{\Box\Box} \cap E^{I_* I_*} = \emptyset$ implies $z^{\Box\Box_*} \cap E^{I_* I_*} = \emptyset$, and in view of Theorem 3.3 this means that E is a box extent of $\mathcal{K}(H_*)$; if $\left(z^{\Box\Box} \cap H_*\right) \setminus \{z\} \subseteq E$, then $z^{\Box\Box_*} \setminus \{z\} \subseteq E$ and by Theorem 3.3 we get that $E \cup \{z\}$ is a box extent of $\mathcal{K}(H_*)$. Hence, all the box extents found belong to $\mathcal{K}(H_*)$. As the successive extensions are ending with (G, M, I), the box extents obtained in the last step are box extents of (G, M, I).

Conversely, let B be a box extent of $\mathcal{K}$, as in Proposition 3.5. Then, $B \cap H = E$ is a box extent of $(H, M, I \cap H \times M)$ and, in view of Lemma 3.1, it can be extended into B by successive one-object extensions. In order to prove that each box extent of $\mathcal{K}$ is found by using our new conditions, assume that neither E nor $E \cup \{z\}$ is decided to be a box extent by checking our conditions for an extension with $z \in G \setminus H$. Then, necessarily, $z^{\Box\Box} \cap E^{I_* I_*} \neq \emptyset$. However, in view of Proposition 3.5, $B \cap H = E$ and $z^{\Box\Box} \cap E^{I_* I_*} \neq \emptyset$ imply $\left(z^{\Box\Box} \cap H_*\right) \setminus \{z\} \subseteq E$ and $\left(E \cup \{z\}\right)^{I_* I_*} = E \cup \{z\}$. The last two relations mean that $E \cup \{z\}$ is recognized as a box extent of the context $\mathcal{K}(H_*)$ by our new method, contrary to the assumption. This contradiction shows that all the box extents of $\mathcal{K}$ are generated by checking the new, modified conditions.

Now, based on Propositions 3.4 and 3.5, and Lemma 3.1, we present an improved method for determining the box extents of a context by successive one-object extensions, eliminating the above-mentioned problems. Its steps are the following:

- First we select atomic objects, i.e., those $g \in G$ from the context $\mathcal{K} = (G, M, I)$ for which (g^{II}, g^I) is an atom and we store them in a list H_0.
- We construct the context $\mathcal{K}_0 := (H_0, M, I \cap H_0 \times M)$ and the concept lattice $\mathcal{L}(H_0, M, I \cap H_0 \times M)$ using some well-known method (see, e.g., [7]). In view of Lemma 2.3, the extent lattice $\mathcal{E}(H_0, M, I \cap H_0 \times M)$ and its box extent lattice $\mathcal{BE}(H_0, M, I \cap H_0 \times M)$ are identical. Thereafter, the finest extent partition $\pi_\Box$ of $\mathcal{K} = (G, M, I)$ is also constructed.
- Now, we add the remaining elements $z_j \in G \setminus H_0$ $(j = 1, ..., n)$ one by one to the actual subcontext $\mathcal{K}_{j-1} := (H_{j-1}, M, I \cap H_{j-1} \times M)$, determining in each step the box extents of the new context $\mathcal{K}_j := (H_j, M, I \cap H_j \times M)$ (where $H_j := H_{j-1} \cup \{z_j\}$) by using the class $z_j^{\Box\Box}$ of z_j in the extent partition $\pi_\Box$ and the list of the box extents of $\mathcal{K}_{j-1}$.

The last inductive step is carried out by the following algorithm:

Algorithm 3.7. Algorithm for determining the list L of the box extents of $\mathcal{K}$ given the set H_0 of atomic objects, the list $\mathcal{E}_0$ of the box extents of $\mathcal{K}_0 := (H_0, M, I \cap H_0 \times M)$, and the list of the box extents contained in $\pi_\Box$.

begin
    $S := \mathcal{E}_0$, $H := H_0$
    while $G \setminus H \neq \emptyset$ do
        $H := H \cup \{z\}$, $z \in G \setminus H$
        $I_* := I \cap H \times M$, $L_* := \{z^{\Box\Box} \cap H\}$
        for $E \in S$ do
            if $E^{I_* I_*} \cap z^{\Box\Box} = \emptyset$ then $L_* := L_* \cup \{E\}$
            if $z^{\Box\Box} \setminus z \subseteq E$ then $E^* := E \cup \{z\}$
                if $(E^*)^{I_* I_*} = E^*$ then $L_* := L_* \cup E^*$
                end if
            end if
        end for
        $S := L_*$
    end while
    $L := S$
end

It is known that the lattice $L_0 := \mathcal{L}(H_0, M, I \cap H_0 \times M)$ can be constructed in $\mathbf{O}(\mid H_0 \mid^2 \mid M \mid\mid L_0 \mid)$ steps (see, e.g., [5]) and $\pi_\Box$ can be computed in $\mathbf{O}(\mid G \mid^2 \mid M \mid)$ (see [2]) steps. In the worst case (when every $g \in G$ is an atomic object) $H_0 = G$ and, hence, $\mathbf{O}(\mid H_0 \mid^2 \mid M \mid\mid L_0 \mid) = \mathbf{O}(\mid G \mid^2 \mid M \mid\mid L_0 \mid)$. As in each step of the above inductive part at most $\mid L_0 \mid$ extents are examined, the box extents of the whole context can be checked in at most $\mathbf{O}(\mid L_0 \mid\mid G \mid)$ steps. Thus, in total, we need at most

$$\mathbf{O}\left(\mid G \mid^2 \mid M \mid\mid L_0 \mid\right) + \mathbf{O}\left(\mid G \mid^2 \mid M \mid\right) + \mathbf{O}\left(\mid L_0 \mid\mid G \mid\right) = \mathbf{O}\left(\mid G \mid^2 \mid M \mid\mid L_0 \mid\right)$$

steps. In the worst case, $\mathbf{O}(\mid L_0 \mid)$ is the same as $\mathbf{O}(\mid \mathcal{L}(G, M, I) \mid)$.

It is also clear that, in each step, the number of the box extents cannot increase, because a box extent of a subcontext $(H, M, I \cap H \times M)$ is either preserved by a one-object extension or extended with the new element or it disappears and these cases exclude each other according to Proposition 3.4. Therefore, (denoting the set of atomic objects by A), we obtain

$$\mid \mathcal{BE}(G, M, I) \mid \leq \mid \mathcal{L}(A, M, I \cap A \times M) \mid .$$

The next theorem states even more:

Theorem 3.8. *Let $\mathcal{K} = (G, M, I)$ be a finite formal context and $S \subseteq G$ a subset containing all the atomic objects of $\mathcal{K}$. Then, the box extent lattice of $\mathcal{K}$ can be order-embedded in the lattice $\mathcal{BE}(S, M, I \cap S \times M)$.*

Proof. Since, in view of [2, Cor. 4.4], for any box extent E of $\mathcal{K}$, $E \cap S$ is a box extent of the subcontext $\mathcal{K}_S := (S,\ M, I \cap S \times M)$, we can define the mapping

$$\varphi\colon \mathcal{BE}(G, M, I) \to \mathcal{BE}(S, M, I \cap S \times M),\ \varphi(E) = E \cap S,\ E \in \mathcal{BE}(G, M, I).$$

Obviously, φ is order-preserving. In order to prove that φ is an order-embedding, suppose that $\varphi(E_1) \subseteq \varphi(E_2)$ for some $E_1, E_2 \in \mathcal{BE}(G, M, I)$. Then, $E_1 \cap S \subseteq E_2 \cap S$, $E_1 \cap E_2 \in \mathcal{BE}(G, M, I)$, and $\varphi(E_1 \cap E_2) \subseteq \varphi(E_1)$. Assume that $E_1 \cap E_2 \neq E_1$. This means that there exists an element $z \in E_1 \setminus (E_1 \cap E_2)$. Now, let us define the sets $H := S \cup (E_1 \cap E_2)$ and $H^* := H \cup \{z\} = S \cup (E_1 \cap E_2) \cup \{z\}$ and consider the corresponding subcontexts $\mathcal{K}_H = (H, M, I \cap H \times M)$ and $\mathcal{K}_{H^*} = (H^*, M, I \cap H^* \times M)$. Clearly, $\mathcal{K}_{H^*}$ is obtained from the subcontext $\mathcal{K}_H$ by a one-object extension using the object z, and $E_1 \cap E_2 = (E_1 \cap E_2) \cap H = (E_1 \cap E_2) \cap H^*$ is a box extent of both contexts.

On the other hand, $E_1 \cap S = (E_1 \cap S) \cap (E_2 \cap S) \subseteq E_1 \cap E_2$ yields

$$\begin{aligned} E_1 \cap H^* &= E_1 \cap (S \cup (E_1 \cap E_2) \cup \{z\}) = (E_1 \cap S) \cup (E_1 \cap E_2) \cup \{z\} \\ &= (E_1 \cap E_2) \cup \{z\}, \end{aligned}$$

and, hence, $(E_1 \cap E_2) \cup \{z\}$ is a box extent of $(H^*, M, I \cap H^* \times M)$. Therefore, after a one-object extension of the context $\mathcal{K}_H$, both $E_1 \cap E_2$ and $(E_1 \cap E_2) \cup \{z\}$ become box extents in the wider context $\mathcal{K}_{H^*}$. Since the set $H \supseteq S$ contains all the atomic objects of the initial context $\mathcal{K}$, this is not possible by Proposition 3.4. This means that $E_1 \cap E_2 = E_1$ must hold and hence $E_1 \subseteq E_2$. Thus, φ is an order-embedding. □

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